Introduction to Graph Coloring

The authoritative reference on graph coloring is probably [Jensen and Toft, 1995]. Most standard texts on graph theory such as [Diestel, 2000, Lovász, 1993, West, 1996] have chapters on graph coloring. Some nice problems are discussed in [Jensen and Toft, 2001].

1 Basic definitions and simple properties

A k-*coloring* of a graph $G = (V, E)$ is a function $c : V \to C$, where $|C| = k$. (Most often we use $C = [k]$.) Vertices of the same color form a *color class*. A coloring is *proper* if adjacent vertices have different colors. A graph is k-*colorable* if there is a proper k-coloring. The *chromatic number* $\chi(G)$ of a graph G is the minimum k such that G is k-colorable.

Let H and G be graphs. The *disjoint union* $G+H$ of G and H is the graph whose vertices and edges are disjoint unions of vertices and edges of G and H, respectively. The *join* $G \vee H$ of simple graphs G and H is obtain from $G + H$ by adding all edges of the form (u, v), with $u \in V(G)$, $v \in V(H)$. The *cartesian product* $G \Box H$ of G and H is the graph with vertex set $V(G) \times V(H)$ and edges of the form $((u, v), (u', v'))$, where either $u = u'$ and $(v, v') \in E(H)$, $v = v'$ and $(u, u') \in E(G)$.

Exercise 1.1. Show that $G \Box H$ is isomorphic to $H \Box G$.

Recall that the *clique number* $\omega(G)$ of a graph G is the maximum clique size; the *independent number* $\alpha(G)$ is the size of a maximum independent set.

We put a few simple observations in the following proposition.

Proposition 1.2. *Let G and H be simple graphs. Then,*

$$\chi(G) \geq w(G) \tag{1}$$
$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \tag{2}$$
$$\chi(G+H) = \max\{\chi(G), \chi(H)\} \tag{3}$$
$$\chi(G \vee H) = \chi(G) + \chi(H) \tag{4}$$
$$\chi(G \Box H) = \max\{\chi(G), \chi(H)\} \tag{5}$$

Proof. Only the last identity deserves discussion. We only need to show that $k = \max\{\chi(G), \chi(H)\}$ is sufficient to color $G \Box H$. We shall color each copy of G in $G \Box H$ using a coloring of G, and then shift all colors up by an amount equal to the color of the vertex of H that this copy of G corresponds to. More precisely, let g and h be optimal colorings of G and H, respectively, then

$$f(u, v) = g(u) + h(v) \pmod k$$

is a proper coloring of $G \Box H$. $\Box$

Exercise 1.3. Let C_n denote a cycle of size n, and $r \geq 2$ and s be positive integers. Show that $\chi(C_{2r+1} \vee K_s) = s + 3$, while $\omega(C_{2r+1} \vee K_s) = s + 2$. (This shows that $\chi(G)$ might be greater than $\omega(G)$.)

Exercise 1.4. Prove that a graph G is m-colorable if and only if $\alpha(G \square K_m) \geq |V(G)|$.

Exercise 1.5. Let G be a graph where every two odd cycles have at least a vertex in common. Prove that $\chi(G) \leq 5$.

Exercise 1.6. Consider the infinite graph G defined as follows. The vertex set V is $\mathbb{R}^2$. Two points in $\mathbb{R}^2$ are adjacent if their Euclidean distance is 1. Show that $4 \leq \chi(G) \leq 7$.

A graph G is k-*critical* if its chromatic number is k, and every proper subgraph of G has chromatic number less than k. Clearly every k-chromatic graph contains a k-critical subgraph. Actually finding a k-critical subgraph is a difficult problem, though.

Theorem 1.7 ([Szekeres and Wilf, 1968]).

$$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H).$$

Proof. Let $k = \chi(G)$, and $\bar{H}$ a critical subgraph of G. It is sufficient to show that $k \leq 1 + \delta(\bar{H})$. Let v be a vertex with degree $\delta(\bar{H})$, then $\chi(\bar{H} - v) = k - 1$, since $\bar{H}$ is k-critical. If $\delta(\bar{H}) \leq k - 2$, then adding v back in does not require the kth color. Thus, $\delta(\bar{H}) \geq k - 1$ as desired. □

2 Greedy Coloring

Let $v_1, \ldots, v_n$ be some ordering of $V(G)$. For i from 1 to n, greedily assign to v_i the lowest indexed color not yet assigned to lower-index neighbor of v_i. This coloring is called the *greedy coloring* with respect to the ordering.

Theorem 2.1 (Welsh-Powell, 1967). *Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degree sequence of a graph G, then*

$$\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\}.$$

Proof. Suppose $\deg(v_i) = d_i$. Apply greedy coloring to the ordering $v_1, \ldots, v_n$. □

Corollary 2.2. $\chi(G) \leq 1 + \Delta(G)$.

The bound in the corollary is not as good as the bound in the theorem. For $r \geq 2$, the graph $G = C_{2r+1} \vee K_s$ has $\Delta(G) = \max\{2r+s, 2+s\} = 2r+s$, and degree sequence $d_1 = \cdots = d_s = 2r+s$, $d_{s+1} = \cdots = d_{s+2r+1} = 2 + s$. Thus, while $1 + \Delta(G) = 1 + 2r + s$,

$$1 + \max_i \min\{d_i, i - 1\} = \max\{\min\{2r + s, s - 1\}, \min\{2 + s, 2r + s\}\} = 3 + s,$$

which is optimal.

Theorem 2.3 ([Brooks, 1941]). *Let G be a connected graph, then $\chi(G) \leq \Delta(G)$, unless G is a complete graph or an odd cycle.*

Proof. Let $k = \Delta(G)$. We can assume $k \geq 3$ and G is neither a complete graph nor an odd cycle. We will try to produce an ordering $v_1, \ldots, v_n$ such that every vertex has at most $k-1$ lower-index neighbors.
Case 1: G is not k-regular. There is a vertex v_n with degree $\leq k - 1$. Consider any spanning tree T of G with v_n as the root. Visit vertices of T level by level, while numbering the vertices in decreasing order starting from v_n at level 0, and we get the desired ordering.
Case 2: G is k-regular.

