# Evaluating Blocking Probability in Generalized Connectors

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*Abstract*— **Generalized connectors provide the capability to connect a single input to one or more outputs. Such networks play an important role in supporting any application that involves the distribution of information from one source to many destinations or many sources to many destinations.**

**We present the first analytic model for evaluating blocking probability in generalized connectors. The model allows flexibility in specifying traffic fanout characteristics and network routing algorithms. Equations are derived for computing blocking probability for the important class of series-parallel networks. We investigate the accuracy of the equations by comparing the blocking probability computed using the equations to results from simulation.**

## I. Introduction

GENERALIZED connectors provide the capability to connect a single input to one or more outputs. Such networks play an important role in supporting any application that involves the distribution of information from one source to many destinations or many sources to many destinations. Examples include television broadcast, teleconferencing and distributed collaboration. Additional applications that rely on generalized connectors are certain to emerge in the future.

While some applications require networks that are non-blocking (i.e., never refuse a "legal" request to set up a call or connection), in many domains some blocking is acceptable, particularly if it occurs infrequently and affords significant savings in network cost. Evaluating the blocking probability of a network under expected traffic conditions is a complex problem. Simulation is one possible approach, however considerable time is required to code and run simulations. Analytic models are another approach, often allowing more efficient calculation of blocking probability. The goal of this work is to develop an efficient, accurate and flexible analytic model for evaluating blocking probability in generalized connectors.

Previous work includes well known models for evaluating blocking probability in *connectors*, networks that support one-to-one transfer of information. Lee's model [5] gives upper bounds on blocking probability in connectors. Pippenger improves upon Lee's model with exact equations for a particular class of networks, those that are *series-parallel* [8]. In the domain of multiprocessor interconnection networks, Patel considers behavior similar to blocking—interference between requests from processors to memory modules [7]. Valdimarsson [11] extends the models of Lee and Pippenger to the multirate environment by associating a fractional weight with each connection, allowing connections to share a link provided the sum of their weights does not exceed 1.0.

In this paper we develop the first analytic model for evaluating blocking probability in generalized connectors. A Basic Model is described first; it is simpler to understand, but places restrictions on the fanout structure and routing in the network. The General Model improves upon the Basic Model by allowing more flexibility in specifying traffic fanout characteristics and routing algorithms. A nice property of the General Model is that it includes as special cases both the Basic Model and Pippenger's exact model for connectors. Preliminary versions of this work appeared in two earlier papers [12], [13].

The paper is organized as follows. In the next section we give formal definitions. In Section III we present a Basic Model for evaluating blocking probability in generalized connectors, by stating the assumptions of the model and deriving equations for computing blocking probability in series-parallel networks. This model is generalized in Section IV to allow more flexibility in specifying traffic fanout characteristics and network routing algorithms. Section V addresses the accuracy of the model by comparing the results of the equations to simulation. We conclude in Section VI with indications of future directions for this work.

## II. Definitions

Informally, the networks we consider are composed of crossbar switches (drawn as rectangles), connected by links (drawn as lines). The crossbar switches can be divided into columns (referred to as *stages*) with links between adjacent stages. The networks will be drawn with the inputs on the left and the outputs on the right. The stages are numbered left to right.

Formally, for network $N$, we associate a quadruple $(S, L, I, O)$, where $S$ is a set of vertices, called *switches*, $L$ is a set of arcs called *links*, $I$ is a set of input terminals and $O$ is a set of output terminals. Each link is an ordered pair $(x, y)$ where $x \in I \cup S$ and $y \in O \cup S$. Each input and output terminal must appear in exactly one link. Links including an input terminal are called *inputs*, those containing an output terminal are called *outputs*. We will also need to refer to

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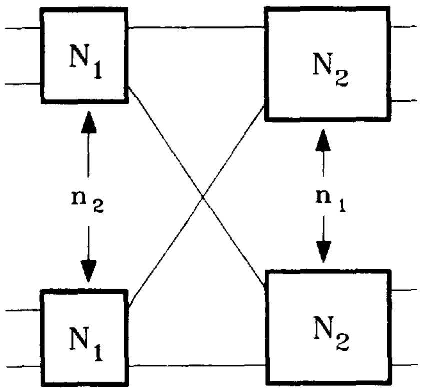



Fig. 1. Series construction $N_1 \times N_2$.

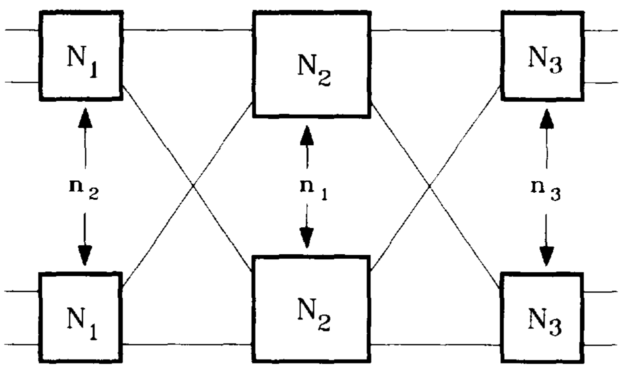



Fig. 2. Parallel construction $N_1 \otimes N_2 \otimes N_3$.

inputs and outputs of particular switches in the network. For $y \in S$, the *switch inputs of* $y$ are all links $(x, y)$. For $x \in S$, the *switch outputs of* $x$ are all links $(x, y)$.

The networks we consider can be divided into a sequence of *stages*, with links allowed only between switches in adjacent stages. The input terminals are in stage 0 and for $i > 0$, a vertex $v$ is in stage $i$ if for all links $(u, v)$, $u$ is in stage $i-1$. A link $(u, v)$ is in stage $i$ if its left endpoint $u$ is in stage $i$. We will consider only networks in which all of the outputs are in the last stage, and no other vertices are in this stage. When the outputs are in stage $k$, the network is called a *k-stage network*.

The topology of many interesting networks can be described by two simple construction operators originally proposed by Cantor [2]. If $N_1$ is a network with $n_1$ outputs and $N_2$ is a network with $n_2$ inputs, then the *series connection* of $N_1$ with $N_2$ is denoted $N_1 \times N_2$ and is constructed as shown in Fig. 1. Informally, this consists of taking $n_2$ copies of $N_1$ in one column and connecting them to $n_1$ copies of $N_2$ in a second column, with one link between each pair of subnetworks. A network constructed using only the series operator has exactly one path between each input-output pair.

The second construction operator combines three networks. If $N_1$ is a network with $n_1$ outputs, $N_2$ is an $(n_2, n_3)$-network and $N_3$ is a network with $n_1$ inputs, then the *parallel connection* of $N_1$, $N_2$ and $N_3$ is denoted $N_1 \otimes N_2 \otimes N_3$ and is constructed as shown in Fig. 2. Informally, this consists of taking $n_2$ copies of $N_1$ in one column, $n_1$ copies of $N_2$ in a second column and $n_3$ copies of $N_3$ in a third column. There is one link between each pair of subnetworks in

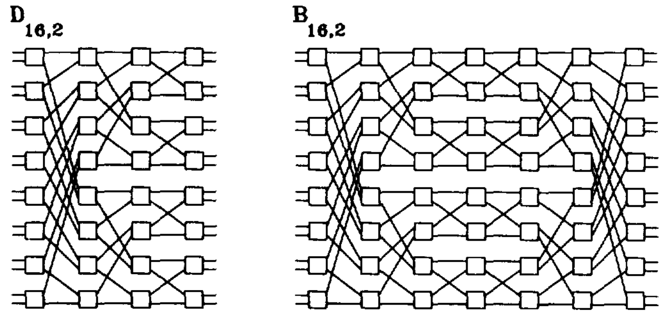



Fig. 3. Delta and Beneš networks.

adjacent columns, providing multiple, disjoint paths between each input-output pair. We will consider the class of networks constructed by repeated applications of the series operator and a restricted version of the parallel operator in which $N_1 = N_3 = X_{d,d}$, where $X_{d,d}$ denotes the $d \times d$ crossbar switch. We refer to such networks as *series-parallel networks*.

We illustrate the construction operators by using them to describe several popular network topologies. A version of the delta network [7] with $n$ inputs constructed of $d \times d$ switches is denoted $D_{n,d}$ and defined recursively as

$$D_{d,d} = X_{d,d} \qquad D_{n,d} = X_{d,d} \times D_{n/d,d}.$$

This version of the delta network is called the *baseline* network and is isomorphic to other popular topologies such as the banyan [6] and omega [4] networks. The Beneš network [1], denoted $B_{n,d}$, is defined using the parallel constructor:

$$B_{d,d} = X_{d,d} \qquad B_{n,d} = X_{d,d} \otimes B_{n/d,d} \otimes X_{d,d}.$$

Examples of the delta and Beneš networks are shown in Fig. 3.

