Apr 30

Satisfiability (SAT) problem

"The NP problem in practice"

Variables: $X = \{X_1, X_2, \ldots X_n\}$

(each $X_i \in \{0,1\} \equiv \{F, T\}$)

Term\Literal: X_i , $\overline{X}_i$

Clause: OR or disjunction of literals

$t_1 \vee t_2 \vee t_3 \vee \ldots \vee t_\ell$

e.g. $n = 3$

$X_1 \vee \overline{X}_2$

SAT formula: AND or conjunction of clauses (CNF)

$C_1, C_2, \ldots, C_m$

$\equiv C_1 \wedge C_2 \wedge C_3 \ldots \wedge C_m$

Eg. (*) $(X_1 \vee \overline{X}_2) \wedge (\overline{X}_1 \vee \overline{X}_3) \wedge (X_2 \vee \overline{X}_3)$ $m = 3$, $n = 3$

Assignment: $v : X \to \{0, 1\}$

(2^n possible assignments)

$$\boxed{\begin{array}{ll} X_1 = 0 & (*) = 1 \\ X_2 = 0 & \\ X_3 = 0 & \end{array}}$$

An assignment satisfies a formula if the formula evaluates to true under the assignment

$\equiv$ the assignment satisfies ALL clauses

→ An assignment satisfies a clause if the clause evaluates to true under the assignment

Assignment: $(1, 0, 0) \Longrightarrow$ satisfies $(*)$

$$x_1 \vee \overline{x_2} = 1 \vee \overline{0} = 1 \vee 1 = 1$$

$$\overline{x_1} \vee \overline{x_3} = \overline{1} \vee \overline{0} = 0 \vee 1 = 1$$

$$x_2 \vee \overline{x_3} = 0 \vee \overline{0} = 0 \vee 1 = 1$$

$$X(1,1,1) = (1 \vee \overline{1}) \wedge (\overline{1} \vee \overline{1}) \wedge (1 \vee \overline{1})$$

$$= 1 \wedge 0 \wedge 1$$

$$= 0$$

Q: Given a SAT formula, does it have a satisfying assignment?

eg. $\{x_1, x_2\}$

$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_2})$$

3-SAT problem: Same as the SAT problem with the extra restriction that each clause has EXACTLY 3 literals.

THM: 3-SAT $\leq_p$ Independent Set

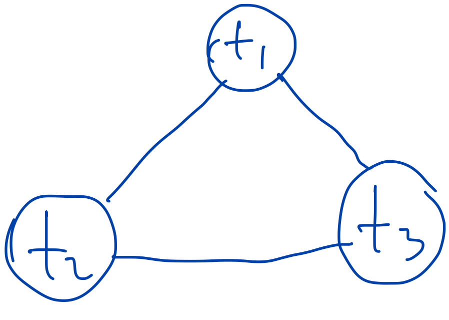
→ Use "gadgets"

2 equivalent ways of looking at 3-SAT:

→ Make independent 0/1 choices for $X_1, \dots, X_n$ s.t. you satisfy at least one literal in each clause

→ Pick one literal for each clause s.t. all picked literals do NOT conflict ← you do NOT pick both X_i and $\overline{X_i}$

gadget: $C = t_1 \vee t_2 \vee t_3$



Each of the 3 IS $\{t_1\}$ or $\{t_2\}$ or $\{t_3\}$ correspond to which literal to choose from C

Redux: Given a 3-SAT formula

$$C_1, \dots, C_m$$

$$\hookrightarrow (G, m)$$

s.t. 3-SAT formula is satisfiable $\iff$ G has an ind. set of size $\geq m$

Claim: Done if we can compute such G in poly time

Redux:

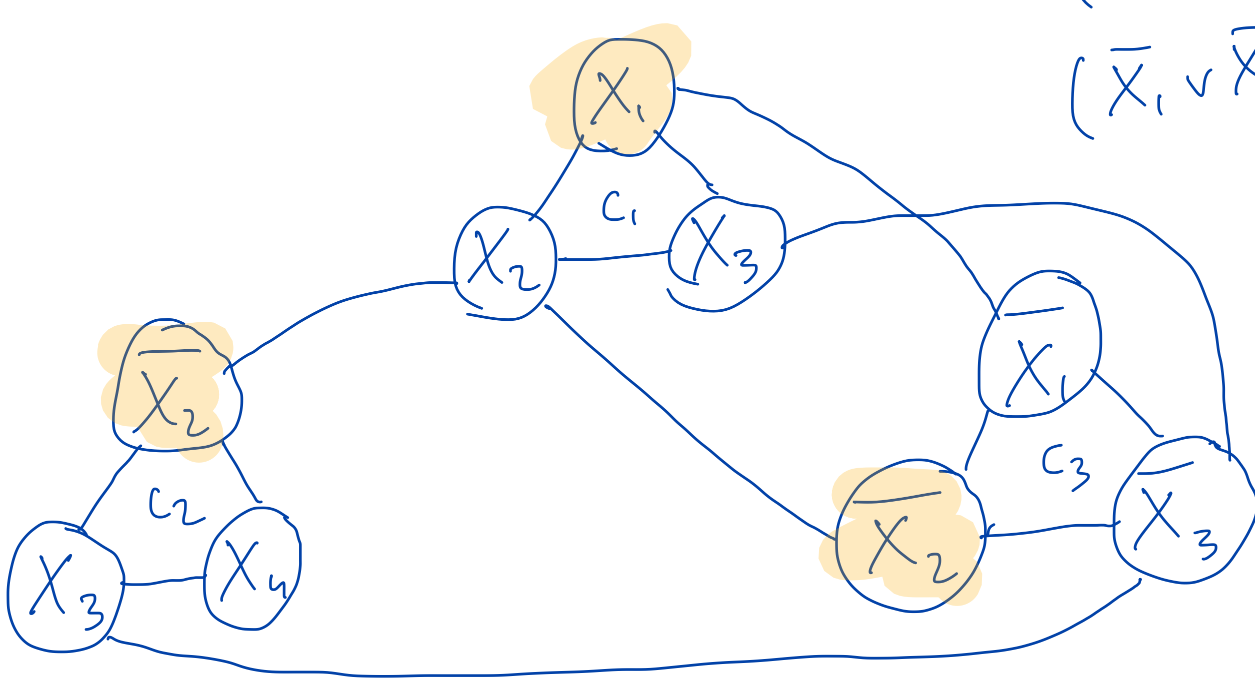
Step 1: Replace each clause C_i by its triangle

Step 2: Add an edge between X_i & $\bar{X}_i$ if they occur in the formula

$(X_1 \vee X_2 \vee X_3) = C_1$

$(\bar{X}_2 \vee X_3 \vee X_4) = C_2$

$(\bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3) = C_3$



$\} G$

To argue: 3-SAT formula is satisfiable

$\Longleftrightarrow$ G has an ind. set of size $\geq m$

$\downarrow$

Since all IS in G have size $\leq m$ $\nearrow$ $= m$

Pf. Sketch:

$\Rightarrow$ Given SAT formula that is satisfiable $\Rightarrow$ G has an IS of size m

$V: X \to \{0,1\}$

$\downarrow$

each clause has at least one literal

$\rightarrow$ pick anyone

Eg. $X_1 = 1$

$X_2 = 0$

$X_3 = 1$

$X_4 = 1$

Claim: The m literals that we pick form IS in G.