Case 2a: G is 1-connected. Let x be a cut vertex of G. Let C be a component of $G - x$. Let C' be the union of the other components of $G - x$. Let H and H' be the subgraphs of G induced by $V(C) \cup \{x\}$

and $V(C') \cup \{x\}$, respectively. The degrees of x in H and H' are less than k, hence both H and H' are k-colorable. Fix a proper k-coloring of H, while permuting the colors of a proper k-coloring of H' to match the two colors of x, we get a proper k-coloring of G.

Case 2b: G is 2-connected. One of the key ideas here is to produce an ordering where v_n has two neighbors of the same color. In particular, suppose G has a vertex v_n with two neighbors v_1 and v_2 which are not adjacent, and $H = G - \{v_1, v_2\}$ is connected. Since the degree of v_n is H is at most $k-2$, we can produce an ordering $v_3, \ldots, v_n$ of vertices of H such that each vertex has at most $k-1$ lower-index neighbors. The greedy coloring applied to $v_1, \ldots, v_n$ then produces a proper k-coloring.

How do we find such a triple v_1, v_2, v_n of vertices? We want non-adjacent v_1, v_2 to be of distance 2 from one another, and $G - \{v_1, v_2\}$ to be connected.

Consider any vertex x. If $\kappa(G - x) \geq 2$, then certainly we can take $v_1 = x$, v_2 any vertex distance 2 away from x (why such a vertex exists?), and v_n be any common neighbor of v_1 and v_2.

Hence, we can assume $\kappa(G - x) = 1$. The graph $G - x$ consists of *blocks*, which are maximal subgraphs of $G - x$ which have no cut-vertex. Since $\kappa(G - x) = 1$, there must be at least 2 such blocks. Also, since G is 2-connected, x must have a neighbor in every block of $G - x$ which is not a cut-vertex of G. Call v_1, v_2 two such neighbors from different blocks. Then, $G - \{x, v_1, v_2\}$ is connected. Since x has degree at least 3, $G - \{v_1, v_2\}$ is connected as well, which completes the proof. □

Exercise 2.4. Show that every graph G has a vertex coloring with respect to which the greedy coloring uses $\chi(G)$ colors.

Exercise 2.5. Consider a set of straight lines on a plane with no three meeting at a point. Form a graph G whose vertices are intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Show that $\chi(G) \leq 3$.

One might ask how bad can greedy coloring be with respect to the optimal coloring. (What is the approximation ratio of greedy coloring?) The following exercise gives a pretty bad lower bound. Note that the chromatic number of any tree is at most 2.

Exercise 2.6. For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering π of $V(T_k)$ such that greedy coloring with respect to π uses $k+1$ colors. (Hint: use inductive construction.)

Exercise 2.7. Show that $\chi(G) = \omega(G)$ when $\bar{G}$ is bipartite.

Exercise 2.8. *Improvement of Brooks' Theorem*

(a) Given a graph G. Let $k_1, \ldots, k_t$ be non-negative integers with $\sum k_i \geq \Delta(G) - t + 1$. Prove that $V(G)$ can be partitioned into sets $V_1, \ldots, V_t$ so that for each i, the subgraph G_i induced by V_i has maximum degree at most k_i.

(b) For $4 \leq r \leq \Delta(G) + 1$, use part (a) to prove that $\chi(G) \leq \left\lceil \frac{r-1}{r}(\Delta(G)+1) \right\rceil$ when G has no r-clique.

Exercise 2.9 ([Albertson, 1998]). Let G be a k-colorable graph, and let S be a set of vertices in G such that $d(x, y) \geq 4$ whenever $x, y \in S$. Prove that every coloring of S with colors from $[k+1]$ can be extended to a proper $(k+1)$-coloring of G.

3 Orientations

An *orientation* of a graph G is a directed graph obtained from G by choosing an orientation $u \to v$ or $v \to u$ for each edge $uv \in E(G)$.

Consider an optimal coloring of G with colors in $[k]$, where $k = \chi(G)$. Suppose we orient each edge $(u, v) \in G$ from the smaller color to the larger color. Then, the longest path length in the orientation is at most $\chi(G) - 1$. What about the reverse, i.e. given an orientation D of G with longest path length $l(D)$, can we produce a coloring which uses at most $l(D) + 1$ colors?

Theorem 3.1 ([Gallai, 1968, Roy, 1967, Vitaver, 1962]). *If D is an orientation of G with longest path length $l(D)$, then $\chi(G) \leq 1 + l(D)$. Moreover, equality holds for some orientation of G.*

Proof. The fact that equality holds was noted above. We now show that there is some proper $(1+l(D))$-coloring.

Consider a maximal acyclic subdigraph D' of D. Note that $V(D') = V(D) = V$ (why?). Color $V(D')$ by assigning to each $v \in V(D')$ the length of a longest path in D' ending at v. It is easy to see that the colors strictly increase along any path P of D'.

Consider any edge $(u, v) \in E(D)$. There must be a path in D' connected u and v, since either (u, v) itself is in D', or adding (u, v) to D' creates a cycle. Since colors strictly increase along any path in D', u and v have different colors. $\square$

Exercise 3.2 (Minty's Theorem [Minty, 1962]). An *acyclic orientation* of a loop-less graph is an orientation having no cycle. For each acyclic orientation D of G, let $r(D) = \max_C \lceil a/b \rceil$, where C is a cycle in G and a, b count the edges of C that are forward in D or backward in D, respectively. Fix a vertex $x \in V(G)$, and let W be a walk in G beginning at x. Let $g(W) = a - b \cdot r(D)$, where a is the number of steps along W that are forward edges in D and b is the number that are backward in D. For each $y \in V(G)$, let $g(y)$ be the maximum of $g(W)$ such that W is an x, y-walk (assume that G is connected).

(a) Prove that $g(y)$ is finite and thus well-defined, and use $g(y)$ to obtain a proper $1 + r(D)$-coloring of G. Thus, G is $1 + r(D)$-colorable.

