May 5

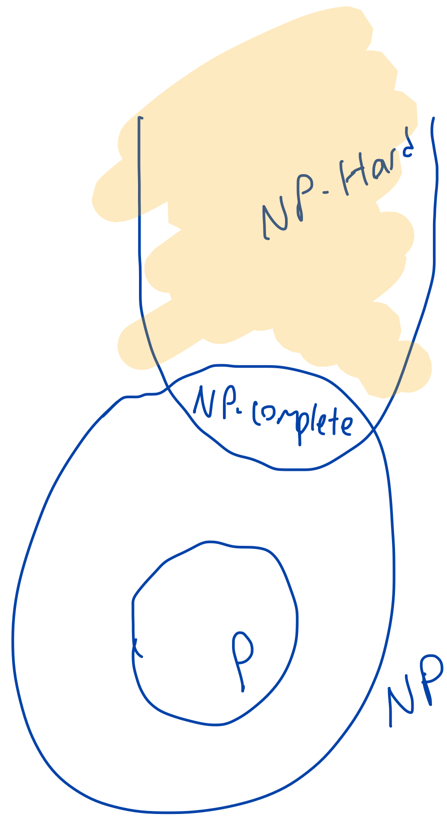
NP-Complete Problems

Def: X is NP-Complete if

(i) $X \in NP$

(ii) $\forall\, Y \in NP$, $Y \le_p X$

$\hookrightarrow$ Just (ii): X is NP-Hard



Lemma 1: Let X be an NP-Complete problem

If $X \in P \Rightarrow P = NP$

Lemma 2: Let Y be an NP-Complete problem

If $Y \le_p X \Rightarrow X$ is also NP-Complete

THM: 3-SAT is NP-Complete $\Rightarrow$ IS is NP-Complete

3-SAT $\le_p$ IS

General strategy: To prove X is NP-Complete

$(X = IS)$

Step 1: $X \in NP$

Step 2: Identify an NP-Complete problem Y $(Y = 3\text{-SAT})$

$(3\text{-SAT} \leq_p IS)$

Step 3: Prove $Y \leq_p X$

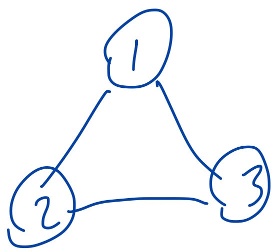
Overview: 3-SAT is NP-Complete

- Define Circuit-SAT
- Prove that Circuit-SAT is NP-Complete
- Circuit-SAT $\leq_p$ 3-SAT

k-colorability (k-coloring)

$G=(V,E)$

Def: A k-coloring of G is $c: V \to \{1,2,\ldots,k\}$
s.t. $\forall (u,w) \in E$, $c(u) \neq c(w)$



← 3-colorable but NOT 2-colorable

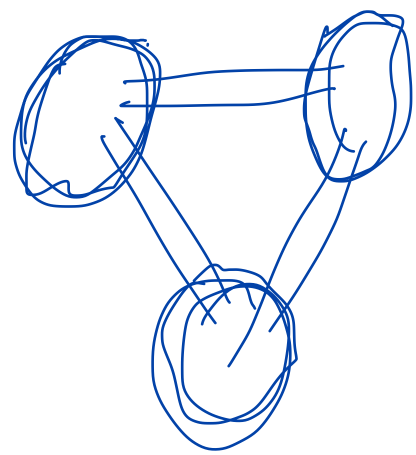
Def: G is k-colorable if $\exists$ a k-coloring for it.

Def: (k-colorabity / k-coloring problem)

Input: G, k

Output: T if G is k-colorable
F o/w

k-partite
k=3



Claim 1: k-colorability $\in$ NP

Claim 2: 2-colorability $\in$ P

THM: 3-SAT $\leq_p$ 3-colorability $\leq_p$ k-colorability

$\underset{\text{Claim 1}}{\Longrightarrow}$ 3-coloring is NP-Complete

Goal: Given any 3-SAT formula $C_1, \ldots, C_m$ on $X = \{X_1, \ldots, X_n\}$

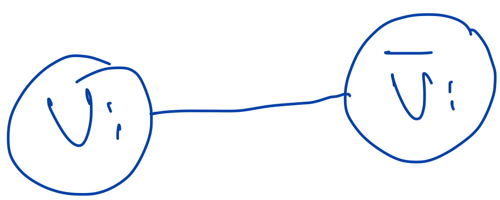
compute a graph G (s.t. $|G| = \text{poly}(n, m)$)

