This Week's Agenda

- Introduction to Queueing Theory
  - Exponential Distribution, Poisson Process, Discrete Time Markov Chain
  - Continuous Time Markov Chain, Birth and Death Process
  - Queueing Networks

Exponential Distribution

T *exponentially distributed* with rate λ if

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1}$$

$f_T(t)$ is the density function of T. We also write

$$T = \text{exponential}(\lambda).$$

The cdf of T is then

$$F_T(t) = \text{Pr}[T \leq t] = \int_{-\infty}^{t} f_T(x)\,dx = 1 - e^{-\lambda t}. \tag{2}$$

Equivalently,

$$F_T(t) = \text{Pr}[T > t] = e^{-\lambda t}. \tag{3}$$

Memoryless Random Variables

T is said to be *memoryless* if

$$\Pr[T > t_1 + t_2 \mid T > t_1] = \Pr[T > t_2]$$

Theorem 1. *A continuous random variable X is memoryless if and only if it has an exponential distribution*

Some facts

Let $T = \text{exponential}(\lambda)$. Then,

$$E[e^{xT}] = \int_{-\infty}^{\infty} e^{xt} f_T(t)\, dt = \cdots = \sum_{n=0}^{\infty} \frac{n!}{\lambda^n} \frac{x^n}{n!}$$

Hence,

$$\begin{aligned} \mathrm{E}[T] &= \frac{1}{\lambda} \\ \mathrm{Var}\,[T] &= \frac{1}{\lambda^2} \end{aligned}$$

Example: Exponential Race Problem

- k lines carrying incoming packet streams are connected to a router.
- The interarrival times $T_1, \dots, T_k$ are independent
- $T_i = \text{exponential}(\lambda_i)$

Questions

- What's the probability that the first packet comes from line 1?
- What's the distribution of the first arrival time $\min\{T_i\}$?
- What's the distribution of the last arrival time $\max\{T_i\}$?

Example: Exponential Race Problem

$$
\begin{aligned}
& \Pr\big[\text{first arrival is from line 1}\big] \\
= \quad & \Pr\big[T_1 = \min\{T_1, \cdots, T_k\}\big] \\
= \quad & \int_0^\infty \Pr\big[T_2 > t\big] \cdots \Pr\big[T_k > t\big] f_{T_1}(t)\, dt \\
= \quad & \int_0^\infty e^{-(\lambda_1 + \cdots + \lambda_k)t} \lambda_1 e^{-\lambda_1 t}\, dt \\
= \quad & \frac{\lambda_1}{\lambda_1 + \lambda_2 + \cdots + \lambda_k}
\end{aligned}
$$

Let $Z = \min\{T_1, \cdots, T_k\}$. The cdf of Z is

$$
\begin{aligned}
F_Z(t) = P[Z \leq t] \quad &= \quad 1 - P[Z > t] \\
= 1 - \prod_1^k P[T_i > t] \quad &= \quad 1 - e^{-(\lambda_1 + \cdots + \lambda_k)t}.
\end{aligned}
$$

Similarly, $W = \max\{T_1, \cdots, T_k\}$, and

$$
F_W(t) = P[W \leq t] = \prod_1^k (1 - e^{-\lambda_i t}).
$$

Intuitively, why is it that Z is exponential but W is not?

The nth packet arrival time

- Packets are arriving at a server.
- Inter-arrival time is $\text{exponential}(\lambda)$.
- T_i is the time that the ith packet arrives.
- $S_n = T_1 + \cdots + T_n$.

Question

Compute the cdf of S_n

The nth packet arrival time

The cdf of S_n can be computed as follows.

$$
\begin{aligned}
F_{S_n}(t) &= \Pr[S_n \leq t] \\
&= \int_0^t (\Pr[S_n - T_1 \leq t - x]) f_{T_1}(x)\, dx
\end{aligned}
$$

Inductively, we get $F_{S_n}(t)$ and

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \tag{4}$$

which is $\text{Gamma}(n, \lambda)$.