(b) Prove that
$$\chi(G) = \min_{D \in \mathcal{D}} (1 + r(D)),$$
where $\mathcal{D}$ is the set of acyclic orientations of G.

Exercise 3.3. Use Minty's Theorem to prove Theorem 3.1

4 Edge Coloring

A k*-edge-coloring* of a graph $G = (V, E)$ is a function $c : E \to C$, where $|C| = k$. (Most often we use $C = [k]$.) An edge-coloring is *proper* if edges incident to the same vertex have different colors. A graph is k*-edge-colorable* if there is a proper k-edge-coloring. The *chromatic index* $\chi'(G)$ of a graph G is the minimum k such that G is k-edge-colorable.

The *line graph* of a graph G, denoted by $L(G)$, is the graph whose vertices are edges of G, and two vertices of $L(G)$ are adjacent iff they are incident to the same vertex of G.

Exercise 4.1. Show that $\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$, for all undirected graphs G,

We already know one result regarding χ':

Theorem 4.2 ([König, 1916]). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

What about cases when G is not bipartite? The following theorem, often referred to as Vizing's theorem, mostly answered this question for simple graphs.

Theorem 4.3 ([Vizing, 1964, Vizing, 1965, Gupta, 1966]). *If G is a simple graph, then*

$$\chi'(G) \leq \Delta(G) + 1.$$

Proof. Let $\Delta = \Delta(G)$. With at most $\Delta + 1$ colors, we shall start from the empty coloring, consider any uncolored edge (u, v) at each step, find a color for (u, v) while possibly altering the current coloring. Repeat the process until all edges are colored.

Suppose (u, v) is not yet colored with the current coloring. Let S be the set of $\Delta + 1$ colors we are going to use. Since the maximum degree is Δ, not all colors appear at any particular vertex.

If there is a color not appearing at both u and v, then we can use that color for (u, v). Otherwise, suppose color c_0 does not appear at u, and c_1 does not appear at V. Note that $c_1 \neq c_0$. Also, let $v_0 = v$.

There must be a neighbor v_1 of u such that (u, v_1) is colored c_1. Let c_2 be a color missing at v_1. If c_2 is also missing at u, then we can "down-shift" the colors from v_1 by coloring (u, v_1) with c_2, (u, v_0) with c_1. Hence, we can assume there is a neighbor v_2 at u such that (u, v_2) gets colored c_2.

This process cannot continue forever, hence either we can down-shift from some v_k and find a color for (u, v), or there is a smallest index $k \geq 1$ such that a color missing at v_k appeared earlier in the list $c_1, \dots, c_{k-1}$, say c_l.

If c_0 is missing at v_k, then we can color (u, v_k) with c_0, and shift the colors down. Otherwise, let P be a maximal path starting from v_k which alternate colors between c_0 and c_l. Note that there is only one such path.

If P goes $v_k, \dots, v_l, u$, then we can switch the two colors on P, and down-shift from v_l. If P goes $v_k, \dots, v_{l-1}$ (and must stop there at color c_0, since c_l is missing at v_{l-1}), then we switch colors on P, assign (u, v_{l-1}) with color c_0, and shift down from v_{l-1}. Lastly, if P does not touch u, v_l, or v_{l-1}, plus the fact that P cannot end at v_k, then we can switch colors on P, assign (u, v_k) with c_0, and down-shift from v_k. □

Unlike the vertex coloring case, the edge multiplicities play an important role in determining $\chi'(G)$. Let $\mu(G)$ denote the maximum edge multiplicity, then one can show the following theorem.

Theorem 4.4 ([Vizing, 1964, Vizing, 1965, Gupta, 1966]). *For any undirected graph G,*

$$\chi'(G) \leq \Delta(G) + \mu(G).$$

The theorem follows from a result stated in exercise 4.5.

Exercise 4.5 ([Andersen, 1977, Goldberg, 1977, Goldberg, 1984]). Let $G = (V, E)$ be an undirected graph. Let $\mathbf{P} := \{x, y, z \in V : y \in \Gamma(x) \cap \Gamma(z)\}$. Then,

$$\chi'(G) \leq \max \left\{ \Delta(G), \max_{\mathbf{P}} \left\lfloor \frac{1}{2}(d(x) + \mu(xy) + \mu(yz) + d(z)) \right\rfloor \right\}. \tag{6}$$

Exercise 4.6. Use the result of Exercise 4.5 to prove Theorem 4.4

Theorem 4.7 ([Shannon, 1949]). *Let G be any graph, then $\chi'(G) \leq \frac{3}{2}\Delta(G)$.*

Proof. Let $k = \chi'(G)$. Let H be a minimal subgraph of G such that $\chi'(H) = k$. Let $e = (u, v)$ be an edge of H with multiplicity $\mu(H)$. Then, $\chi'(H - e) = k - 1$.

Consider a proper $(k-1)$-edge-coloring of $H - e$ with a set C of $k - 1$ colors. Let X and Y be the sets of colors of edges around u and v, respectively. Clearly, $(C - X) \cap (C - Y) = \emptyset$, $|X| \leq \Delta(H) - 1$, and $|Y| \leq \Delta(H) - 1$. Hence,

$$\begin{aligned} k - 1 = |C| &= |C - X| + |C - Y| + |X \cap Y| \\ &\geq 2(k-1) - (|X| + |Y|) + \mu(H) - 1 \\ &\geq 2(k-1) - 2(\Delta(H) - 1) + \mu(H) - 1. \end{aligned}$$

which implies $k \leq 2\Delta(H) - \mu(H)$. This inequality along with the fact that $k \leq \Delta(H) + \mu(H)$ lead to the desired result. □

Exercise 4.8. For any simple graph G which is k-connected, show that $L(G)$ is k-connected and $(2k-2)$-edge-connected.

Exercise 4.9. Give an explicit edge-coloring to show that $\chi'(K_{m,n}) = \Delta(K_{m,n})$, where $K_{m,n}$ is the complete bipartite graph whose color classes have sizes m and n.

Exercise 4.10. Let D be a directed graph (possibly with loops), where the in-degrees and the out-degrees are at most d. Show that there is a d-edge-coloring of D such that all in-edges at a particular vertex have different colors, so do all out-edges at a vertex.

