We've done

- Administrative aspects
- A brief overview of the course

Now

- Growth of functions
- Asymptotic notations
- Scare some people off

Next

- Recurrence relations & ways to solve them

Some conventions

$$
\begin{aligned}
\lg n &= \log_2 n \\
\log n &= \log_{10} n \\
\ln n &= \log_e n
\end{aligned}
$$

Growth of functions

Consider a Pentium-4, 1 GHz, i.e. roughly 10^{-9} second for each basic instruction.

	10	20	30	40	50	1000
$\lg\lg n$	1.7 ns	2.17 ns	2.29 ns	2.4 ns	2.49 ns	3.3 ns
$\lg n$	3.3 ns	4.3 ns	4.9 ns	5.3 ns	5.6 ns	9.9 ns
n	10 ns	20 ns	3 ns	4 ns	5 ns	1 μs
n^2	0.1 μs	0.4 μs	0.9 μs	1.6 μs	2.5 μs	1 ms
n^3	1 μs	8 μs	27 μs	64 μs	125 μs	1 sec
n^5	0.1 ms	3.2 ms	24.3 ms	0.1 sec	0.3 sec	277 h
2^n	1 μs	1 ms	1 s	18.3 m	312 h	$3.4 \cdot 10^{282}$ centuries
3^n	59 μs	3.5 s	57.2 h	386 y	227644 c	$4.2 \cdot 10^{458}$ centuries

1.6^{100} ns is approx. 82 centuries (Recall FibA).

$$\lg 10^{10} = 33, \quad \lg\lg 10^{10} = 4.9$$

Some other large numbers

- The age of the universe ≤ 13 G-Years $= 13 \cdot 10^7$ centuries.
- $4 * 10^{78} \leq$ Number of atoms is the universe $\leq 6 * 10^{79}$.
- Number of seconds since big-bang $\approx 10^{17}$.
- The probability that a monkey can compose Hamlet is $\frac{1}{10^{40}}$ (what's the philosophical implication of this?).

Talking about large numbers

Just for fun:

What's the largest number you could describe using thirteen English words?

How about:

"Nine trillion trillion ... trillion"

(The *trillion* is repeated 12 times)

Note:

A googol $= 10^{100}$.

Dominating Terms

Compare the following functions:

$$\begin{aligned} f_1(n) &= 2000n^2 + 1,000,000n + 3 \\ f_2(n) &= 100n^2 \\ f_3(n) &= n^5 + 10^7 n \\ f_4(n) &= 2^n + n^{10,000} \\ f_5(n) &= 2^n \\ f_6(n) &= \frac{3^n}{10^6} \end{aligned}$$

when n is "large" (we often say "sufficiently large")

Behind comparing functions

Mathematically, $f(n) \gg g(n)$ for "large enough" n means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

We also say $f(n)$ is asymptotically larger than $g(n)$. They are "comparable" if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$$

and $f(n) \ll g(n)$ for large n means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

In this case, $f(n)$ is asymptotically smaller than $g(n)$.

Asymptotic notations give a convenient way to formalize these concepts.

Asymptotic Notations

$$
\begin{aligned}
f(n) = O(g(n)) &\quad \text{iff} \quad \exists c > 0, n_0 > 0 : f(n) \leq cg(n), \text{ for } n \geq n_0 \\
f(n) = \Omega(g(n)) &\quad \text{iff} \quad \exists c > 0, n_0 > 0 : f(n) \geq cg(n), \text{ for } n \geq n_0 \\
f(n) = \Theta(g(n)) &\quad \text{iff} \quad f(n) = O(g(n)) \;\&\; f(n) = \Omega(g(n)) \\
f(n) = o(g(n)) &\quad \text{iff} \quad \forall c > 0, \exists n_0 > 0 : f(n) \leq cg(n), \text{ for } n \geq n_0 \\
f(n) = \omega(g(n)) &\quad \text{iff} \quad \forall c > 0, \exists n_0 > 0 : f(n) \geq cg(n), \text{ for } n \geq n_0
\end{aligned}
$$

Note:

- we shall be concerned only with functions f of the form $f : \mathbb{N}^+ \to \mathbb{R}^+$, unless specified otherwise.
- $f(n) = \mathbf{x}(g(n))$ isn't mathematically correct