We are interested in *generalized connection networks* or *generalized connectors*, which operate in the multipoint environment. Generalized connectors are also called *distribution networks* or *broadcast networks* elsewhere in the literature. A *generalized connection request* is a pair $(x, Y)$ where $x$ is an input and $Y$ is a nonempty set of outputs. A *generalized connection assignment* is a set of requests in which every input and output appears at most once. A *generalized connection route* is a list of links forming a tree whose root is an input and whose leaves are outputs. A route *realizes* a request $(x, Y)$ if its root is $x$ and its leaves are exactly the set $Y$. There is a second type of request in a generalized connection network. An *augmentation request* in a state $s$ is a pair $(r, y)$ where $r = (x, Y)$ is a request in the assignment realized by $s$ and $y$ is an output not in $Y$. An augmentation request is compatible with $s$ if $y$ is idle in $s$. An augmentation request can be satisfied in $s$ if the route realizing $r$ can be extended by adding unused links so that $y$ becomes a leaf of the route.

We restrict our attention to requests of the form $(x, y)$ where $x$ may be busy or idle. If $x$ is idle, the request is to create a new connection. If $x$ is busy, the request is an augmentation request. This represents the worst case for routing, since the outputs of a multipoint connection are added one at a time. If a compatible request is made, but cannot be satisfied, we say the request is *blocked*. We define the *generalized connection* *blocking probability* $P_{GC}(N,p)$ for network $N$ with input $x$, output $y$ and carried load $p$ as follows:

$$P_{GC}(N,p) \equiv \Pr\{\text{route}(x) \leftrightarrow y \text{ blocked} \mid y \text{ idle}\}$$

where "route$(x) \leftrightarrow y$ blocked" means that every path from $y$ to the route originating at $x$ is blocked. The input $x$ is chosen at random from the network inputs. If $x$ is busy, the generalized connection route is a tree; $y$ may join the connection at any point in the tree. We refer to the complementary probability as the *generalized connection linking probability* $Q_{GC}(N,p) = 1 - P_{GC}(N,p)$.

We also define two variants on the generalized connection blocking probability that will be useful in developing equations. The *generalized connection blocking probability to a busy input* $P_B(N,p)$ is defined as:

$$P_B(N,p) \equiv \Pr\{\text{route}(x) \leftrightarrow y \text{ blocked} \mid x \text{ busy}, y \text{ idle}\}$$

while the *generalized connection blocking probability to an idle input* $P_I(N,p)$ is defined as:

$$P_I(N,p) \equiv \Pr\{x \leftrightarrow y \text{ blocked} \mid x,y \text{ idle}\}.$$

Each of these variants has a complementary probability denoted $Q_B(N,p)$ and $Q_I(N,p)$, respectively. The following equation holds for all networks $N$.

$$\begin{aligned}P_{GC}(N,p) = {} & \Pr\{x \text{ busy} \mid y \text{ idle}\}P_B(N,p) \\ & + \Pr\{x \text{ idle} \mid y \text{ idle}\}P_I(N,p). \qquad (1)\end{aligned}$$

It should be clear that this model, like those of Lee [5] and Pippenger [8], assumes a circuit-switching abstraction; a single connection requires the full bandwidth of a link, and sharing of links by different connections is not allowed. This work would serve as a good starting point for a model that reflects a multirate environment (e.g., ATM), where connections can share link bandwidth. The extension is beyond the scope of this paper, however, and is likely to be nontrivial, as indicated by the complexity of similar efforts for the point-to-point case [3], [11].

## III. Basic Model

The models we consider have two components. The first is a set of assumptions that define a probability distribution on states of the network. The second component is equations for calculating blocking probability based on the probability distribution of network states. In this section we develop a probabilistic model for connections in a generalized connector with the following assumptions:

1) Every output is busy with probability $p$.
2) The conditions of different outputs (busy or idle) are independent.
3) If a given $d \times d$ switch has $r$ busy outputs, all of the $d^r$ ways in which they may be connected to the inputs are equally likely. That is, each busy switch output independently selects a switch input to be connected to.

The first two assumptions are straightforward. The third assumption captures the multipoint nature of the networks we are considering; multiple busy switch outputs may select the same switch input, creating a multipoint connection. A consequence of Assumption 3 is that if the links in stage $i+1$ are independently busy with probability $p_{i+1}$, then a link in stage $i$ is busy with probability $p_i = 1-(1-(p_{i+1}/d))^d$. A link in stage $i$ is busy if any of the busy links in stage $i+1$ select it. Each link in stage $i+1$ is busy with probability $p_{i+1}$ and, if busy, selects from the $d$ inputs with equal probability. If the conditions of the links in stage $i+1$ are not independent, then the equation can be used to approximate $p_i$. Later in the paper we consider the error resulting from the approximation.

In a network with $s$ stages, $p_s = p$. We let $q_i = 1 - p_i$ denote the probability a stage $i$ link is idle. For simplicity we have assumed that all switches are $d \times d$; it is straightforward to generalize the results to nonsquare switches.

As stated earlier, we consider the class of uniform series-parallel networks. The blocking probability for an arbitrary network in this class can be determined from the blocking probability for a crossbar network and the transformation of the blocking probability under the series and parallel construction operators. The equations for the crossbar network are trivial: $P_{GC}(X_{d,d},p) = P_B(X_{d,d},p) = P_I(X_{d,d},p) = 0$. Before developing the equations for the series and parallel construction, we consider several fundamental probabilities that will appear throughout the derivation.

### A. Fundamental Probabilities

We derive expressions for the following conditional probabilities relating input $x$ and output $y$ of network $N$ with output load $p$:

$$\begin{aligned}Q^+(N,p) &\equiv \Pr\{x \text{ idle} \mid y \text{ idle}\},\\ P^-(N,p) &\equiv \Pr\{x \text{ busy} \mid y \text{ idle}\},\\ Q^-(N,p) &\equiv \Pr\{x \text{ idle} \mid y \text{ busy}\},\\ P^+(N,p) &\equiv \Pr\{x \text{ busy} \mid y \text{ busy}\}\end{aligned}$$

by introducing a quantity which covers all four conditional probabilities:

$$C(N,p,\hat{p}) \equiv \Pr\{x \text{ busy} \mid y \text{ busy with probability } \hat{p}\}.$$

Notice that $Q^+(N,p) = 1 - C(N,p,0)$. $P^-(\cdot)$, $Q^-(\cdot)$ and $P^+(\cdot)$ can be similarly expressed as functions of $C(\cdot)$, with $\hat{p} = 0$ corresponding to $y$ idle and $\hat{p} = 1$ corresponding to $y$ busy.

We can express $C(N,p,\hat{p})$ on a crossbar and give the transformed probability under the series and parallel construction operators:

$$C(X_{d,d},p,\hat{p}) = 1 - \left(1-\frac{p}{d}\right)^{d-1}\left(1-\frac{\hat{p}}{d}\right).$$

The intuition is that $x$ is busy if some busy output selects $x$. Each output except $y$ is busy with probability $p$ and, if busy, selects independently from the $d$ inputs. Output $y$ is busy with probability $\hat{p}$ and, if busy, selects independently from the $d$ inputs.

Turning to the series construction operator, consider the network $N = N_1 \times N_2$, with input $x$ and output $y$. We let $w$ denote the link connecting the copy of $N_1$ containing $x$ with the copy of $N_2$ containing $y$. We let $s_1$ denote the number of stages in $N_1$. (See Fig. 4.) Let "wp" denote "with probability".

$$\begin{aligned} &C(N_1 \times N_2, p, \hat{p}) \\ &\quad = \Pr\{w \text{ busy} \mid y \text{ busy wp } \hat{p}\} \Pr\{x \text{ busy} \mid w \text{ busy}\} \\ &\quad + \Pr\{w \text{ idle} \mid y \text{ busy wp } \hat{p}\} \Pr\{x \text{ busy} \mid w \text{ idle}\} \\ &\quad = C(N_2, p, \hat{p}) C(N_1, p_{s_1}, 1) \\ &\quad + (1 - C(N_2, p, \hat{p})) C(N_1, p_{s_1}, 0). \end{aligned}$$

For the series transformation, the condition (busy or idle) of $y$ affects the condition of $x$ through the link $w$. Given $C(\cdot)$ for both $N_1$ and $N_2$ we can express the probability $x$ is busy for the two cases of $w$ busy and $w$ idle.