Exercise 4.11. Let G be a regular graph with a cut-vertex. Show that $\chi'(G) > \Delta(G)$.

Exercise 4.12 ([de Werra, 1971, McDiarmid, 1972]). Let G be a simple graph. Let $k = \Delta(G)$. Show that G has a proper $(k+1)$-edge-coloring in which each color is used $\lceil |E(G)|/(k+1) \rceil$ or $\lfloor |E(G)|/(k+1) \rfloor$ times.

Exercise 4.13. Let G and H be simple graphs which have more than one vertex. Suppose $\chi'(H) = \Delta(H)$. Show that $\chi'(G \square H) = \Delta(G \square H)$.

5 List Coloring

Suppose there is a color list $L(v)$ at each vertex v of a graph G. A *list coloring* (also called a *choice function*) is a proper coloring c such that $c(v) \in L(v), \forall v \in V(G)$. The graph G is *k-choosable* (or *k-list-colorable*) is there is a list coloring for any assignment of size k color lists to vertices of G. The *choosability* of G (also called *list chromatic number* or *choice number*), denoted by $\chi_l(G)$, is the minimum k such that G is k-choosable. It should be clear that

$$\chi(G) \leq \chi_l(G) \leq \Delta(G) + 1.$$

An upper bound of $\chi_l(G)$ in terms of $\chi(G)$ does not exist, since there are bipartite graphs ($\chi = 2$) with arbitrarily large choice number:

Theorem 5.1 ([Alon, 1992]). *There are positive constants c_1 and c_2, such that for any integers $p, q \geq 2$, we have*

$$c_1 p \log q \leq \chi_l(K_{p,q}) \leq c_2 p \log q.$$

There is, however, an analog of Brooks theorem:

Theorem 5.2 ([Vizing, 1976, Erdős et al., 1980]). *Let G be a simple connected graph. Let $d = \Delta(G)$. Then, $\chi_l(G) \leq d + 1$, and $\chi_l(G) = d + 1$ if an only if G is an odd cycle or a complete graph.*

Let us first give a simple example from [Erdős et al., 1980] which gives some intuition on the choice number.

Proposition 5.3. *$K_{m,m}$ is not k-choosable if $m \geq \binom{2k-1}{k}$.*

Proof. We use a pigeonhole-like idea. We only need to consider $m = \binom{2k-1}{k}$. Consider a list assignment where the vertices on each color class of $K_{m,m}$ get distinct k-subsets of $[2k-1]$. Suppose there exists a proper k-list-coloring for $K_{m,m} = (A, B; A \times B)$. Let S be the set of colors the coloring assigns to vertices of A. If $|S| \geq k$, then a vertex $v \in B$ with $L(v) \subseteq S$ cannot be colored. If $|S| \leq k - 1$, then a vertex $v \in A$ with $L(v) \subseteq [2k-1] - S$ cannot be colored. □

Heawood (1890) showed that every planar graph is 5-colorable (the five color theorem) using an argument earlier by Kempe (1879) who gave a flawed proof of the four color theorem. [Vizing, 1976] and [Erdős et al., 1980] conjectured that planar graphs are 5-choosable. [Voigt, 1993] and [Mirzakhani, 1996] constructed examples of planar graphs which are not 4-choosable. [Thomassen, 1994] proved the conjecture, giving another entirely different proof of the five coloring theorem. For good introductions to topological graph theory from combinatorial perspectives, see [Mohar and Thomassen, 2001, Bonnington and Little, 1995].

Theorem 5.4 ([Thomassen, 1994]). *Every planar graph is* 5*-choosable.*

Proof. Consider a plane graph G. Keep adding edges so that every bounded face is a triangle, the outer face is a cycle $C = (v_1, \dots, v_m)$ (in clock-wise order), while keeping the planarity of G. Doing so does not reduce the choice number of G. We will show that all such plane graphs are 5-choosable by inductively showing the following statement:

> Suppose v_1 and v_m have been assigned with two different colors, $v_2, \dots, v_{m-1}$ have color lists of size 3, and the rest of the vertices have lists of size 5. Then it is possible to extend this coloring to a proper list coloring of G.

The statement is trivially true for $|V(G)| = 3$, since G is just the triangle C. Suppose $|V(G)| \geq 4$.

If C has a chord $(v_i, v_j) \in E(G)$, where $1 \leq i \leq j-2 \leq m-2$, and $(v_i, v_j) \neq (v_1, v_m)$. Then, we can apply induction first on the plane graph whose unbounded face is $v_1, \dots, v_i, v_j, \dots, v_m$; and then on the plane graph whose unbounded face is $v_i, v_{i+1}, \dots, v_j$.

Otherwise, let $v_1, u_1, \dots, u_p, v_3$ be the neighbors of v_2, such that $v_1, u_1, \dots, u_p, v_3, \dots, v_m$ is the unbounded face of $G - \{v_2\}$. Choose two colors $\{c_1, c_2\}$ from the list (of size three of v_2) other than the assigned color of v_1. Remove c_1 and c_2 (if any) from the lists (of size 5) of $u_1, \dots, u_p$. Then, our induction hypothesis ensures that $G - \{v_2\}$ is still list colorable. Moreover, none of $u_1, \dots, u_p$ gets either c_1 or c_2. It is possible for v_3 to get c_1 or c_2, in which case we assign v_2 with c_2 or c_1, respectively. $\square$

The edge version is somewhat more interesting. We can define the concepts of *list edge-coloring*, k*-edge-choosable*, and *edge-choosability* $\chi'_l(G)$ in the same fashion. The outstanding open problem is the so-called *list coloring conjecture*, observed by many researchers, including Vizing, Gupta, Albertson, Collins, and most likely appeared first in print in [Bollobás and Harris, 1985].

Conjecture 5.5 (List Coloring Conjecture). For any graph G, we have $\chi'_l(G) = \chi'(G)$.

