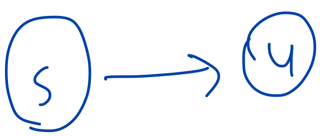
Apr 21

Solution: Introduce an implicit parameter in sub-problems

Attempt 4: $OPT(s, E') \to$ cost of shortest s-t path only using edges in E'

(2) Recursion



If a shortest path uses s-u edge

$$OPT(s,E) = c_{s,u} + OPT(u, E \setminus \{s,u\})$$

More generally: $$OPT(s,E) = \min_{w:(s,w)\in E} \{ c_{s,w} + OPT(w, E \setminus \{s,w\}) \}$$

(3) Ordering? Increasing order of $|E'|$

(1) How many sub-problems? $n \cdot 2^m \to$ exponential!

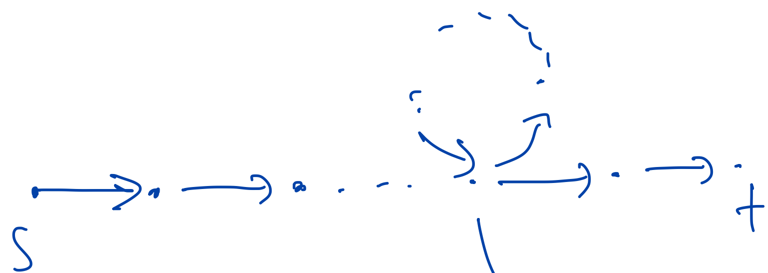
Attempt 5 : Bellman-Ford Algo

$OPT(s, i)$ = cost of a shortest-path with $\leq i$ edges

$s \in V \;, i \geq 0$

Prop: If G has no negative cycle $\Rightarrow \forall s \;\exists$ shortest s-t path that is simple

Pf(idea):

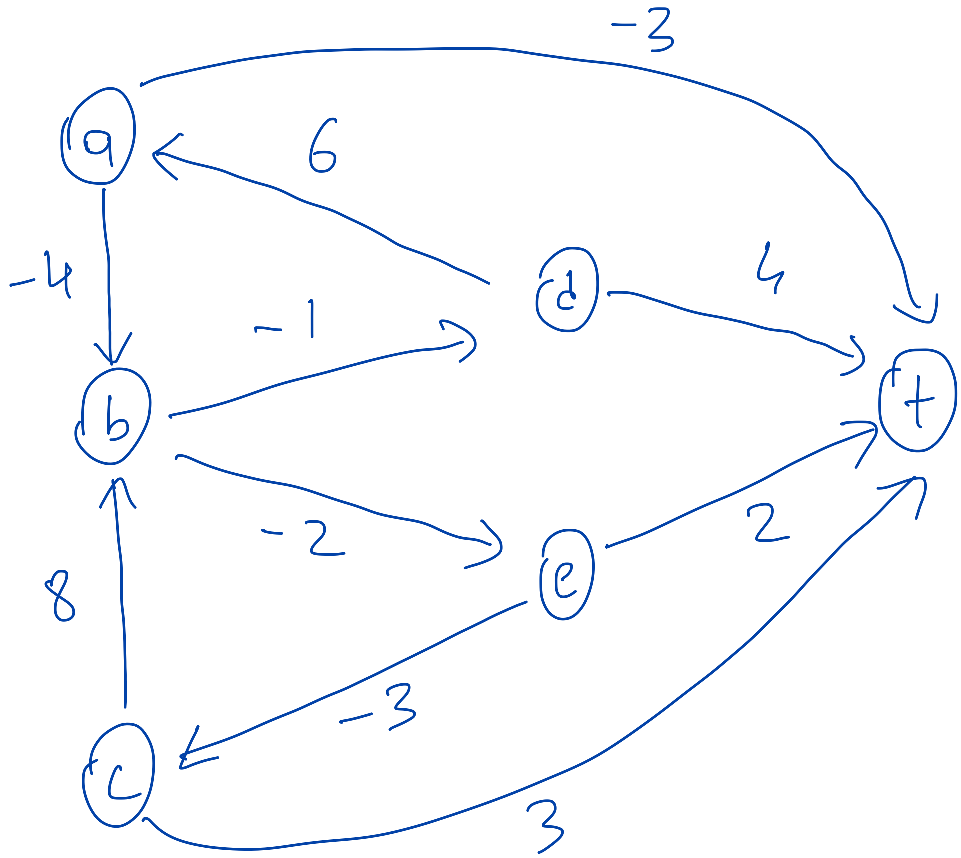


cost cannot go up

$OPT(s, i) \;:\; 0 \leq i \leq n-1$

$\Rightarrow OPT(s, n-1)$ = cost of a shortest s-t path

Goal: Compute $OPT(s, n-1) \;\forall s \in V$



look at d

OPT(d,0) = ∞ (as $d \neq t$)

