Lecture 6: STRONG MARKOV PROPERTY

1 Stopping Time

A random variable T is called a stopping time *iff* the following event $T = n$ is only dependent on $X_0, \dots, X_n$. That is it is only dependent on the past random variables and not on the future random variables. Now let us look at some examples which will make the defination more clear.

eg 1: First Passage Time

$$T_i = min\{n \geq 1 : X_n = i\}$$
$$i \in I$$

$\{T_i = n\} = \{X_1 \neq i, \dots, X_{n-1} \neq i, X_n = i\}$ So T_i is a stopping time

To make it more clear, the minimum step when the random variable takes the value i is in the n^{th} step. So in the previous steps the random variable can not take the value i. So we see that the minimum step does not depend on the future steps.

eg 2: Hitting Time

$$H_A = min\{n \geq 0; X_n \in A\}$$

where $A \in I$. H_A is an example of a stopping time. This represents the minimum step when the random variable in the sequence gets a value in the set hitting set A. So all the previous random variables in the previous steps should not get a value in this set.

eg 3: Last Exit Time

$$L_A = max\{n \geq 0 : X_n \in A\}$$

Here L_A is not a stopping time. This is because L_A depends on the future random variables.That is the variables in the future like n+1, n+2... should not have a value that is in A.

$$\{L_A = n\} = \forall m \text{ greater than } n\{X_m \text{ does not take a value in } A\}$$

Some Notations

$(X_n)_{n\geq 0}$ is $Markov(\lambda, P)$ if X_0 has distribution λ and the markov chain has transition probability matrix P.

1_F for any event F is defined as having values

$$1_F = 1 \text{ if F happens}$$

$$1_F = 0 \text{ if F does not happen}$$

2 Strong Markov Property

Suppose $(X_n)_{n\geq 0} = Markov(\lambda, P)$ and T is a stopping time. Then given that $T = m, X_T = i_o$ then the markov chain $(X_{T+n})_{n\geq 0}$ behaves like the markov $(X_n)_{(n\geq 0)}$ with $X_0 = i_0$. i.e

$$1.\mathbf{P}\left[X_{T+0} = i_0 | T = m \text{ and } X_T = i_0\right] = 1$$
$$2.\mathbf{P}\left[X_{T+n+1} | X_{T+n} = i, X_{T+n-1} = i_{n-1}, \ldots, X_T = i_0, T = m\right] = P_{ij}$$

proof:

The proof of the first one is trivial. Let us look at the proof of the second one.

$$\mathbf{P}\left[X_{T+1} = j | X_T = i_0, T = m\right] = p_{i_0 j}$$

let $V = \{(x_0, \ldots, x_m)$ for which if $X_0 = x_0, \ldots, X_m = x_m$ then $T = m$ and $X_T = i_0\}$

$$\begin{aligned}
\mathbf{P}\left[X_{T+1} = j, X_T = i_0, T = m\right] &= \sum_{(x_0,\ldots,x_m)\epsilon V} (P)\left[X_{m+1} = j, X_m = x_m, \ldots, X_0 = x_0\right] \\
&= \sum_{x\epsilon V} \mathbf{P}\left[X_{m+1} = j | X_m = i_0, \ldots, X_0 = x_0\right] \mathbf{P}\left[x_m = i_0, \ldots, X_0 = x_0\right] \\
&= \mathbf{P}_{i_0 j} \sum_{x\epsilon V} \mathbf{P}\left[X_m = i_0, \ldots, X_0 = x_0\right] \\
&= \mathbf{P}_{i_0 j} \mathbf{P}\left[T = m \& X_T = i_0\right]
\end{aligned}$$

The above statements thus imply :

$$\frac{\mathbf{P}\left[X_{T+1} = j, X_T = i_0, T = m\right]}{\mathbf{P}\left[T = m \& X_T = i_0\right]} = \mathbf{P}_{i_0 j}$$

Recurrence issue of Infinite Closed Classes

We already know from previous lectures that finite closed classes are recurrent. But in the case of infinite closed classes we cannot say so. We show by an example that it could be either recurrent or transitive depending on the arc probabilities. An example of this is the random walk example. Let p be the probability for the arcs toward the right and q for those to the left.

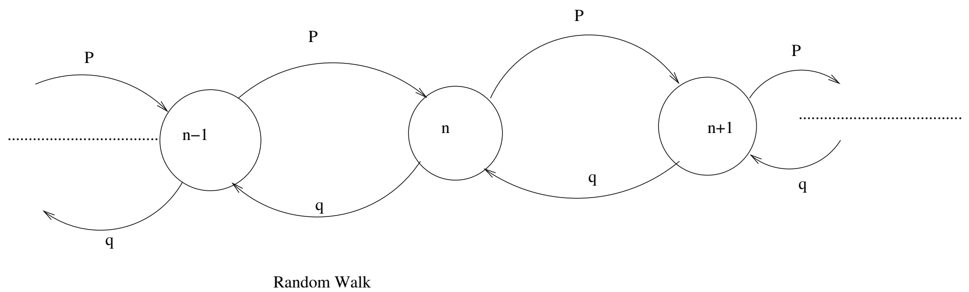

Figure 1: A Random Walk

Let i be a state.

$$\mathbf{P}_{ii}^{2n+1} = 0 \text{ beacause } (2n+1) \text{ is odd}$$

Therefore the probability of ever coming back to the state i is

$$\sum_{\text{all even k}} p^{\frac{k}{2}} q^{\frac{k}{2}}$$

By sterling's approximation it can be shown that the above formula is equal to infinity when p=q=.5 and is less than inifinity otherwise. So in the former case the chain is recurrent and in the second case it is transcient.