For simple graphs G, if the conjecture is true, then we get $\chi'_l(G) \leq \Delta(G) + 1$. There have been a series of results aiming at showing this bound: $\chi'_l(G) < c\Delta(G), c > 11/6$ and large $\Delta(G)$ [Bollobás and Harris, 1985], $\chi'_l(G) < (1 + o(1))\Delta(G)$ [Kahn, 1996], plus a few minor improvements [Häggkvist and Janssen, 1997, Molloy and Reed, 2000].

The List Coloring Conjecture for $G = K_{n,n}$ is related to (partial) *Latin squares*: given an n by n matrix where each entry has a list of n symbols, then it is possible to assign to each entry a symbol from its list such that symbols on the same row or same column are distinct. This is known as the *Dinitz problem* or *Dinitz conjecture* (1979). In 1993, Galvin found a proof of the list coloring conjecture for bipartite graphs, settling Dinitz problem as a special case. The proof is not published until 1995 [Galvin, 1995].

We need to set a few things up before proving Galvin's theorem. We have seen how path lengths in orientations relate to graph coloring. There is different relation. In a listing $v_1, \dots, v_n$ of vertices of a graph G for greedy coloring, let us orient edges from higher indexed to lower indexed vertices. Then, the set S_1 of vertices which got colored 1 satisfies the following conditions: (a) S_1 is independent, (b) each vertex $v \in V - S_1$ has an out-edge pointing into S_1. The set S_2 of vertices which got colored 2 satisfies the same conditions with respect to $G - S_1$. Note also that the greedy algorithm produces a k-coloring if the out-degree of each vertex is less than k.

Let D be an orientation of a graph G. For each $v \in V(D)$, let $d^+(v), d^-(v), \Gamma^+(v)$, and $\Gamma^-(v)$ denote the out-degree, in-degree, set of neighbors v is pointing to, and set of neighbors pointing to v, respectively. A set $K \subseteq V(D)$ is called a *kernel* of D if K is independent and $\Gamma^+(v) \cap K \neq \emptyset$ for each $v \in V(D) - K$.

Lemma 5.6. *Let D be an orientation of a graph G, which has a list assignment $L(v)$ for each $v \in V(G)$. Suppose $d^+(v) < |L(v)|$ for all $v \in V(D)$, and every induced subgraph of D has a kernel. Then, there is proper list coloring for G.*

Proof. We show this by induction. For $|V(G)| = 1$, the assertion clearly holds. Suppose $|V(G)| \geq 2$. Let c be any color in some $L(v)$, $v \in V(D)$. Consider the subgraph H of D induced by the set of vertices having c in their color lists. Let U be a kernel of H, and $D' = D - U$. Remove c from the color lists of vertices of D'. We claim that D' still satisfies the condition of the lemma. Consider any vertex $v \in V(D')$. If $v \in V(D) - V(H)$, then $d^+_{D'}(v) < |L(v)|$ as before, as $c \notin L(v)$. If $v \in V(H) - U$, then $|L(v)|$ is reduced by one but $d^+_{D'}$ is also reduced by at least one as v has an out-edge pointing into U. Hence, there is a list coloring of D' which does not involve c. We can then color all vertices in U with c. □

Theorem 5.7 ([Galvin, 1995]). *Every bipartite graph G satisfies $\chi'_l(G) = \chi'(G) = \Delta(G)$.*

Proof. The second equality is the content of König's theorem. We show the first equality here. The idea is to find an orientation of $L(G)$ satisfying the conditions of Lemma 5.6. Let $c : E(G) \to [k]$ be an edge coloring of G, where $k = \chi'(G) = \Delta(G)$.

Suppose $G = (A \cup B; E)$. We refer to vertices in A as the *left vertices* and B as *right vertices*. Thus edges may be adjacent on the left or on the right. Consider an orientation D of $L(G)$, where $(e, e') \in E(D)$ if either e and e' are adjacent on the left and $c(e) > c(e')$, or e and e' are adjacent on the right and $c(e) < c(e')$.

We first verify that $d^+(e) < k$, for all $e \in V(D)$. If $c(e) = i$, then the number of e' incident to e on the left with $c(e') < c(e)$ is at most $i-1$, and the number of e' incident to e on the right with $c(e') > c(e)$ is at most $k - i$, for a total of at most $k - 1$. Hence, $d^+(e) \leq k - 1 < k$.

Secondly, we show that every induced subgraph of D has a kernel by induction on the size of the induced subgraph. Consider a subset S of $V(D)$. The base case is trivial. We assume $|S| \geq 2$. Let $D' = D[S]$. Define

$$U := \{e \mid e = (u, v) \in S, c(e) \leq c(e'), \forall e' = (u, v') \in S.\}$$

It is clear that for every $e' \in S - U$, there is some $e \in U$ such that $(e', e) \in E(D)$. Otherwise e' would have been in U. If U is a matching of G, then we are done. Otherwise, there are $e, e' \in U$ which share a common right end point. Without loss of generality, assume $c(e) < c(e')$, so that $(e, e') \in E(D')$. By induction hypothesis, $D' - \{e\}$ has a kernel U'. If $e' \in U'$, then U' is also a kernel of D'. If $e' \notin U'$, then there is some $e'' \in U'$ such that $(e', e'') \in E(D')$. Clearly $c(e') < c(e'')$ and e' and e'' are incident on the right. Which means $(e, e'') \in E(D')$, and U' is also a kernel for D'. □

Exercise 5.8 ([Erdős et al., 1980]). Show that $K_{k,m}$ is k-choosable if and only if $m < k^k$.

Exercise 5.9. Given color lists for vertices of a graph G such that $|L(v)| \geq d(v)$, for all $v \in V(G)$. And, there is some v_0 such that $|L(v_0)| > d(v)$. Show that there is a proper list coloring for G.

Exercise 5.10. A *total coloring* of a graph G assigns a color to each vertex and each edge of G, such that elements of $V(G) \cup E(G)$ have different colors when they are adjacent or incident. Show that every graph G has a total coloring with at most $\chi'_l(G) + 2$ colors.

Exercise 5.11. Show that $\chi_l(K_2^m) = m$.

Exercise 5.12 (Richardson's theorem). Show that every directed graph without odd directed cycle has a kernel.

