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THEOREM: For any input (M, W, $2n$ pref. lists) the GS algorithm outputs a stable matching.

$\implies$ every input has a stable matching.

LEMMA 1: For every input, GS algo. terminates in $\leq n^2$ iters.

LEMMA 2: The output of GS algo (S) is a perfect matching

LEMMA 3: S has no instability (from w to m) ~~(from m to w)~~

Lemmas 1+2+3 $\implies$ Theorem

Pf Idea Lemma 1: In each iteration, a new proposal is made

$\implies$ # iters = # proposals $\leq$ # pairs (w, m) $= |W \times M|$

$= |W| \times |M| = n \cdot n = n^2$

(Pf details are on pg 7 in textbook)

Obs 0: S is a matching.

Obs 1: Once a man gets engaged, he keeps getting engaged to better woman.

Obs 2: If w proposes to m after m' $\Rightarrow m' > m$ in L_w

LEMMA 4: If at the end an iteration, w is free $\Rightarrow$ w has not proposed to all men

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Pf of Lemma 2 (Pf idea) Proof by contradiction (use Obs 0, Lemma 1+4, algo. def)

(Pf details): Assume S is not a perfect matching.

$\xrightarrow{\text{(Obs 0, Algo def)}}$ $\exists$ a free woman w

$\xrightarrow{\text{[Lemma 4)}}$ $\exists$ a man m that w has not proposed to. ($\star$)

By Lemma 1, algo has terminated $\xrightarrow{\text{(Alg defn)}}$ All free woman have proposed to all men $\Rightarrow$ contradicts ($\star$)

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Pigeon-hole principle

If $\leq n-1$ pigeons are put in n holes

$\Longrightarrow$ $\exists$ at least one empty hole.

Lemma 4: At the end of any iteration (of GS alg)

if woman w is free $\Longrightarrow$ w has NOT proposed to all men

Pf. Idea: Pf by contradiction (Pigeon-hole principle + Obs 1 + Alg. defn)

Pf. details: Assume free woman w has proposed to all men

$\xrightarrow[\text{Obs 1, Alg defn}]{}$ all n men are engaged (*)

Since w is free $\Longrightarrow$ n-1 women are engaged

$\xrightarrow[\text{PHP: hole::men, pigeon::women, assign::engaged}]{}$ n-1 men are engaged

$\Longrightarrow$ contradicts with (*)