For the parallel construction operator, consider the network $M = X_{d,d} \otimes N \otimes X_{d,d}$. We let $s$ denote the number of stages in $M$. (See Fig. 6.) To compute $C(M, p, \hat{p})$, we introduce an approximation. Although the states of the links out of the switch containing $x$ are not independent, we assume that they are. Without this assumption we have been unable to develop a useful analytic expression for the desired probability. With the assumption, we approximate $C(\cdot)$ by

$$1 - \left(1 - \frac{C(N, p_{s-1}, C(X_{d,d}, p, \hat{p}))}{d}\right)^d.$$

Input $x$ is busy if some busy link out of the switch containing $x$ selects $x$. The probability each of these links out of the switch containing $x$ is busy is determined recursively with $p = p_{s-1}$ and $\hat{p} = C(X_{d,d}, p, \hat{p})$.

The expression for $C(\cdot)$ on the crossbar switch, together with the series and parallel transformations, gives us the desired conditional probabilities $Q^+(N,p)$, $P^-(N,p)$, $Q^-(N,p)$ and $P^+(N,p)$ for any uniform series-parallel network $N$. From the conditional probabilities we can express any joint probability on the condition of $x$ and $y$. For example, $\Pr\{x \text{ idle}, y \text{ busy}\} = \Pr\{x \text{ idle} \mid y \text{ busy}\} \Pr\{y \text{ busy}\} = Q^-(N,p)p$. We can now develop equations showing how the generalized connection blocking probability is transformed under the series and parallel construction operators.

### B. Derivation of Equations

Using the conditional probabilities of the previous section, the generalized connection blocking probability of (1) becomes

$$P_{GC}(N,p) = P^-(N,p) P_B(N,p) + Q^+(N,p) P_I(N,p).$$

It suffices to show how the blocking probabilities $P_B(\cdot)$ and $P_I(\cdot)$ are transformed under the series and parallel construction operators. We begin with the simpler case—the series operator—then tackle the parallel operator.

*1) Series Construction Operator:* Consider the network $N = N_1 \times N_2$ in Fig. 4. We show how the probabilities $Q_B(\cdot) = 1 - P_B(\cdot)$ and $Q_I(\cdot) = 1 - P_I(\cdot)$ are transformed by the construction operators.

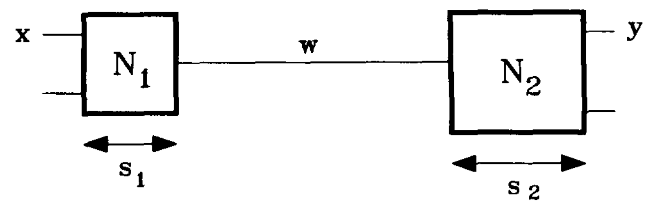



Fig. 4. Relevant part of series construction.

The generalized connection linking probability to a busy input is

$$\begin{aligned} Q_B(N_1 \times N_2, p) &= \Pr\{w \text{ busy} \mid x \text{ busy}, y \text{ idle}\} \\ &\cdot \Pr\{w \text{ connected to } x \mid x, w \text{ busy}\} Q_B(N_2, p) \\ &+ \Pr\{w \text{ idle} \mid x \text{ busy}, y \text{ idle}\} Q_I(N_2, p) Q_B(N_1, p_{s_1}). \end{aligned} \quad (2)$$

Informally, an augmentation request to join $y$ to a busy $x$ succeeds if and only if one of two conditions holds. First, $w$ is busy and connected to $x$ and $y$ can be connected to route($w$). Second, $w$ is idle, there is an idle path from $y$ to $w$ and $w$ can be connected to route($x$).

The generalized connection linking probability to an idle input is

$$\begin{aligned} &Q_I(N_1 \times N_2, p) \\ &\quad = \Pr\{w \text{ idle} \mid x, y \text{ idle}\} Q_I(N_2, p) Q_I(N_1, p_{s_1}). \end{aligned} \quad (3)$$

A new connection request to join $y$ to an idle $x$ succeeds if and only if $w$ is idle, there is an idle path from $y$ to $w$ and there is an idle path from $w$ to $x$.

The conditional probabilities regarding the state (busy or idle) of $w$ given the state of $x$ and $y$ can be computed using Bayes' theorem, the fact that the states of $x$ and $y$ are independent given the condition of $w$, and the joint probability regarding the states of $x$ and $y$. Recall that Bayes' theorem states

$$\Pr\{S \mid T\} = \frac{\Pr\{T \mid S\} \Pr\{S\}}{\Pr\{T\}}.$$

Then, for example,

$$\begin{aligned} &\Pr\{w \text{ idle} \mid x \text{ busy}, y \text{ idle}\} \\ &\quad = \frac{\Pr\{x \text{ busy} \mid w \text{ idle}\} \Pr\{w \text{ idle} \mid y \text{ idle}\}}{\Pr\{x \text{ busy} \mid y \text{ idle}\}} \\ &\quad = \frac{P^-(N_1, p_{s_1}) Q^+(N_2, p)}{P^-(N_1 \times N_2, p)}. \end{aligned}$$

The expression for $Q_B(\cdot)$ also requires the probability $w$ is connected to $x$ given $x$ and $w$ are busy. For this we need the following lemma on series-parallel networks under our model.

*Lemma 3.1:* A busy output is equally likely to be connected to any input.

*Proof:* The lemma holds for $X_{d,d}$ by Assumption 3. We show it is preserved by the series and parallel construction operators. Let $y$ be a busy output and $x$ be an arbitrary input. Assume the lemma holds for $N_1$ with $n_1$ inputs and $N_2$ with $n_2$ inputs. In $N_1 \times N_2$, $y$ is connected to $x$ if and only if $y$

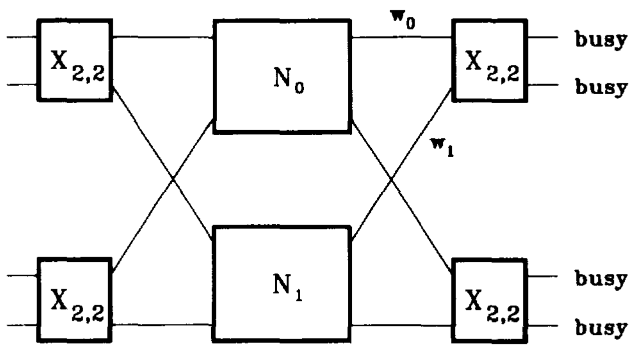



Fig. 5. Dependence of subnetwork states.

is connected to $w$ and $w$ is connected to $x$. This occurs with probability $1/(n_1 n_2)$.

Assume the lemma holds for $N$ with $n$ inputs and consider $X_{d,d} \otimes N \otimes X_{d,d}$. (See Fig. 6.) $y$ is connected to some $w'$ into one of the $d$ subnetworks $N$. Given this, $y$ is connected to $x$ if and only if this $w'$ is connected to $w$ and $w$ is connected to $x$. This occurs with probability $1/(nd)$. □

Using Lemma 1 and the definition of conditional probability we obtain

$$\begin{aligned} &\Pr\{w \text{ connected to } x \mid x, w \text{ busy}\} \\ &= \frac{\Pr\{w \text{ connected to } x \text{ and } x \text{ busy} \mid w \text{ busy}\}}{\Pr\{x \text{ busy} \mid w \text{ busy}\}} \\ &= \frac{1}{n_1 P^+(N_1, p_{s_1})}. \end{aligned}$$

Substituting into (2) and (3) we get the equations at the bottom of this page. Those familiar with Pippenger's model for blocking in connectors will recognize the close correspondence between $Q_I(N_1 \times N_2, p)$ and the connection linking probability under the series construction.

*2) Parallel Construction Operator:* We now consider networks constructed using the parallel construction operator. Consider the network $M = X_{d,d} \otimes N \otimes X_{d,d}$. It appears that deriving an exact equation for the blocking probability of $M$ in this model is quite difficult. The conditions of different subnetworks $N$ are not independent, as they are in Pippenger's model for connectors. This can be demonstrated by a simple example in which $d = 2$ and $p = 1.0$. Consider a last stage switch where $w_0$ denotes the link into the top subnetwork and $w_1$ denotes the link into the bottom subnetwork. (See Fig. 5.)