Exercise 5.13. Show that every bipartite planar graph is 3-choosable.

6 Perfect Graphs

A graph G is *perfect* if every induced subgraph H of G has $\chi(H) = \omega(H)$. (Recall that $\omega(H)$ is the *clique number* of H.) This concept dated back to the works in [Gallai, 1958, Gallai, 1959, Gallai, 1962, Hajnal and Surányi, 1958, Berge, 1960, Berge, 1966, Dirac, 1961]. The concept is closely related to computational complexity and linear programming, but far from being understood. For example, even though computing the chromatic number of a general graph is NP-hard [Karp, 1972], computing the chromatic number of a perfect graph can be done in polynomial time as shown by [Grötschel et al., 1981]. More information on perfect graphs and connections to optimization can be found in [Lovász, 1983, Berge and Chvátal, 1984, Lovász, 1994].

Berge (1960) made two conjectures on perfect graphs, called the *weak perfect graph conjecture* and the *strong perfect graph conjecture*. The weak perfect graph conjecture states that G is perfect if and only if $\overline{G}$ is perfect. Lovász, then 22 year-old, proved the weak perfect graph conjecture [Lovász, 1972a, Lovász, 1972b], turning it into the *perfect graph theorem*. We shall prove the perfect graph theorem later in this section.

The strong perfect graph conjecture was one of the most outstanding and challenging open problems in graph theory, until Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas announced a proof in May 2002. The four joint authors presented their work at a workshop held from October 30 to November 3, 2002 at the American Institute of Mathematics in Palo Alto, California. The proof needs about 150 pages, and the paper is still under reviewed. Some sketches of the proof can be found in [Cornuéjols, 2002]. Graphs which have neither C_{2k+1} nor $\overline{C}_{2k+1}$ ($k \geq 2$) as induced subgraphs are called *Berge graphs*.

Conjecture 6.1 (Strong Perfect Graph Conjecture). A graph is perfect iff it is a Berge graph.

Relevant information on perfect graphs, including many open problems can be found at Chvátal's perfect graph webpage: `http://www.cs.rutgers.edu/~chvatal/perfect/problems.html`

Many classes of graphs are known to be perfect, including *bipartite graphs*, *chordal graphs* (also called *rigid circuit graphs*, or *triangulated graphs*), *comparability graphs*, and *interval graphs*. By the perfect graph theorem, the complements of these graphs are also perfect.

Exercise 6.2. Determine the smallest imperfect graph G such that $\chi(G) = \omega(G)$.

Let us give a sample class of perfect graphs. A simple graph is *chordal* if every cycle of length 4 or more has a chord. In other words, every induced cycle of a chordal graph is a triangle. A vertex v of a graph is *simplicial* if $G[\Gamma(v)]$ is a clique. A *simplicial elimination ordering* (also called *perfect elimination ordering*) of a graph G is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that v_i is simplicial with respect to $G[\{v_i, \ldots, v_n\}]$.

Lemma 6.3 ([Voloshin, 1982, Farber and Jamison, 1986]). *For every vertex v of a chordal graph G, there is a simplicial vertex among the vertices farthest away from v.*

Proof. We induct on $|V(G)|$. If G is not connected, then we can apply induction on the component of G containing v. Thus, we can assume G is connected.

Claim: *if S is a minimal separating set of G, then $G[S]$ is a clique.*

We can assume $|S| \geq 2$. Consider any $u, v \in S$. We show that $uv \in E(G)$. Suppose $uv \notin E(G)$. Both u and v must have a neighbor in every component of $G - S$. Let C_1, C_2 be two components of $G - S$. For $i = 1, 2$, let P_i be a shortest path from u to v in $G[V(C_i) \cup \{u, v\}]$. Then P_1 and P_2 form a cycle of length at least 4, in which the only possible chord is uv. This shows the claim.

Now, back to our lemma. Let d be the longest distance from v. If $d = 1$, then we apply induction on $G - \{v\}$ and any vertex $v' \in V(G - v)$. A simplicial vertex of $G - \{v\}$ is also simplicial for G.

Assume $d \geq 2$. Let A be the set of all vertices of distance d from v, B the set of vertices of distance $d-1$ from v. Clearly B is a separating set of G. Let S be a minimal subset of B which is also separating. Then, S is a clique because of the previous claim. Let C be some component of $G[A]$ which is separated from v by S. Note that vertices in C cannot have neighbors outside of $S \cup V(C)$, and every vertex of C has a neighbor in S.

Let $H = G[S \cup V(C)]$. If there is some vertex in $V(C)$ of distance at least 2 from a vertex u in S, then we can apply induction on u and H to find a simplicial vertex w of H. The vertex w must be in $V(C)$, and w is also be a simplicial vertex of G, as desired. Otherwise, we can assume S and $V(C)$ are completely connected. If C is complete, then any vertex of C would do. Otherwise, we can apply induction on H and any $u \in V(C)$ which has some vertex of distance at least 2 away from it (in H). The resulting simplicial vertex of H must be in C. □

Theorem 6.4 ([Dirac, 1961]). *A graph is chordal iff it has a perfect elimination ordering.*

Proof. Suppose G is chordal. Applying Lemma 6.3 inductively we get a perfect elimination ordering for G. Conversely, suppose G has such an ordering. Consider any cycle C of length at least 4 in G. By the time we "remove" the first vertex v from the cycle according to the perfect elimination ordering, v's neighbors form a clique, which means C must have a chord. □

Theorem 6.5 ([Berge, 1960]). *Chordal graphs are perfect.*

Proof. Since every subgraph of a chordal graph G is also chordal, we only need to show that $\chi(G) = \omega(G)$ for a particular chordal graph G. Since $\chi(G) \geq \omega(G)$, it is sufficient to find a coloring of G with as many colors as some clique of G. By Theorem 6.4, there is a perfect elimination ordering $v_1, \ldots, v_n$ for G. We apply greedy coloring on the sequence $v_n, \ldots, v_1$. Let $k = \chi(G)$. Suppose some vertex v_i gets colored k. Since the neighbor set $\Gamma(v_i)$ of v_i in $G[\{v_n, \ldots, v_i\}]$ forms a clique, $G[\Gamma_{v_i} \cup \{v_i\}]$ is a clique of size k. □

Exercise 6.6. Let G be a chordal graph. Show that

(a) G has at most n maximal cliques, and G has exactly n maximal cliques iff G has no edges.