OPT(d,1) = 4 [d, t]

OPT(d,2) = 3 [d, a, t]
(6 + -3)

OPT(d,3) = 3 [d, a, t]

OPT(d,4) = 6 − 4 − 2 + 2 = 2 [d, a, b, e, t]

OPT(d,5) = 6 − 4 − 2 − 3 + 3 = 0 [d, a, b, e, c, t]

OPT(d,6) = 0

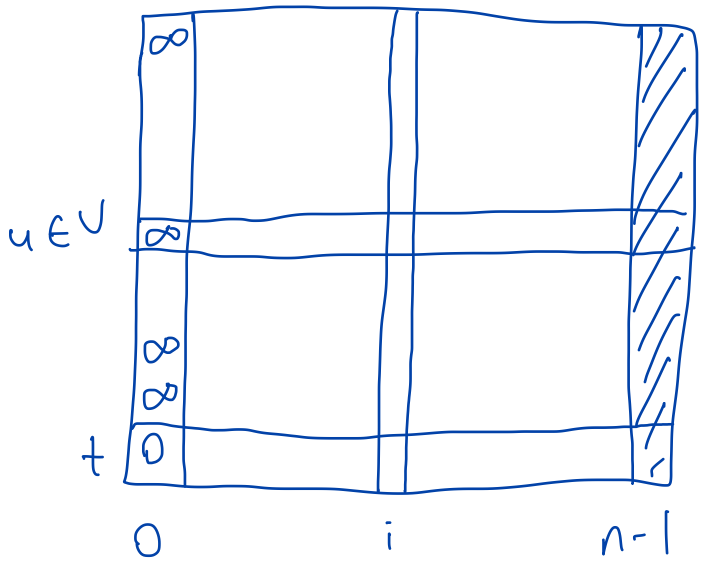
OPT(d,7) = 0

OPT(d,6) = OPT(d,5) = OPT(d,7)

By Prop n = 6 ⇒ OPT(d,5) is the cost of the shortest path

$OPT(s,i)$ = cost of a shortest s-t path with $\leq i$ edges

$s \in V \quad 0 \leq i \leq n-1$



<u>Goal:</u> $M[u,i] = OPT(u,i)$

\# subproblems $= n^2$ poly-many ✓

→ <u>Output:</u> $M[u, n-1] \; \forall u \in V$

<u>Recurrence:</u> $OPT(t,0) = 0$

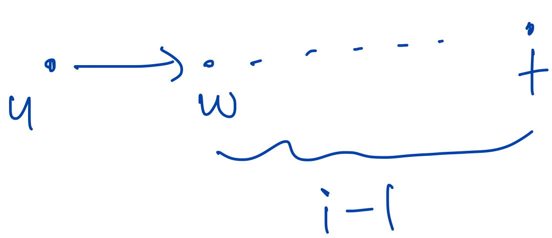
$OPT(u,0) = \infty \quad \forall u \neq t$

$OPT(u,i)$ for $i > 0$

<u>Case 1:</u> ∃ a shortest u-t path with $\leq i$ edges that actually uses $\leq i-1$ edges

$$OPT(u,i) = OPT(u,i-1)$$

<u>Case 2:</u> All shortest u-t paths $\leq i$ edges uses EXACTLY i edges

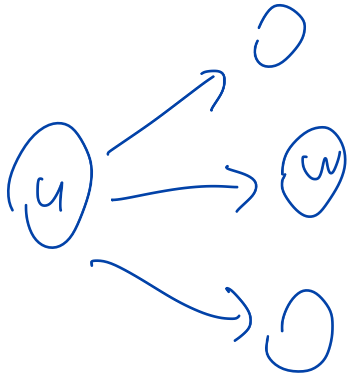


Know 1st edge is (u,w)

$$OPT(u,i) = c_{u,w} + OPT(w,i-1)$$

↓

$\Rightarrow$ shortest w-t path with $\leq i-1$ edges



Generally, 1st edge has to be from u to one of its neighbors

$$OPT(u,i) = \min_{\substack{w: \\ (u,w)\in E}} \{c_{u,w} + OPT(w,i-1)\}$$

Overall //

$$OPT(u,i) = \min\left\{OPT(u,i-1), \min_{\substack{w \\ (u,w)\in E}} \{c_{u,w} + OPT(w,i-1)\}\right\}$$