Since $p = 1.0$, it is easy to see that if $w_1$ is idle, then $w_0$ must be busy. That is, $\Pr\{w_0 \text{ idle} \mid w_1 \text{ idle}\} = 0.0$. If, however, $w_1$ is busy, then $w_0$ has a nonzero probability of being idle.

$$\Pr\{w_0 \text{ idle} \mid w_1 \text{ busy}\} = \frac{\Pr\{w_0 \text{ idle, } w_1 \text{ busy}\}}{\Pr\{w_1 \text{ busy}\}} = \frac{1/4}{3/4} = \frac{1}{3}.$$

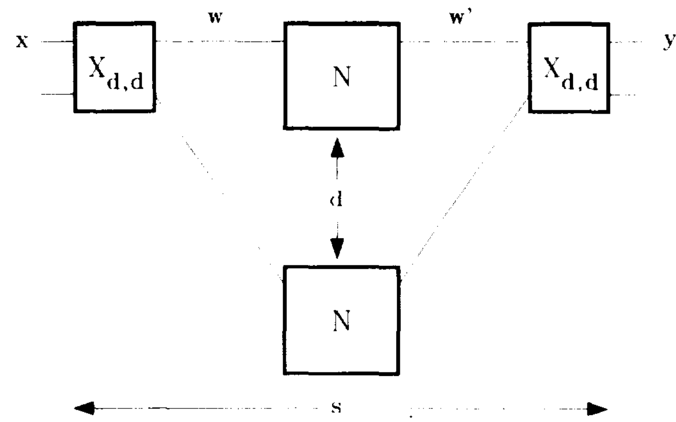



Fig. 6. Relevant part of parallel construction.

The conditions of the subnetworks are not independent since the condition of $w_0$ (a part of subnetwork $N_0$) depends on the condition of $w_1$ (a part of subnetwork $N_1$). While the example assumed $d = 2$ and $p = 1.0$, the dependence also exists for general $d$ and $p$.

Since it appears difficult to derive exact equations for the parallel operator, we derive approximate equations by making the simplifying (but inaccurate) assumption that the conditions of the different subnetworks are independent. In Section V we investigate the accuracy of the approximation.

The quantities $P_B(\cdot)$ and $P_I(\cdot)$ can be calculated by introducing several random variables to account for the conditions of the links into and out of each subnetwork $N$ needed to create a connection between $x$ and $y$, similar to Pippenger's approach for blocking in connectors. Specifically, each subnetwork $N$ has a link $w$ connecting it to the first stage switch containing $x$ and a link $w'$ connecting it to the last stage switch containing $y$. (See Fig. 6.) Let the random variable $J$ denote the number of subnetworks for which $w$ is busy and $w'$ is idle. Let the random variables $K$ and $L$ denote the number of subnetworks for which $w$ and $w'$ are both idle, and $w$ and $w'$ are both busy, respectively. Let $A$ denote $(J = j, K = k, L = l)$.

$$\begin{aligned} P_I(M,p) = \sum_{k=0}^{d} &\Pr\{K = k \mid x, y \text{ idle}\} \\ &\cdot \Pr\{x \leftrightarrow y \text{ blocked} \mid x, y \text{ idle}, K = k\} \end{aligned} \tag{4}$$

$$\begin{aligned} P_B(M,p) = \sum_{\substack{j,k,l \geq 0 \\ j+k+l \leq d}} &\Pr\{A \mid x \text{ busy}, y \text{ idle}\} \\ &\cdot \Pr\{\text{route}(x) \leftrightarrow y \text{ blocked} \mid x \text{ busy}, y \text{ idle}, A\}. \end{aligned} \tag{5}$$

Intuitively, if $x$ is idle, the only way to satisfy a new connection request to connect $y$ to $x$ is by way of an idle path from $y$ to $x$. Only those subnetworks which have both $w$ and $w'$ idle are candidates to realize the request. If $x$ is busy, an

$$Q_I(N_1 \times N_2, p) = \frac{Q^+(N_1, p_{s_1}) Q^+(N_2, p)}{Q^+(N_1 \times N_2, p)} Q_I(N_1, p_{s_1}) Q_I(N_2, p)$$

$$Q_B(N_1 \times N_2, p) = \frac{P^-(N_2, p)}{n_1 P^-(N_1 \times N_2, p)} Q_B(N_2, p) + \frac{P^-(N_1, p_{s_1}) Q^+(N_2, p)}{P^-(N_1 \times N_2, p)} Q_I(N_2, p) Q_B(N_1, p_{s_1}).$$

augmentation request to connect $y$ to $x$ may be satisfied in several ways. In fact, only those subnetworks which have $w$ idle and $w'$ busy are immediately ruled out as candidates for realizing the request.

*1) Idle Input:* First we consider calculating the generalized connection blocking probability to an idle input. Using the assumption of subnetwork independence, $P_I(\cdot)$ in (4) is approximated as

$$\sum_{k=0}^{d} \Pr\{K = k \mid x, y \text{ idle}\}(P_I(N, p_{s-1}))^k.$$

Determining $\Pr\{K = k \mid x, y \text{ idle}\}$ requires the random variables $J$, $K$ and $L$. Let $G(a,b,c)$ be defined as follows:

$$G(a,b,c) \equiv \sum_{j,k,l} \Pr\{A \mid x, y \text{ idle}\} a^j b^k c^l \tag{6}$$

with the same conditions on $j$, $k$ and $l$ as in (5). Notice that $P_I(M,p) = G(1, P_I(N, p_{s-1}), 1)$, therefore if we can evaluate $G(\cdot)$, we will have the desired blocking probability. To evaluate the conditional probability in this equation, we introduce a similar equation with an unconditional probability:

$$H(a,b,c) \equiv \sum_{j,k,l} \Pr\{A\} a^j b^k c^l. \tag{7}$$

The links at the outputs of the subnetworks $N$ are each busy with probability $p_{s-1}$ and idle with probability $q_{s-1}$. A particular subnetwork is of type $J$ with probability $q_{s-1}P^-(N, p_{s-1})$, of type $K$ with probability $q_{s-1}Q^+(N, p_{s-1})$, of type $L$ with probability $p_{s-1}P^+(N, p_{s-1})$, and of none of these types with probability $p_{s-1}Q^-(N, p_{s-1})$. With the assumption of subnetwork independence, we approximate $H(a,b,c)$ by

$$\begin{aligned} H(a,b,c) \approx (&p_{s-1}Q^-(N,p_{s-1}) + q_{s-1}P^-(N,p_{s-1})a \\ &+ q_{s-1}Q^+(N,p_{s-1})b + p_{s-1}P^+(N,p_{s-1})c)^d. \end{aligned}$$

We now relate the conditional and unconditional probabilities. By Bayes' theorem,

$$\Pr\{A \mid x, y \text{ idle}\} = \frac{\Pr\{x, y \text{ idle} \mid A\} \Pr\{A\}}{\Pr\{x, y \text{ idle}\}}. \tag{8}$$

The conditions of $x$ and $y$ are independent given $A$, thus we can take the product of $\Pr\{x \text{ idle} \mid A\}$ and $\Pr\{y \text{ idle} \mid A\}$ to get the joint probability. Given $A$, the switch containing $x$ has exactly $j+l$ busy outputs. The probability $x$ is idle under these conditions is

$$\Pr\{x \text{ idle} \mid A\} = \left(1 - \frac{1}{d}\right)^{j+l} = \left(\frac{d-1}{d}\right)^{j+l}$$

since $x$ is idle if none of the $j+l$ busy switch outputs selects $x$.

