We've done

- Dynamic Programming
  - Matrix Chain Multiplication
  - Longest Common Subsequence

Now

- Dynamic Programming
  - Assembly-line scheduling
  - Optimal Binary Search Trees

Next

- Shortest paths algorithms

Assembly Line Scheduling (ALS)

- A factory has two assembly lines with n stations each
  - Line 1: $S_{1,1}, S_{1,2}, \dots, S_{1,n}$
  - Line 2: $S_{2,1}, S_{2,2}, \dots, S_{2,n}$
- Automobile chassis enter one of the lines, have parts added at n stations, and exit at the end of a line
- Enter time for line i is e_i
- Exit time for line i is x_i
- $S_{1,j}$ and $S_{2,j}$ perform the same function (thus, a chassis goes through exactly one of $S_{1,j}$ or $S_{2,j}$)
- Time required at station $S_{i,j}$ is $a_{i,j}$
- Time required to move from $S_{i,j}$ to the other line is $t_{i,j}$
- Time required to move from $S_{i,j}$ to the next station on the same line is 0.

$$\boxed{\text{Find a fastest schedule to complete one auto.}}$$

The recurrence

- f^* : the optimal time
- $f_i[j]$: the fastest time to get through $S_{i,j}$

Then,

$$f^* = \min\{f[1,n] + x_1, f[2,n] + x_2\}.$$

$$f_1[j] = \begin{cases} e_1 + a_{1,1} & \text{if } j = 1 \\ \min\{f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}\} & \text{if } j > 1 \end{cases}$$

$$f_2[j] = \begin{cases} e_2 + a_{2,1} & \text{if } j = 1 \\ \min\{f_2[j-1] + a_{2,j}, f_1[j-1] + t_{1,j-1} + a_{2,j}\} & \text{if } j > 1 \end{cases}$$

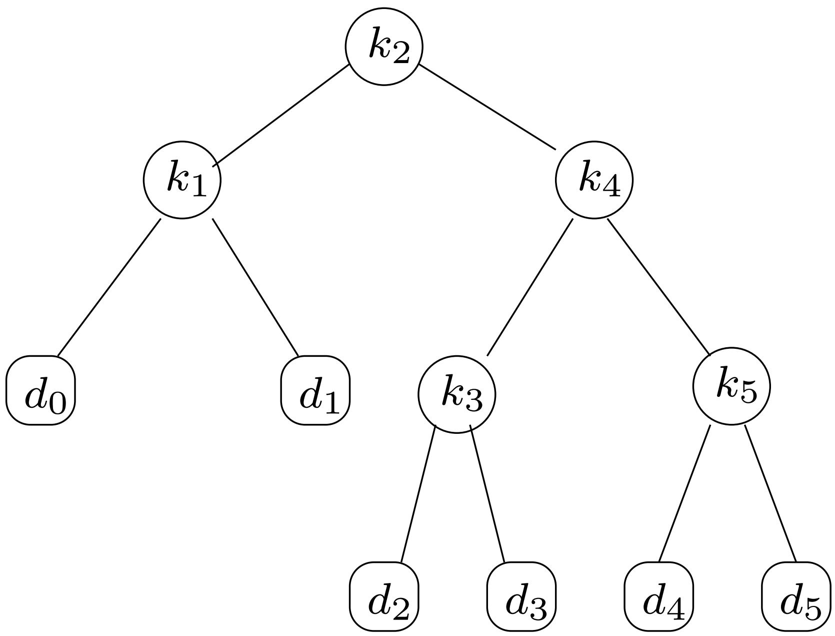
The rest (pseudo code and stuff)

- Straightforward, see text for details
- Running time is linear
- Space used is also linear

Binary search trees

Keys $k_1, k_2, \dots, k_n$, and **dummy keys** $d_0, d_1, \dots, d_n$.
Given an ordering:

$$d_0 < k_1 < d_1 < k_2 < d_2 < \cdots < k_n < d_n.$$



A BST on these keys is a tree satisfying

- For every node v, keys on the left are less than v
- For every node v, keys on the right are greater than v
- Dummy keys are leaf nodes (representing NOT FOUND)