Stochastic Processes

- A *stochastic process* is a collection of random variables indexed by some set T:

$$\{X(t), t \in T\}$$

- Elements of T are often throught of as points in time
- The set of all posible values of the $X(t)$ are called the *state space* of the process
- When T is countable the process is said to be *discrete-time*
- When T is an interval of the real line, then the process is called a *continuous-time* process

Example: Bernoulli Process

A *B*ernoulli process is a sequence $\{X_1, X_2, \dots, \}$ of independent Bernoulli random variables with parameter p, i.e.

$$\begin{aligned} \Pr[X_i = 1] &= p \\ \Pr[X_i = 0] &= 1 - p \end{aligned}$$

We are interested in the following quantities

$$\begin{aligned} S_n &= X_1 + \cdots + X_n \\ T_n &= \text{number of slots from the } (n-1)\text{th } 1 \text{ to the } n\text{th } 1 \\ Y_n &= T_1 + \cdots + T_n \end{aligned}$$

Questions

- Compute the probability mass functions of S_n, T_n, Y_n
- Compute the expectations and variances of S_n, T_n, Y_n

Example: Bernoulli Process

$$\begin{aligned} p_{S_n}(k) &= \binom{n}{k} p^k (1-p)^{n-k}, 0 \le k \le n \\ p_{T_n}(k) &= (1-p)^{k-1} p \\ p_{Y_n}(k) &= \binom{k-1}{n-1} p^n (1-p)^{k-n}, k \ge n. \end{aligned}$$

$$\begin{aligned} \mathrm{E}[S_n] &= np \\ \mathrm{Var}\,[S_n] &= np(1-p) \end{aligned}$$

$$\begin{aligned} \mathrm{E}[T_n] &= \frac{1}{p} \\ \mathrm{Var}\,[T_n] &= \frac{1-p}{p^2} \end{aligned}$$

$$\begin{aligned} \mathrm{E}[Y_n] &= \frac{n}{p} \\ \mathrm{Var}\,[Y_n] &= \frac{n(1-p)}{p^2} \end{aligned}$$

Poisson Distribution

X has Poisson distribution with parameter μ, written as $X = \text{Poisson}(\mu)$ if

$$\Pr[X = n] = e^{-\mu}\frac{\mu^n}{n!}$$

We have $\mathrm{E}[X] = \mu$ and Var $[X] = \mu$.

Theorem 2. *If $X = \text{Poisson}(\lambda)$ and $Y = \text{Poisson}(\mu)$, then $X + Y = \text{Poisson}(\lambda + \mu)$, given that X and Y are independent*

Poisson Process

- There are several equivalent definitions. We give the most intuitive here.
- Let $T_1, \dots, T_n, \dots$ be i.i.d. random variables which are all exponential(λ)
- Think of T_i as the inter-arrival time between the $(i-1)$th event and the ith event
- Let $S_n = T_1 + \cdots + T_n$
- Define the random process $\{N(t), t \geq 0\}$ by

$$N(t) = \max\{n \;:\; S_n \leq t\}.$$

  Then, the process is called a *Poisson process with rate* λ

It is easy to see that

$$\begin{aligned}
\Pr[N(t) = n] &= \Pr[S_n \leq t, S_{n+1} > t] \\
&= \int_0^t (P[T_{n+1} > t - x]) f_{s_n}(x)\, dx \\
&= \dots \\
&= e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \\
&= \text{Poisson}(\lambda t)
\end{aligned}$$

Merging Independent Poisson Processes

Let $N_1(t), \dots, N_k(t)$ be independent Poisson processes.

Then, $N(t) = N_1(t) + \dots N_k(t)$ is called the *merging*, or the *superposition* of Poisson processes.

Proposition 3. *The merging, or superposition of independent Poission processes $N_1(t), N_2(t), \cdots, N_k(t)$ with rates $\lambda_1, \lambda_2, \cdots, \lambda_k$ is a new Poisson process $N(t)$ with rate*

$$\lambda = \sum_{i=1}^{k} \lambda_i.$$

Splitting Independent Poisson Processes

We can split $N(t)$ with rate λ into $N_i(t)$ with probability p_i, where $1 \leq i \leq k$ and $p_1 + \cdots + p_k = 1$.

This act is called *splitting* or *thinning* the Poisson process.

Theorem 4. *$N_i(t)$ is a Poisson process with rate λp_i.*

Proof.

$$\Pr[N_1(t) = n_1, \cdots, N_k(t) = n_k] = \prod_{j=1}^{k} e^{-\lambda p_j t} \frac{(\lambda p_j t)^{n_j}}{n_j!}$$

Thus,

$$\begin{aligned}
& \Pr[N_1(t) = n] \\
= & \sum_{n_2,\cdots,n_k} \Pr[N_1(t) = n, N_2(t) = n_2, \cdots, N_k(t) = n_k] \\
= & \cdots \\
= & e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^n}{n!} \prod_{j=2}^{k} \sum_{n_j=0}^{\infty} e^{-\lambda p_j t} \frac{(\lambda p_j t)^{n_j}}{n_j!} \\
= & e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^n}{n!}
\end{aligned}$$

$\square$