(b) Every maximal clique of G containing no simplicial vertex of G is a separating set.

Exercise 6.7. Show that G is chordal iff for every induced subgraph H of G we have

$$\omega(H) = 1 + \max_{K \subseteq H} \delta(K).$$

The number $1 + \max_{K \subseteq H} \delta(K)$ is called the *Szekeres-Wilf number* of H.

Exercise 6.8. Let $n_k(G)$ be the number of k-cliques in a connected chordal graph G. Show that

$$\sum_{k \geq 1} (-1)^{k-1} n_k(G) = 1.$$

(Hint: induction.)

Exercise 6.9. Let e be an edge of a cycle C in a chordal graph G. Show that there is a triangle containing e.

Exercise 6.10. Let Q be a maximal clique in a chordal graph G. Show that if $G - Q$ is connected, then Q contains a simplicial vertex.

A *transitive orientation* D of a graph G is an orientation such that $xy, yz \in E(D)$ imply $xz \in E(D)$. A graph that has a transitive orientation (such as a bipartite graph) is called a *comparability graph*.

Theorem 6.11 ([Berge, 1960]). *Comparability graphs are perfect.*

Proof. Let D be a transitive orientation of a comparability graph G, then D is also an acyclic orientation. Vertices on every directed path in D form a clique. By Theorem 3.1, $\chi(G) \leq 1 + l(D) \leq \omega(G)$. $\square$

Exercise 6.12. A graph G is an *interval graph* if there exists a set of (closed) real intervals $\{I_v \mid v \in V(G)\}$ such that uv is an edge iff $I_u \cap I_v \neq \emptyset$. Let G be an interval graph. Show that G is a chordal graph and $\bar{G}$ is a comparability graph.

The following theorem settles the weak perfect graph conjecture by Berge, and thus it is called the perfect graph theorem. It was shown by [Lovász, 1972b] in the context of normal hypergraphs.

Theorem 6.13 (Perfect Graph Theorem). *A graph is perfect iff its complement is perfect.*

[Lovász, 1972a] proved another characterization of perfect graph which also proves the weak perfect graph conjecture. [Gasparian, 1996] found a simple and elegant proof of this theorem. We shall present Gasparian's proof.

Theorem 6.14 ([Lovász, 1972a]). *A graph G is perfect if and only if, for all induced subgraphs H of G,*

$$|V(H)| \leq \alpha(H) \cdot \omega(H).$$

Exercise 6.15. Show that Theorem 6.14 implies Theorem 6.13

Before showing these theorems, let us try to find out as much as possible the properties of perfect graphs and their complements. Let $\theta = \theta(G)$ be the minimum size of a partition of $V(G)$ into $V_1, \ldots, V_\theta$ such that $G[V_i]$ is a clique, for all $i = 1, \ldots, \theta$. Then, $\chi(G) = \theta(\bar{G})$ and $\omega(G) = \alpha(\bar{G})$. The relations $\chi(G) \geq \omega(G)$ and $\theta(G) \geq \alpha(G)$ are obvious, since every color class of a coloring of G intersects a clique at at most one vertex, and since every V_i in a clique partition intersects an independent set at at most one vertex. Thus, in order to show that a graph G is perfect, we only need to show $\chi(H) \leq \omega(H)$ for every induced subgraph H of G; in other words, we can show that there is an $\omega(H)$-coloring for H.

Suppose we want to show the perfect graph theorem by induction. At the induction step, if G is perfect but $\bar{G}$ is not, then $\bar{G}$ is a graph all of whose proper induced subgraphs are perfect, but $\bar{G}$ itself is imperfect. In other words, $\bar{G}$ is a minimal imperfect graph. What can we say about a minimal imperfect graph? Clearly, $\chi(\bar{G}) > \omega(\bar{G})$, but $\chi(H) \leq \omega(H)$ for every proper induced subgraph H of $\bar{G}$. Let S be any independent set of $\bar{G}$, then

$$\omega(\bar{G} - S) = \chi(\bar{G} - S) \geq \chi(\bar{G}) - 1 > \omega(\bar{G}) - 1.$$

Hence, $\omega(\bar{G} - S) = \omega(\bar{G})$. Consequently, we get the following lemma.

Lemma 6.16. *Every independent set of a minimal imperfect graph misses at least one maximum clique.*

Let v be a vertex of a perfect graph G. Let $G \circ v$ be the graph obtained from G by *duplicating* v, namely $G \circ v$ is obtained from v by creating another copy v' of v, where v' is adjacent to every neighbor of v, and $vv' \in E(G \circ v)$. (Basically, we replaced v by an edge vv'.) The next lemma is a crucial observation by Lovász.

Lemma 6.17. *If G is perfect and $v \in V(G)$, then $G \circ v$ is perfect.*

Proof. We induct on $|V(G)|$. The base case is trivial. Consider a perfect graph G of size at least 2, and $v \in V(G)$. Let $G' = G \circ v$, with the extra vertex v'. Every proper induced subgraph H' of G' is either isomorphic to an induced subgraph of G or equal to $H \circ v$ for some induced subgraph H of G. Thus, $\chi(H') = \omega(H')$ by induction hypothesis. We only need to show that $\chi(G') \leq \omega(G')$. Let $\omega = \omega(G) = \chi(G)$. It is easy to see that

$$\omega = \omega(G) \leq \omega(G') \leq \chi(G') \leq \chi(G) + 1 = \omega + 1.$$

Hence, we can assume $\omega(G') = \omega$; otherwise, we would be done. This means that v is not part of any maximum clique in G. Let's color G with ω colors. Suppose v gets colored c. Let S be the set of all vertices which got colored c excluding v. Any ω-clique of G must intersect S and miss v. Hence, $\chi(G - S) = \omega(G - S) < \omega$, which means we can color $G - S$ with at most $\omega - 1$ colors, then color $S \cup \{v'\}$ with one additional color for a total of at most ω color. This shows $\chi(G') \leq \omega$ as desired. $\square$

A proof of Theorem 6.13. We only need to show that if G is perfect, then $\bar{G}$ is also perfect. Again, induction is the most natural tool for dealing with perfect graphs. For the induction step, it is sufficient to prove that $\omega(\bar{G}) = \chi(\bar{G})$. Assume otherwise, then $\bar{G}$ is a minimal imperfect graph, which means that every independent set of $\bar{G}$ misses at least one maximum clique of $\bar{G}$. In other words, every clique of G misses at least one maximum independent set of G.