s.t. G is 3-colorable $\iff$ $C_1, \ldots, C_m$ is satisfiable

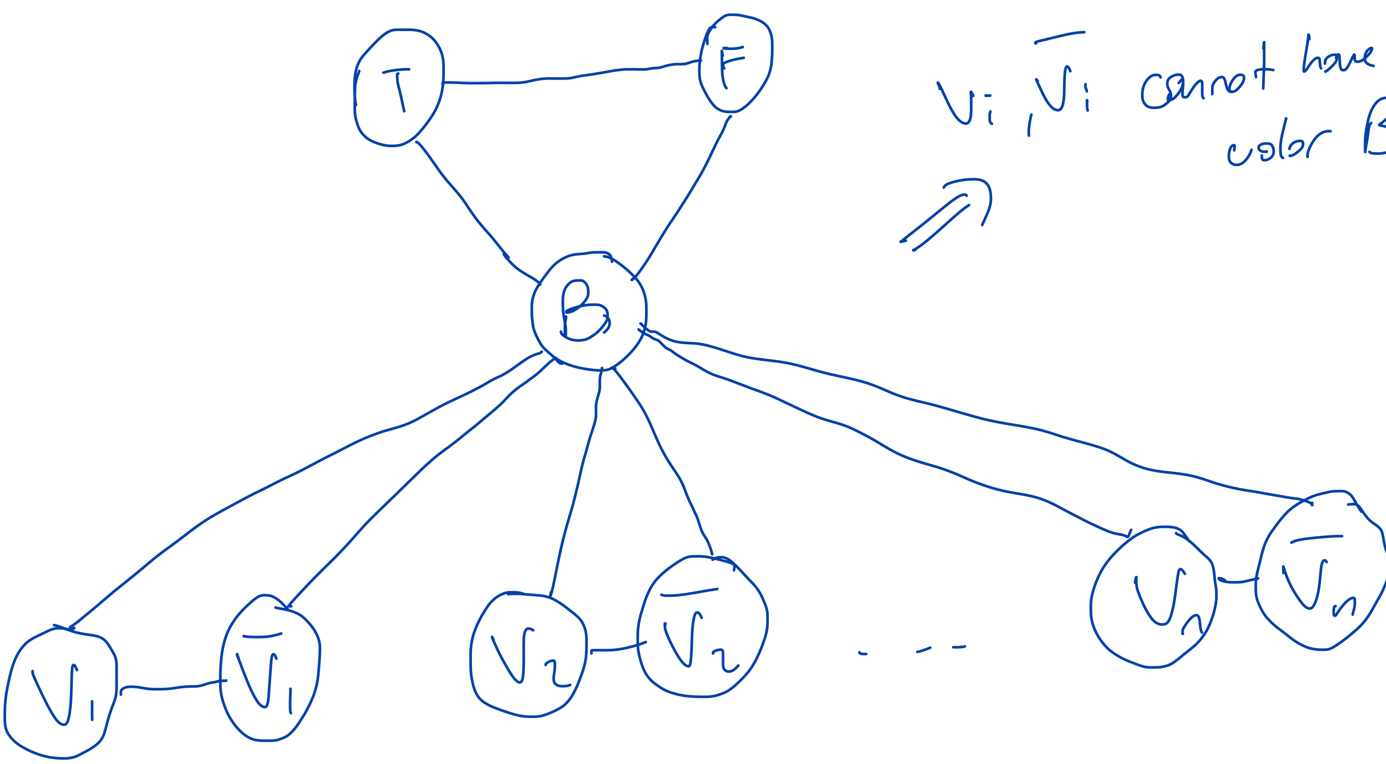
Step 1: One node $V_i \equiv X_i$

$\overline{V_i} \equiv \overline{X_i}$

$\forall i$:



$\Rightarrow$ V_i & $\overline{V_i}$ need to be assigned different colors



$V_i, \overline{V_i}$ cannot have color B

Claim: If $c(V_i) = c(T)$ $\Rightarrow c(\bar{V_i}) = c(F)$

$c(V_i) = c(F)$ $\Rightarrow c(\bar{V_i}) = c(T)$

Claim: 3-coloring of $\Longleftrightarrow$ valid assignments to $X_1, \dots, X_n$