Given $A$, the switch containing $y$ has exactly $d-j-k$ busy inputs. Let $r = d-j-k$. The probability $y$ is idle given $r$ busy inputs can be evaluated using Bayes' theorem. We use this to approximate the probability $y$ is idle given $A$. The dependence of subnetwork states makes this an approximation, rather than an exact computation.

$$\begin{aligned} \Pr\{y \text{ idle} \mid A\} &\approx \Pr\{y \text{ idle} \mid r \text{ busy in}\} \\ &= \frac{\Pr\{r \text{ busy in} \mid y \text{ idle}\} \Pr\{y \text{ idle}\}}{\Pr\{r \text{ busy in}\}}. \end{aligned} \tag{9}$$

We evaluate $\Pr\{r \text{ busy in}\}$ as follows:

$$\begin{aligned} &\Pr\{r \text{ busy in}\} \\ &= \sum_{v=r}^{d} \Pr\{v \text{ busy out}\} \Pr\{r \text{ busy in} \mid v \text{ busy out}\} \\ &= \sum_{v=r}^{d} \binom{d}{v} p^v q^{d-v} \Pr\{r \text{ busy in} \mid v \text{ busy out}\}. \end{aligned} \tag{10}$$

The expression $\Pr\{r \text{ busy in} \mid v \text{ busy out}\}$ can be evaluated recursively. Let $P_{u,v}$ denote $\Pr\{u \text{ busy in} \mid v \text{ busy out}\}$ on a $d \times d$ crossbar. The base cases are

$$\begin{aligned} &P_{0,0} = 1.0, \\ &P_{0,v} = P_{v,0} = 0.0 \qquad \text{for } v \neq 0. \end{aligned}$$

The general case of $u$ busy inputs given $v$ busy outputs can occur in two ways. First, it occurs if there are $u-1$ busy inputs with $v-1$ busy outputs and the next busy output connects to one of the idle inputs. Second, it occurs if there are $u$ busy inputs with $v-1$ busy outputs and the next busy output connects to one of the busy inputs. The general expression is

$$P_{u,v} = \frac{(d-u+1)}{d} P_{u-1,v-1} + \frac{u}{d} P_{u,v-1}.$$

Using $P_{u,v}$, (10) becomes

$$\Pr\{r \text{ busy in}\} = \sum_{v=r}^{d} \binom{d}{v} p^v q^{d-v} P_{r,v}.$$

The evaluation of $\Pr\{r \text{ busy in} \mid y \text{ idle}\}$ is similar, except that there are only $d-1$ possible busy outputs.

$$\Pr\{r \text{ busy in} \mid y \text{ idle}\} = \sum_{v=r}^{d-1} \binom{d-1}{v} p^v q^{d-1-v} P_{r,v}.$$

The expressions for $\Pr\{r \text{ busy in} \mid y \text{ idle}\}$ and $\Pr\{r \text{ busy in}\}$ can be substituted into (9), giving

$$\Pr\{y \text{ idle} \mid A\} \approx \frac{\sum_{v=r}^{d-1} \binom{d-1}{v} p^v q^{d-v} P_{r,v}}{\sum_{v=r}^{d} \binom{d}{v} p^v q^{d-v} P_{r,v}}.$$

Returning to (8), we now have

$$\Pr\{A \mid x, y \text{ idle}\} = \Pr\{y \text{ idle} \mid A\} \left(\frac{d-1}{d}\right)^{j+l} \frac{\Pr\{A\}}{qQ^+(M,p)}.$$

Substituting this into (6) gives

$$\begin{aligned} G(a,b,c) = &\frac{1}{qQ^+(M,p)} \\ &\cdot \sum_{j,k,l} \Pr\{y \text{ idle} \mid A\} \left(\frac{d-1}{d}\right)^{j+l} \Pr\{A\} a^j b^k c^l \end{aligned}$$

where $\Pr\{A\} \approx$

$$\binom{d}{jkl}(p_{s-1}Q^-(N,p_{s-1}))^{d-j-k-l}(q_{s-1}P^-(N,p_{s-1}))^j (q_{s-1}Q^+(N,p_{s-1}))^k(p_{s-1}P^+(N,p_{s-1}))^l.$$

This expression can be used to compute $P_I(M,p)$.

*2) Busy Input:* We now consider the generalized connection blocking probability to a busy input, $P_B(M = X_{d,d} \otimes N \otimes X_{d,d}, p)$. Using subnetwork independence, $P_B(\cdot)$ in (5) is approximated as

$$\sum_{j,k,l}\Pr\{J=j,K=k,L=l \mid x \text{ busy}, y \text{ idle}\} \cdot (P_1(N,p_{s-1}))^j(P_2(N,p_{s-1}))^k(P_3(N,p_{s-1}))^l \tag{11}$$

where

$$P_1(N,p_{s-1}) = 1 - \frac{Q_B(N,p_{s-1})}{dP^+(X_{d,d},p_1)}$$
$$P_2(N,p_{s-1}) = P_I(N,p_{s-1})$$
$$P_3(N,p_{s-1}) = 1 - \frac{1}{ndP^+(N,p_{s-1})P^+(X_{d,d},p_1)}.$$

Informally, we block in each of the $j$ subnetworks with $w$ busy and $w'$ idle if we cannot realize the augmentation request in $N$ to join $w'$ to $w$ or $w$ is not connected to $x$ (occurs with probabability $P_1(\cdot)$). We block in each of the $k$ subnetworks with $w$ and $w'$ idle if we cannot realize the augmentation request in $N$ to connect $w'$ to $w$ (occurs with probabability $P_2(\cdot)$). We block in each of the $l$ subnetworks with $w$ and $w'$ busy if $w$ is not connected to $w'$ in $N$ or $w$ is not connected to $x$ (occurs with probabability $P_3(\cdot)$).

To determine the conditional probability, we apply a similar technique to the one introduced in deriving the expression for $P_I(\cdot)$. Specifically, define $F(a,b,c)$ as follows:

$$F(a,b,c) \equiv \sum_{j,k,l}\Pr\{A \mid x \text{ busy}, y \text{ idle}\}a^jb^kc^l \tag{12}$$

and notice that $F(P_1(\cdot),P_2(\cdot),P_3(\cdot))$ is exactly the generalized connection blocking probability to a busy input in (11). As before, we use the function $H(\cdot)$ in (7) and $\Pr\{A \mid x \text{ busy}, y \text{ idle}\}$ with Bayes' theorem to determine $F(\cdot)$. The result is

$$\Pr\{A \mid x \text{ busy}, y \text{ idle}\} = \Pr\{y \text{ idle} \mid A\}\left(1-\left(\frac{d-1}{d}\right)^{j+l}\right)\frac{Pr\{A\}}{qP^-(M,p)}.$$

This can be used in (12) to give an expression for $P_B(M,p)$.

Combining the blocking probabilities $P_I(M,p)$ and $P_B(M,p)$ completes the transformation of the generalized connection blocking probability under the parallel construction operator:

$$\begin{aligned}P_{GC}(M,p) &= Q^+(M,p)P_I(M,p) + P^-(M,p)P_B(M,p)\\ &= Q^+(M,p)G(1,P_I(N,p_{s-1}),1)\\ &\quad + P^-(M,p)F(P_1(N,p_{s-1}),P_2(N,p_{s-1}),P_3(N,p_{s-1})).\end{aligned}$$

## IV. General Model

For a model to be useful, it must do a reasonable job of representing real traffic. With the set of assumptions in the Basic Model, the only parameter by which traffic can be specified is the output load $p$. The fanout distribution in each stage, and thus in the entire network, is fixed by Assumption 3. In the General Model we modify Assumption 3 to allow more variety in specifying the fanout characteristics of the multipoint traffic and the routing of connections. Another criteria for usefulness of the model is the accuracy of the equations, that is, how faithful are they to the underlying probability distribution implied by the assumptions. In both models an approximation is made that causes an inaccuracy in the equations, however in some cases the General Model is considerably more accurate than the Basic Model.

In the General Model, we retain Assumptions 1 and 2 and modify Assumption 3 as follows.

3′ If a given $d \times d$ switch has $r$ busy outputs, the state of the switch is determined by a local fanout function $f(\cdot)$. The busy outputs are considered in random order. When an output is considered, it is connected to an arbitrary *busy* input with probability $f(u)$, where $u$ is the current number of busy inputs. Otherwise it is connected to an arbitrary *idle* input.

We call $f(u)$ the *fanout function* since it controls the fanout structure of the multipoint connections. To see how this generalizes the previous model, note that if $f(u) = u/d$ then the probability that a busy switch output connects to a busy switch input is exactly equal to the fraction of busy switch inputs. This matches Assumption 3 of the previous model in which the busy switch output chooses an input at random. Another special case of particular interest is $f(u) = 0.0$. This gives Pippenger's model for connectors, in which each busy switch output is connected to an arbitrary idle switch input. The resulting network will have only point-to-point connections.

The modification to the model allows considerably more control over fanout structure and routing scheme. To further increase the flexibility of the model, it is possible to specify different fanout functions for different stages of the network. This allows one to model, for example, a network in which copies are made in the early stages and point-to-point routing occurs in later stages, a technique which has been considered for some ATM networks [9].

The equations derived in the previous section can be modified to give the blocking probability under the General Model. We highlight only the changes to the earlier equations.