We finish the proof by a counting argument. Let $\mathcal{K}$ be the collection of all cliques of G, and $\mathcal{I}$ be the collection of all maximum independent sets of G. Then, for every $K \in \mathcal{K}$, there is some $I_K \in \mathcal{I}$ such that $K \cap I_K = \emptyset$.

To this end, create a graph G' from G by duplicating each vertex v t_v times, where

$$t_v = |\{K \in \mathcal{K} \mid v \in I_K\}|.$$

(Note that $t_v > 0$.) By Lemma 6.17, G' is a perfect graph. It is easy to see that for some $K_0 \in \mathcal{K}$, we have

$$\omega(G') = \sum_{v \in K_0} t_v = \sum_{v \in K_0} |\{K \in \mathcal{K} \mid v \in I_K\}| = \sum_{K \in \mathcal{K}} |I_K \cap K_0| \leq |\mathcal{K}| - 1.$$

This is because $|I_K \cap K_0| \leq 1, \forall K \in \mathcal{K}$, and $I_{K_0} \cap K_0 = \emptyset$. On the other hand,

$$|V(G')| = \sum_{v \in V(G)} t_v = \sum_{v \in V(G)} |\{K \in \mathcal{K} \mid v \in I_K\}| = \sum_{K \in \mathcal{K}} |I_K| = \alpha(G) \cdot |\mathcal{K}|.$$

Thus, $\omega(G') = \chi(G') \geq |V(G')|/\alpha(G') = |\mathcal{K}|$, a contradiction. $\square$

A proof of Theorem 6.14. Necessity is obvious. We show sufficiency by induction on $|V(G)|$. For the induction step, suppose G is a graph of order at least 2 such that $|V(H)| \leq \alpha(H) \cdot \omega(H)$ for every induced subgraph H of G. If G is not perfect, then by induction hypothesis G is a minimal imperfect graph. Thus, every independent set in G misses some maximum clique of G. Consequently, for any vertex $v \in V(G)$, we have

$$\chi(G - v) = \omega(G - v) = \omega(G).$$

Let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Let $I_0 = \{v_1, \dots, v_\alpha\}$ be an independent set of G. Let $I_{(j-1)\omega+1}, \dots, I_{j\omega}$ be the color classes of an ω-coloring of $G - v_j$, for $j = 1, \dots, \alpha$. We get $\alpha\omega + 1$ independent sets from I_0 to $I_{\alpha\omega}$. There is some maximum clique K_i for which $I_i \cap K_i = \emptyset$, for each $i \leq \alpha\omega$. It is easy to see that $|K_i \cap I_j| \leq 1$. We claim that $|K_i \cap I_j| = 1$, whenever $i \neq j$. In fact, a stronger statement holds: every ω-clique K is disjoint from at most one I_j. If K is disjoint from I_0, then it must intersect all other I_j, since vertices of K take all ω colors in every ω-coloring of $G - v_j$. If K is not disjoint from I_0, then

K intersects I_0 at some v_j. In this case, vertices of K get ω colors in every ω-coloring of $G - v_{j'}, j' \neq j$. For the ω-coloring of $G - v_j$, vertices of $K - \{v_j\}$ gets $\omega - 1$ colors, missing at most one color class.

Let $n = |V(G)|$. Let A be the $(\alpha\omega + 1) \times n$ matrix whose ith row is a characteristic vector of I_i. Let B be the $n \times (\alpha\omega + 1)$ matrix whose jth column is a characteristic vector of K_j. Then, $AB = \mathbf{J} - \mathbf{I}$, where $\mathbf{J}$ is the all-one matrix and $\mathbf{I}$ is the identity matrix. Since rank$(A) \geq$ rank$(\mathbf{J} - \mathbf{I}) = \omega\alpha + 1$ (it is easy to see that $\det(\mathbf{J} - \mathbf{I}) = (n-1)(-1)^{n-1}$), we have $n \geq \omega\alpha + 1$, a contradiction. □

Exercise 6.18. Show that the complement of every bipartite graph is perfect without using Theorems 6.13 and 6.14.

Exercise 6.19. Show that $\omega(G) \leq \chi(G) \leq \omega(G) + 1$ if G is a line graph.

Exercise 6.20. Let G be a graph for which every induced subgraph H has the property that every maximal independent set of H intersects every maximal clique of H. Prove that G is perfect. Prove also that such graphs G are precisely the graphs with no P_3 as an induced subgraph. (P_3 is a path of length 3.)

Exercise 6.21. Let G be a perfect graph. Prove that we can find a collection $\mathcal{I}$ of independent sets and a collection $\mathcal{K}$ of cliques such that $V(G) = \cup_{I \in \mathcal{I}} I = \cup_{K \in \mathcal{K}} V(K)$, and that $I \cap K \neq \emptyset$ for every pair $(I, K) \in \mathcal{I} \times \mathcal{K}$. (Hint: apply Lemma 6.17.)

Exercise 6.22. In the proof of Theorem 6.14, we basically replaced each vertex v by a clique of order t_v. Show that if G is a perfect graph, then replacing each vertex of G by a perfect graph yields another perfect graph.

Exercise 6.23. Let $G \diamond v$ be the graph obtained from G by *duplicating* v, namely $G \diamond v$ is obtained from v by creating another copy v' of v, where v' is adjacent to every neighbor of v. Show that if G is perfect then $G \diamond v$ is perfect for any $v \in V(G)$. (There is a short proof using Lemma 6.17 and Theorem 6.13.)

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