We've done

- Growth of functions
- Asymptotic Notations $(O, o, \Omega, \omega, \Theta)$

Now

- Recurrence relations, solving them, Master theorem

Next

- Sorting

Recurrence Relations: Examples

FibA

$$T(n) = T(n-1) + T(n-2) + \Theta(1)$$

Binary search

$$T(n) \leq T(\lceil n/2 \rceil) + \Theta(1)$$

Merge sort

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n) \tag{1}$$

and tons of others

$$\begin{aligned} T(n) &= 4T(n/2) + n^2 \lg n \\ T(n) &= 3T(n/4) + \lg n \\ T(n) &= T(n/a) + T(a) \end{aligned}$$

As we've done earlier, equations like (1) means:

"$T(n)$ is $T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil)$ plus some function $f(n)$ which is $\Theta(n)$"

Recurrent Relations: Methods

- Substitution method
- Recurrence tree
- Generating functions and others (take CSE 594 next sem)
- Master Theorem

The Substitution Method

Mostly to show O and Ω relations

- Guess a solution
- Use induction to show it works

Recall the first lecture

$$T(n) = \begin{cases} a & \text{if } n \leq 1 \\ T(n-1) + T(n-2) + b & \text{if } n \geq 2 \end{cases}$$

then we "guess"

$$T(n) = (a+b)F_{n+1} - b \tag{2}$$

where F_n is the nth Fibonacci number

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right) = \Theta(\phi^n) \end{aligned}$$

Hence (why?),

$$T(n) = \Theta(\phi^n) \tag{3}$$

We can show both (2) & (3) by induction.

More examples

- Given

$$
\begin{aligned}
T(1) &= \Theta(1) \\
T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n),
\end{aligned}
$$

  which means

$$
\begin{aligned}
T(1) &= c_0 \\
T(n) &\geq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_1 n \\
T(n) &\leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + c_2 n
\end{aligned}
$$

- Guess: $T(n) = \Theta(n \lg n)$.
- Need to show by induction that there are $a, b > 0$ s.t.

$$an \lg n \leq T(n) \leq bn \lg n.$$

Now try

- $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$.
- Guess $T(n) = \Theta(n)$.

Misc. Notes

1. It is often safe to ignore integrality issues. For example:

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n,$$

can be solved by saying

$$T(n) = 2T(n/2) + n$$

which gives the exact same solution.

2. To see why integrality isn't important, consider

$$T(n) = 2T(\lfloor n/2 \rfloor + 17) + n.$$

It is still $O(n \lg n)$. (Induction)

3. Avoid doing

$$\begin{aligned} T(n) &\leq 2c\lfloor n/2 \rfloor + n \\ &\leq cn + n \\ &= O(n) \end{aligned}$$

which is dead wrong!!

Change of variable

Solve

$$T(n) = 2T(\sqrt{n}) + 1$$

Let $m = \lg n$, then

$$T(2^m) = 2T(2^{m/2}) + 1$$

Let $S(m) = T(2^m)$, then

$$S(m) = 2S(m/2) + 1$$

Hence,

$$S(m) = O(m)$$

Thus,

$$T(n) = S(\lg n) = O(\lg n)$$

How to guess?

Iterating the recurrence a few times

Like we did with the Fibonacci case.

Use recursion-tree method

Let's try

$$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2).$$

Recursion tree suggests $T(n) = O(n^2)$. Now rigorously shows it by induction.

Sick of induction already?

Now try

$$T(n) = T(n/3) + T(2n/3) + O(n)$$

Master Theorem

Let $a \geq 1$, $b > 1$ be constants. Suppose

$$T(n) = aT(n/b) + f(n),$$

where n/b could either be $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$. Then

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then

$$T(n) = \Theta(n^{\log_b a})$$

2. If $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \lg n)$$

3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ for all sufficiently large n, then

$$T(n) = \Theta(f(n))$$

Examples

- $T(n) = 8T(n/2) + n^5$
- $T(n) = 8T(n/2) + n^4$
- $T(n) = 8T(n/2) + n^3$
- $T(n) = 3T(n/2) + n^2$
- $T(n) = 3T(n/2) + n$

Notes

- There is a gap between case 1 & case 2
- There is a gap between case 2 & case 3
- There is a gap within case 3