### A. Fundamental Probabilities

In the last section we defined the quantity $P_{u,v}$ to be the probability a $d \times d$ switch has $u$ busy inputs given that it has $v$ busy outputs. This quantity was used for one particular calculation. In the General Model it is ubiquitous. Recall the recursive expression for $P_{u,v}$ in the model from the previous section:

$$P_{u,v} = \frac{(d-u+1)}{d}P_{u-1,v-1} + \frac{u}{d}P_{u,v-1}.$$

The quantity $(d - u + 1)/d$ is the probability the next busy output selects from the idle inputs when there are currently $u - 1$ busy inputs. The quantity $u/d$ is the probability the next busy output selects from the busy inputs when there are currently $u$ busy inputs. In the General Model, the probability the next busy output selects from the idle inputs when there are $u - 1$ busy inputs is $1 - f(u - 1)$. The probability the next busy output selects from the busy inputs when there are $u$ busy inputs is $f(u)$. The expression for $P_{u,v}$ in the General Model is modified accordingly:

$$P_{u,v} = (1 - f(u-1))P_{u-1,v-1} + f(u)P_{u,v-1}.$$

The base cases remain the same, that is, $P_{0,0} = 1.0$ and $P_{0,v} = P_{v,0} = 0.0$ for $v \neq 0$.

Another fundamental probability needed throughout the equation derivation is the probability a link in stage $i$ is busy. In the model from the previous section, a link in stage $i$ is busy with probability $p_i = 1 - (1 - (p_{i+1}/d))^d$, assuming the links in stage $i + 1$ are independently busy with probability $p_{i+1}$. In the General Model, $p_i$ is more complicated to evaluate. Consider a switch in stage $i + 1$. The inputs are in stage $i$ and the outputs are in stage $i + 1$. Assume the outputs are each independently busy with probability $p_{i+1}$. If there are $j$ busy inputs, a particular input is busy with probability $j/d$:

$$\begin{aligned} p_i &= \sum_{j=1}^{d} \frac{j}{d} \Pr\{j \text{ busy inputs}\} \\ &= \sum_{j=1}^{d} \frac{j}{d} \sum_{k=j}^{d} \Pr\{k \text{ busy outputs}\} P_{j,k} \\ &= \sum_{j=1}^{d} \frac{j}{d} \sum_{k=j}^{d} \binom{d}{k} p_{i+1}^{k} q_{i+1}^{d-k} P_{j,k}. \end{aligned}$$

We now consider modifications to the quantity $C(N, p, \hat{p})$, the probability input $x$ to network $N$ is busy given that output $y$ is busy with probability $\hat{p}$ and all other outputs are busy with probability $p$. When $N$ is a crossbar, the equation is similar to the one above for $p_i$, except that output $y$ is busy with probability $\hat{p}$ rather than $p$:

$$\begin{aligned} &C(X_{d,d}, p, \hat{p}) \\ &= \sum_{j=1}^{d} \frac{j}{d} \Pr\{j \text{ busy inputs} \mid y \text{ busy wp } \hat{p}\} \\ &= \sum_{j=1}^{d} \frac{j}{d} \sum_{k=j}^{d} \Pr\{k \text{ busy outputs} \mid y \text{ busy with prob } \hat{p}\} P_{j,k} \\ &= \sum_{j=1}^{d} \frac{j}{d} \sum_{k=j}^{d} \left( \hat{q} \binom{d-1}{k} p^k q^{d-1-k} \right. \\ &\left. + \hat{p} \binom{d-1}{k-1} p^{k-1} q^{d-k} \right) P_{j,k}. \end{aligned}$$

Under the series construction operator, the transformation of $C(N_1 \times N_2, p, \hat{p})$ derived previously also holds for the General Model. Under the parallel construction operator, we saw previously that each link out of the switch containing $x$ is busy with probability $C(N, p_{s-1}, C(X_{d,d}, p, \hat{p}))$. This is true in the General Model as well. Let $\hat{p}_1$ denote $C(N, p_{s-1}, C(X_{d,d}, p, \hat{p}))$. Input $x$ is busy if at least one of the busy links selects $x$:

$$\begin{aligned} &C(X_{d,d} \otimes N \otimes X_{d,d}, p, \hat{p}) \\ &= \sum_{j=1}^{d} \frac{j}{d} \Pr\{j \text{ busy inputs at switch containing } x\} \\ &= \sum_{j=1}^{d} \frac{j}{d} \sum_{k=j}^{d} \binom{d}{k} \hat{p}_1^k \hat{q}_1^{d-k} P_{j,k}. \end{aligned}$$

### *B. Series and Parallel Equations*

The transformation derived previously for the series operator holds for the General Model. Of course, since the transformation involves various instances of $Q^+(\cdot)$ and $P^-(\cdot)$, the modifications to $C(N, p, \hat{p})$ result in underlying modifications to the transformation.

The majority of the derivation under the parallel operator carries over to the General Model. The only modification is in the expression for the probability input $x$ is idle given $A$ (needed to calculate $P_I(\cdot)$) and the complementary probability that $x$ is busy given $A$ (needed to calculate $P_B(\cdot)$). Recall that when $A$ holds, the switch containing $x$ has exactly $j + l$ busy outputs. Input $x$ is idle if none of these connects to $x$.

$$\Pr\{x \text{ idle} \mid A\} = 1 - \sum_{k=1}^{d} \frac{k}{d} P_{k,j+l}.$$

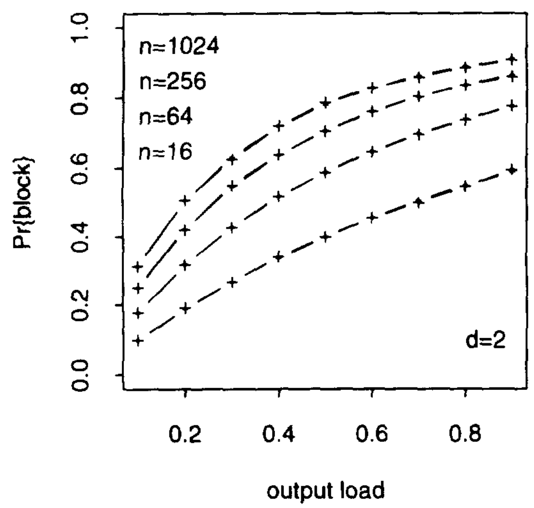
## V. Application of the Model

In this section we apply the model to some specific series-parallel network topologies, addressing two issues of interest. Primarily, we examine the effect of the approximation in the equations for the parallel construction operator, by comparing the equation results to results obtained by a simulation designed to exactly match the assumptions of the model. We also examine the fanout distribution obtained by various fanout functions. Before discussing the results, we detail the simulation methodology.

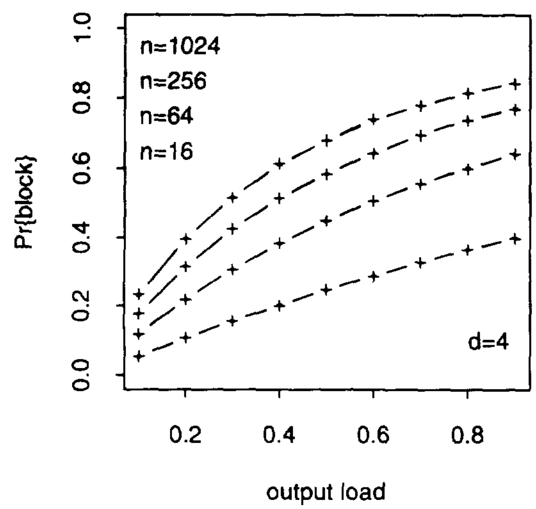
### *A. Simulation Methodology*

In designing the simulator, we created a traffic environment that matches the assumptions of the model. The simulator operates as follows. In the first phase, a background of multipoint connections is established working from the outputs of the network toward the inputs. For a specified network of size $n$ and output load $p$, one output (call it $y$) is selected to be idle, and each other output is independently made busy with probability $p$. Each last stage switch is configured by allowing each busy switch output to select from the $d$ switch inputs according to the fanout function. If multiple busy switch outputs select the same input, a multipoint connection is formed. This continues, one stage at a time, throughout the network, until the inputs are reached.

In the second phase, an input $x$ is chosen at random. If we are measuring the blocking probability to an idle input ($P_I(\cdot)$), we check to see if $x$ is idle. If so, an attempt is made to connect $y$ to $x$. The success or failure of the attempt is recorded, and the background connections are cleared. If $x$ is




(Top)




(Bottom)

Fig. 7. Blocking probability in Delta network. (Top) $d = 2$. (Bottom) $d = 4$.

busy, the background setup is discarded. This does not count as an attempt. If we are measuring the blocking probability to a busy input ($P_B(\cdot)$), the procedure is similar except that the background is discarded when $x$ is idle.