Optimal binary search tree problem

Inputs

- $\{k_1, \dots, k_n\}$
- $\{d_0, \dots, d_n\}$
- $d_0 < k_1 < d_1 < k_2 < d_2 < \cdots < k_n < d_n$
- $p_i = \text{Prob}[k_i \text{ is queried}]$
- $q_i = \text{Prob}[d_i \text{ is queried}]$
  (i.e. the probability that a query is in between k_i and k_{i+1})

Clearly, it is necessary that

$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1.$$

The COST of a query is the number of nodes visited.

$$\begin{aligned} &\text{Expected query cost} \\ = \quad & \sum_{i=1}^{n} (\text{DEPTH}_T(k_i) + 1) \cdot p_i + \sum_{i=0}^{n} (\text{DEPTH}_T(d_i) + 1) \cdot q_i \end{aligned}$$

Construct a binary search tree which minimizes expected query cost.

Step 1: Identify subproblems

- Structure of a BST:
  - A root containing k_r, for some $r \in \{1, \dots, n\}$.
  - Left subtree consists of keys $k_1, \dots, k_{r-1}$, and dummy keys $d_0, \dots, d_{r-1}$
  - Right subtree consists of keys $k_{r+1}, \dots, k_n$, and dummy keys $d_r, \dots, d_n$
  - Subtle point: if $r = 1$, then the left has only d_0; if $r = n$, then the right has only d_n.
- Optimal substructure:
  - For a BST to be optimal, it is necessary that the left subtree and the right subtree are optimal (why?)
- Sub problems:
  - Given $k_i, \dots, k_j$ ($1 \leq i \leq j+1 \leq n+1$) and $d_{i-1}, \dots, d_j$ (no key k_x if $i = j+1$)
  - Construct a BST T_{ij} on these keys that minimizes

$$\sum_{x=i}^{j} (\text{DEPTH}_{T_{ij}}(k_x)+1) \cdot p_x + \sum_{x=i-1}^{j} (\text{DEPTH}_{T_{ij}}(d_x)+1) \cdot q_x$$

Step 2: The recurrence relation

- Given $1 \leq i \leq j+1 \leq n+1$.
- Let $e[i,j]$ denote the expected query cost of an optimal BST on keys $k_i, \dots, k_j$ and dummy keys $d_{i-1}, \dots, d_j$.
- Define

$$w(i,j) := \sum_{x=i}^{j} p_x + \sum_{x=i-1}^{j} q_x.$$

Noting that, if k_r was the root of T_{ij}, then

$$\begin{aligned} e[i,j] &= p_r + (e[i,r-1] + w(i,r-1)) \\ &\quad +(e[r+1,j] + w(r+1,j)) \\ &= e[i,r-1] + e[r+1,j] + w(i,j) \end{aligned}$$

Hence,

$$e[i,j] = \begin{cases} q_{j-1} & \text{if } j = i-1 \\ \min\limits_{i \leq r \leq j} \{e[i,r-1] + e[r+1,j] + w(i,j)\} & \text{if } i \leq j \end{cases}$$

Step 3: How to fill out the table?

- This is very similar to Matrix Chain Multiplication
- Entries $e[i, j]$ are dependent on other $e[x, y]$ with $y - x < j - i$.

Let $root[i, j]$ denote the r for which k_r is at the root of T_{ij}.

We can record $root[i, j]$ while updating $e[i, j]$

The rest is similar to MCM

Step 4: Pseudo code

OPTIMAL-BST(p, q, n)

```
for i = 1 to n + 1 do
    e[i, i − 1] ← q_{i−1} // base cases
end for
for l = 1 to n do
    for i ← 1 to n − l + 1 do
        j ← i + l − 1; // not really needed, just to be clearer
        e[i, j] ← ∞;
        w[i, j] ← w[i, j − 1] + p_j + q_j; // save some time
        for r ← i to j do
            t ← e[i, r − 1] + e[r + 1, j] + w[i, j];
            if e[i, j] > t then
                e[i, j] ← t;
                root[i, j] ← r;
            end if
        end for
    end for
end for
return e[1, n];
```

Step 5: Analysis of time and space

- Time: $\Theta(n^3)$
- Space: $\Theta(n^2)$

Constructing the tree from the table $root$ is easy.