Some examples

$$\begin{aligned} a(n) &= \sqrt{n} \\ b(n) &= n^5 + 10^7 n \\ c(n) &= (1.3)^n \\ d(n) &= (\lg n)^{100} \\ e(n) &= \frac{3^n}{10^6} \\ f(n) &= 3180 \\ g(n) &= n^{0.0000001} \\ h(n) &= (\lg n)^{\lg n} \end{aligned}$$

A few properties

$$f(n) = o(g(n)) \quad \Rightarrow \quad f(n) = O(g(n)) \;\&\; f(n) \neq \Theta(g(n)) \tag{1}$$

$$f(n) = \omega(g(n)) \quad \Rightarrow \quad f(n) = \Omega(g(n)) \;\&\; f(n) \neq \Theta(g(n)) \tag{2}$$

$$f(n) = O(g(n)) \quad \Leftrightarrow \quad g(n) = \Omega(f(n)) \tag{3}$$

$$f(n) = \Theta(g(n)) \quad \Leftrightarrow \quad g(n) = \Theta(f(n)) \tag{4}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty \quad \Leftrightarrow \quad f(n) = \omega(g(n)) \tag{5}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0 \quad \Rightarrow \quad f(n) = \Theta(g(n)) \tag{6}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \quad \Leftrightarrow \quad f(n) = o(g(n)) \tag{7}$$

Remember: we only consider functions from $\mathbb{N}^+ \to \mathbb{R}^+$.

A reminder: L'Hôpital's rule

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

if

$$\lim_{n\to\infty} f(n) \text{ and } \lim_{n\to\infty} g(n) \text{ are both 0 or both } \pm\infty.$$

Examples:

$$\begin{aligned} \lim_{n\to\infty} \frac{\lg n}{\sqrt{n}} &= 0 \\ \lim_{n\to\infty} \frac{(\lg n)^{\lg n}}{\sqrt{n}} &= ? \end{aligned}$$

Stirling's approximation

For all $n \geq 1$,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n},$$

where

$$\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}.$$

It then follows that

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

The last formula is often referred to as the "Stirling's approximation"

More examples

$$
\begin{aligned}
a(n) &= \lfloor \lg n \rfloor ! \\
b(n) &= n^5 + 10^7 n \\
c(n) &= 2^{\sqrt{\lg n}} \\
d(n) &= (\lg n)^{100} \\
e(n) &= 3^n \\
f(n) &= (\lg n)^{\lg \lg n} \\
g(n) &= 2^{n^{0.001}} \\
h(n) &= (\lg n)^{\lg n} \\
i(n) &= n!
\end{aligned}
$$

Special functions

Some functions cannot be compared, e.g. $n^{1+\sin(n\frac{\pi}{2})}$ and n.

$$\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\},$$

where for any function $f : \mathbb{N}^+ \to \mathbb{R}^+$,

$$f^{(i)}(n) = \begin{cases} n & \text{if } i = 0 \\ f(f^{(i-1)}(n)) & \text{if } i > 0 \end{cases}$$

Intuitively, compare

$$\begin{aligned} \lg^* n \quad & \text{vs} \quad \lg n \\ \lg^* n \quad & \text{vs} \quad (\lg n)^\epsilon,\ \epsilon > 0 \\ 2^n \quad & \text{vs} \quad n^n \\ \lg^*(\lg n) \quad & \text{vs} \quad \lg(\lg^* n) \end{aligned}$$

How about rigorously?

Asymptotic notations in equations

$$5n^3 + 6n^2 + 3 = 5n^3 + \Theta(n^2)$$

means "the LHS is equal to $5n^3$ plus "some" function which is $\Theta(n^2)$."

$$o(n^6) + O(n^5) = o(n^6)$$

means "for any $f(n) = o(n^6)$, $g(n) = O(n^5)$, the function $h(n) = f(n) + g(n)$ is $o(n^6)$." Why?

Be very careful!! For example:

$$O(n^5) + \Omega(n^2) \stackrel{?}{=} \Omega(n^2)$$
$$O(n^5) + \Omega(n^2) \stackrel{?}{=} O(n^5)$$

Tight and not tight

$n \log n = O(n^2)$ is not tight

$n^2 = O(n^2)$ is tight

What to focus on when comparing functions asymptotically?

- Determine the dominating term
- Use intuition first
- Transform intuition into rigorous proof.