The two phases are repeated multiple times. The blocking probability is the ratio of the number of failed attempts to the total number of attempts. An important issue is the number of total attempts to perform. For large networks with low output load, the blocking probability can be very low, requiring many simulation attempts to get statistically valid results. We have kept confidence interval information for all of the simulations and ran them long enough so that, except at very low loads, the 95% confidence interval was within 10% of the simulation average.

### B. Delta Network

We first consider the delta network under the Basic Model. As described in Section II, a delta network is constructed purely from the series construction operator. We made no simplifying assumptions in deriving the equation for blocking probability under the series operator, thus we expect the equation results to match simulation results. Fig. 7 shows the generalized connection blocking probability computed using the equation and measured by simulation. The plot on the top is for $d = 2$; the plot on the bottom is for $d = 4$. Each plot contains simulation and equation curves for network sizes $n = 16, 64, 256, 1024$. In all of the plots, the simulation curves are marked by a plus (+) and the equation curves are marked by a minus (−). The blocking probability is plotted as a function of the output load, i.e., the value $p$ from the model.

As expected, the equation results agree with the simulation results. (In fact, it's difficult to discern both the plusses and minuses in the plots.) The blocking probability increases with network size $n$ and decreases with switch size $d$. Both an increase in $n$ for fixed $d$ and a decrease in $d$ for fixed $n$ have the effect of adding more stages to the network. Intuitively, it becomes more difficult to satisfy a request when the number of stages increases. Not surprisingly, this indicates that a delta network is a poor choice as a generalized connector, exhibiting a high rate of blocking even at moderate loads.

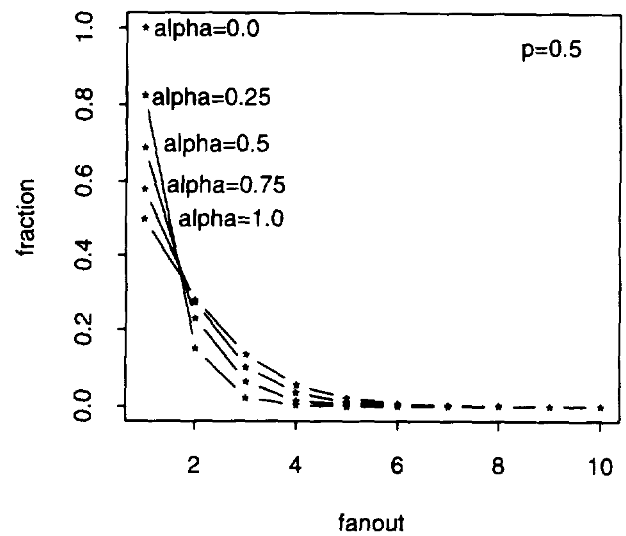


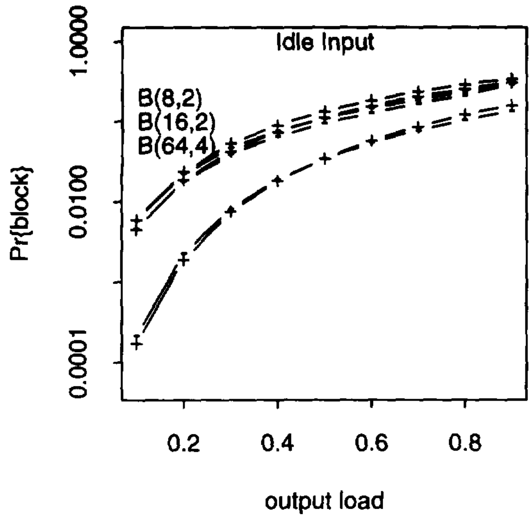

Fig. 8. Fanout distribution for $B_{16,2}$.

### C. Fanout Functions

If the network designer has knowledge about the fanout distribution expected for connections through the network, that can be modeled by the fanout function to give blocking probability estimates that better reflect expected traffic. We restrict this discussion to fanout functions of the form $f(u) = \alpha u/d$, where $d$ is the number of inputs to the given switch and $0 \leq \alpha \leq 1$. In the conclusions, we mention other interesting fanout functions.

Consider the fanout distribution for the fanout function $f(u) = \alpha u/d$. Intuitively, the effect of varying $\alpha$ is straightforward. As $\alpha$ increases, busy outputs are more likely to choose the same input, leading to background connections with high fanout. To get a more detailed view, we consider the fanout distribution on a specific network.

Fig. 8 shows the fanout distribution for the network $B_{16,2}$ with an output load of 0.5. Five values of $\alpha$ are considered, ranging from 0.0 to 1.0. The plot shows the fraction of

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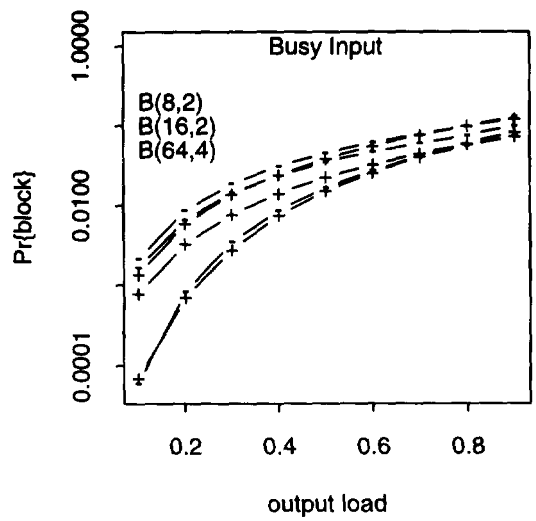
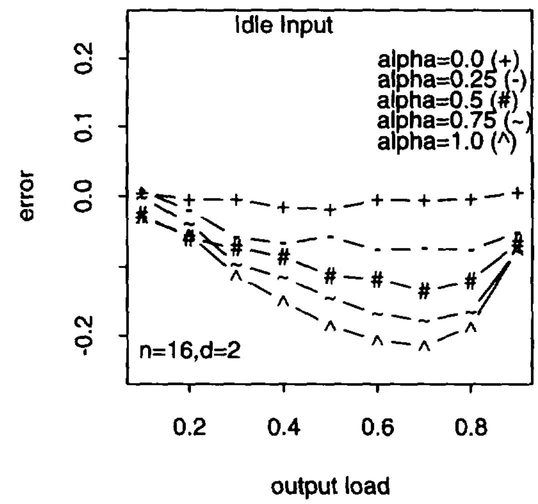



(Bottom)

Fig. 9. Blocking probability in Benešnetworks.

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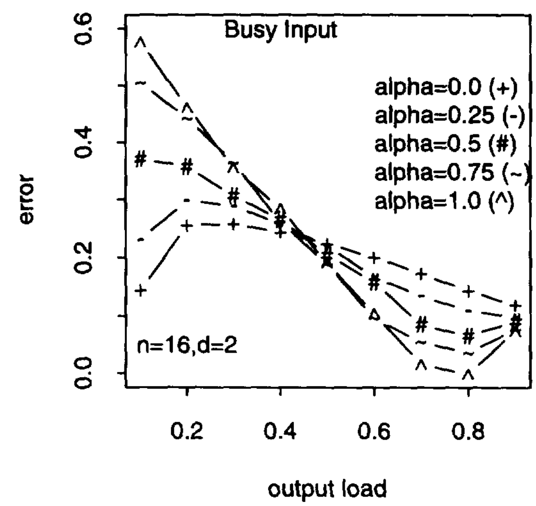


(Bottom)

Fig. 10. Accuracy of model for $B_{16,2}$.

connections with a particular fanout, for all fanout values that appeared at least once. When $\alpha$ is 0.0, all of the connections are point-to-point, thus all of the connections have fanout of one. As $\alpha$ increases, the connections tend to be larger, although even at $\alpha = 1.0$, a significant fraction (about half) are point-to-point connections.

Additional experiments have been performed to study the fanout distribution on different networks and with different loads. These experiments support the intuition that increasing the output load results in more larger connections. Furthermore, decreasing the number of stages by increasing $d$ results in less opportunity for combination of busy links, leading to more smaller connections. The fanout distribution study also motivates considering fanout functions that allow for more larger connections; this will require a different form of fanout function.