$\rightarrow$ Encode the clauses $C_1, \dots, C_m$ in this graph

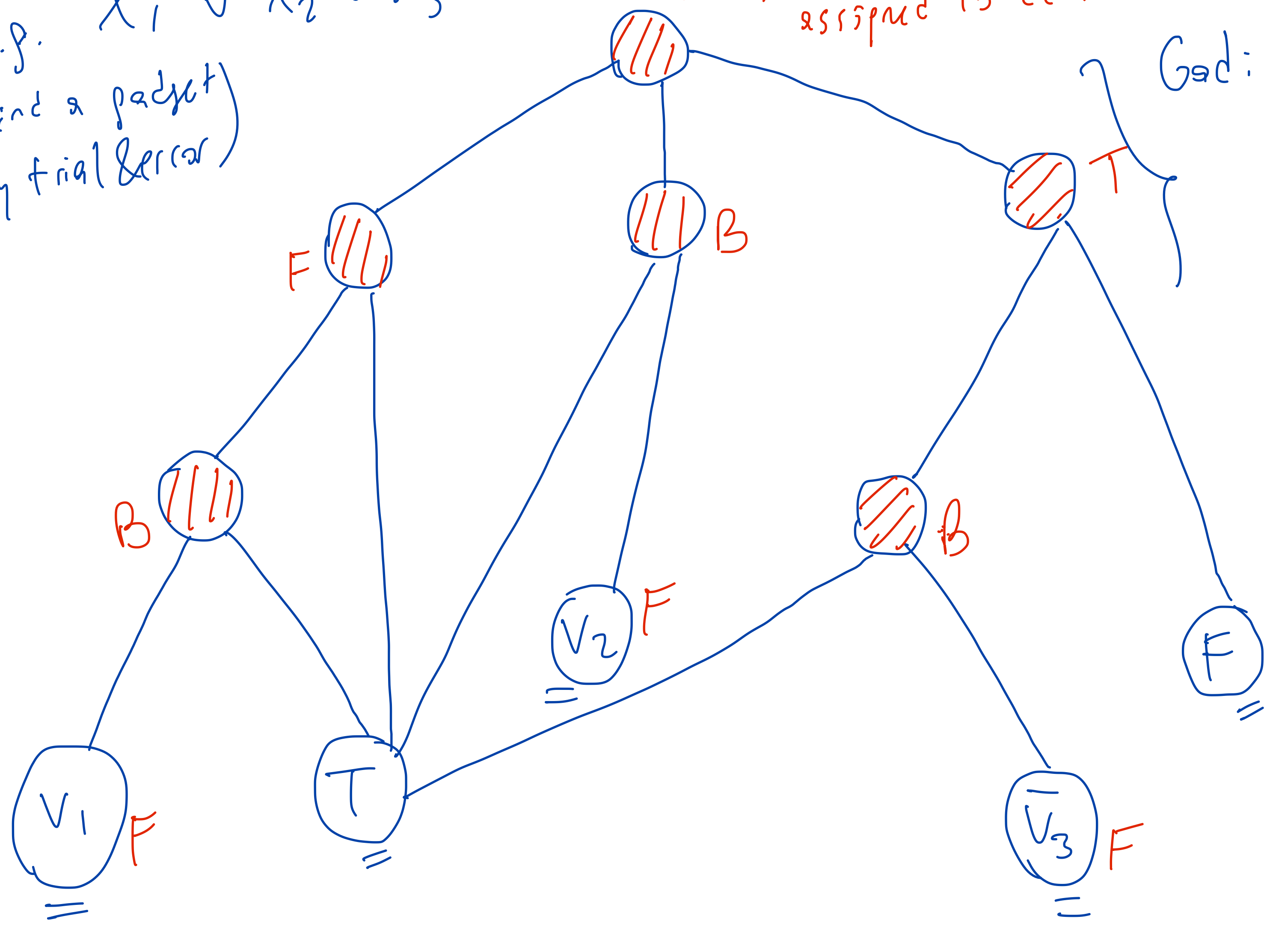
Step 2: (adding more vertices & edges in G)

e.g. $X_1 \vee X_2 \vee \bar{X_3}$

(Find a gadget by trial & error)

$w \Rightarrow$ cannot be assigned to $c(T), c(F), c(B)$

Goal:



Claim: 3-coloring of $Gad_i \implies$ at least one of the three literals in Gad_i is assigned $c(T)$

Pf idea: Assume all $V_1, V_2, \bar{V}_3$ are assigned $c(F)$

$\implies$ w cannot be assigned any of $c(T), c(F), c(B)$

$\implies$ we do not have a valid 3-coloring!

Final reduction: Given $C_1, \dots, C_m$ on X

1. Compute G from X
2. Add Gad_i to G $\forall$ clauses $C_i \implies G'$
3. Output G' / Feed G' to an algo for 3-coloring

Claim: $C_1, \dots, C_m$ is satisfiable $\iff G'$ is 3-colorable