### D. Accuracy of the General Model

In this section we investigate the accuracy of the General Model. The series equations are exact, but the parallel equations contain an approximation, with accuracy that varies with $\alpha$. We already know something about what occurs at one extreme value of $\alpha$: when $\alpha = 0.0$ and we are connecting to an idle input, the model is equivalent to Pippenger's exact model for blocking in connectors.

We first look at the values of blocking probability predicted by the equations and measured during simulation when $\alpha = 1.0$. Fig. 9 shows the simulation results marked by plus (+) and the equation results marked by minus (-) for three networks, $B_{8,2}$, $B_{16,2}$ and $B_{64,4}$. Consistent with the observations regarding the delta network, the blocking probability is smaller for the larger networks and smaller for an idle input than for a busy input. Less clear from this plot is the accuracy of the equations, beyond the qualitative observation that the curves are relatively close, but not exact. To better explore accuracy, we turn to plots of the *error* in the equation, rather than the value of blocking probability.

Consider the Beneš network $B_{16,2}$. Fig. 10 shows the error in the equation for blocking to an idle input (on the top) and to a busy input (on the bottom), as a function of output load,

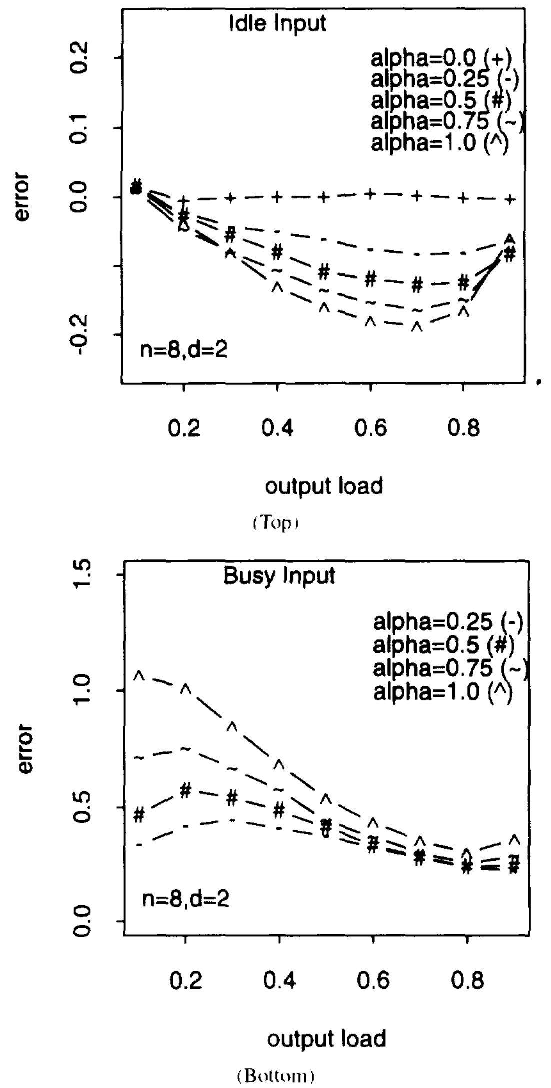


Fig. 11. Accuracy of model for $B_{8,2}$.

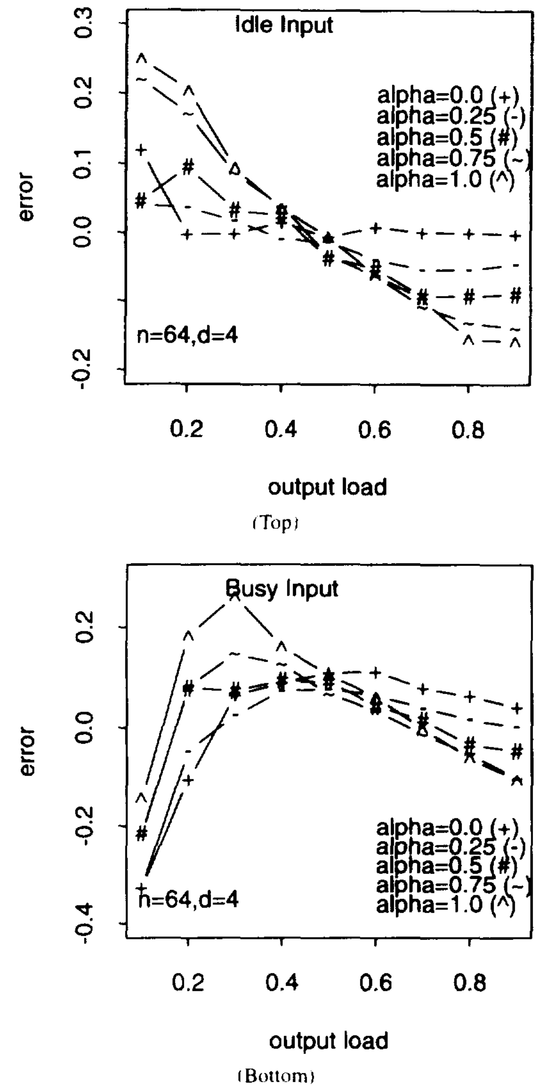


Fig. 12. Accuracy of model for $B_{64,4}$.

for values of $\alpha$ ranging from 0.0 to 1.0. Error is defined as the difference between the equation value and the simulation value, divided by the simulation value. For an idle input, the accuracy improves as $\alpha$ decreases, approaching an exact match between the equation and simulation when $\alpha$ is 0.0. The curves all have the same general shape, that is, the accuracy decreases as output load increases until moderately high output load is reached. The intuition is that at very high loads, the blocking probability is close to 1.0 and the dependencies across subnetworks are less important since nearly all subnetworks block. The blocking probability (and accuracy) are higher when $\alpha$ is small since most of the connections are point-to-point.

For a busy input, the accuracy has somewhat different characteristics than was observed for creating a new connection. When the output load is less than 0.5, the equation is more accurate for smaller $\alpha$. When the output load exceeds 0.5, the equation is more accurate for larger $\alpha$. When $\alpha = 0.0$ and we are connecting to a busy input, we have the same distribution on states as in Pippenger's exact model, but the blocking probability calculation contains approximations. The intuition for why the error is greater for a busy input than for an idle input is that the equation for blocking probability to a busy input is more complex. The equation must take into account a wider variety of situations and thus there are more opportunities for error in the equation.

To determine how the accuracy varies with the size of the network, we also examined a smaller network, $B_{8,2}$ (Fig. 11) and a larger network $B_{64,4}$ (Fig. 12). In each case the top plot is for an idle input and the bottom plot is for a busy input. The accuracy for an idle input is similar in all three networks for moderate to high output load and worse for the larger network at low output load. As network size increases, the accuracy for a busy input improves. Two trends are at work here: first, as the network size increases, the blocking probability decreases; second, as the network size increases, the equations involve additional levels of recursive evaluation of blocking probability in smaller subnetworks. Errors in the recursive evaluation are manifest in the overall equation in complex ways, leading to the somewhat unintuitive shape of the error plots.

## VI. Conclusion

In this paper we have developed a model for evaluating blocking probability in generalized connectors and derived equations for the class of series-parallel networks. The model allows a flexible specification of the fanout structure of connections, including a specification which corresponds to the point-to-point case and agrees with Pippenger's exact model for connectors. The ability to specify fanout structure allows more accurate modeling of expected traffic characteristics and examination of the effects of various routing schemes.

An investigation of the accuracy of the model indicates that the approximations in the equations for parallel networks are generally within 20% of the simulation value for the blocking probability to an idle input. The accuracy is not as good for a busy input, particularly for small networks and low values of output load, however the equations are relatively accurate—within 20%—for larger networks with load of 0.5 or higher. While exact results would certainly be desirable, the approximations are accurate enough for the intended use of such a model—to get an idea of the blocking characteristics of a network and make broad design decisions. Building on the foundation of this model, there are several possible directions for future work: upper and lower bounds on blocking probability, exact equations for a limited class of networks (e.g., three stage networks), partial or full elimination of the assumption that subnetworks are independent, and extensions to the multirate environment.

We restricted attention to fanout functions that were uniform across all stages of the network and had the form $f(u) = \alpha u/d$. It would certainly be interesting to look at some other fanout functions, particularly those which could be tailored to reflect routing algorithms. For example, using $f(u) = 0.0$ in the first half of the network and $f(u) = u/d$ in the second half of the network would model the type of routing found in Turner's cascaded Clos networks [10] in which branching is restricted to the second half of the network.

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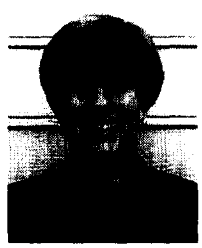

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