This Week's Agenda

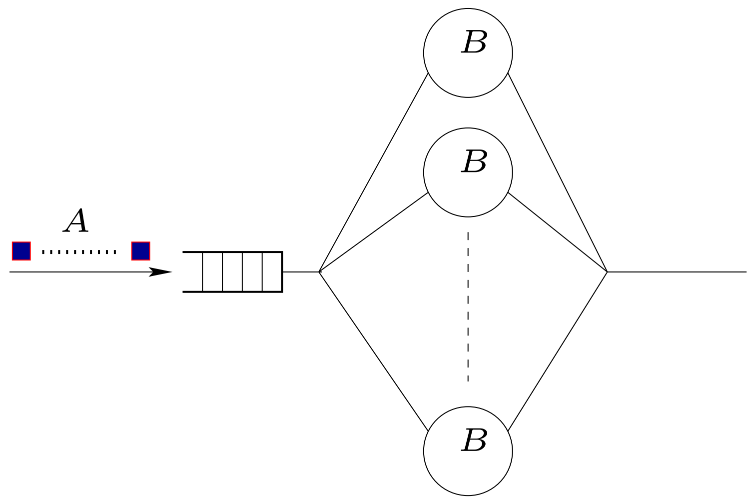
Last Time

- CTMC

Today

- A Primer on Queueing Theory

Basic Model and Kendall Notation



Denote the system by $A/B/k/c - S$, where

- A is the inter-arrival time distribution
- B is the service time distribution at each server
- k is the number of servers
- c is the queue capacity (omit if ∞)
- S is the service discipline (omit if FIFO)

Some common distributions of inter-arrival and service times:

- M: Markov, i.e. exponential; e.g. $M/M/1$, $M/M/k$
- D: Deterministic; e.g. $M/D/k$
- E_m: Erlang-m (common with phone call arrivals at toll offices)
- H_m: Hyper-m
- G: a general distribution

Basic Parameters and Performance Measures

Input Parameters

- Packet arrival processes (from the outside)
- Distributions of service times at servers
- Service disciplines at servers: FIFO, LIFO, Random, Priorities (rush order first, shortest processing time first, ...)
- Service capacity (nodes have single or multiple servers)
- Queue capacity (finite or infinite)

Output Performance Measures

- Distributions of waiting times at nodes
- Distributions of sojourn times (wait + service times)
- Distributions of number of packets at nodes
- Distributions of busy times at servers
- Packet loss rates
- Throughput
- ...

Common Parameters and Measures for $A/B/k/c$ Systems

The averages

- λ: mean arrival rate
- μ: mean service time at each server
- $\rho = \lambda/\mu$: traffic intensity
- $\bar{N}$: mean number of packets in the system
- $\bar{N}_q$: mean number of packets in the queue
- $\bar{W}$: mean waiting time overall, also called response time
- $\bar{W}_q$: mean waiting time in the queue
- r: throughput (departure rate)
- u: utilization

The random variables

- $N(t)$: number of packets in the system at time t
- N: number of packets in the system at steady state
- τ: invariant distribution (if any) of $\{N(t)\}_{t\geq 0}$

Little's Formulas

Think: each packet pays money to wait and/or to be serviced.

$$\boxed{\text{Mean rate system earns} = \lambda \times \text{ Mean amount a packet pays}}$$

The following are called Little's Formulas

- Each packet pays 1$ per unit time in the system, then

$$\bar{N} = \lambda \cdot \bar{W}$$

- Each packet pays 1$ per unit time in the queue, then

$$\bar{N}_q = \lambda \cdot \bar{W}_q$$

The $M/M/1$ Queue

- A BDP with constant birth rate λ, constant death rate μ
- Stability condition

$$C = \sum_{n=0}^{\infty} \rho^n < \infty \text{ iff } \rho < 1 \text{ iff} \lambda < \mu$$

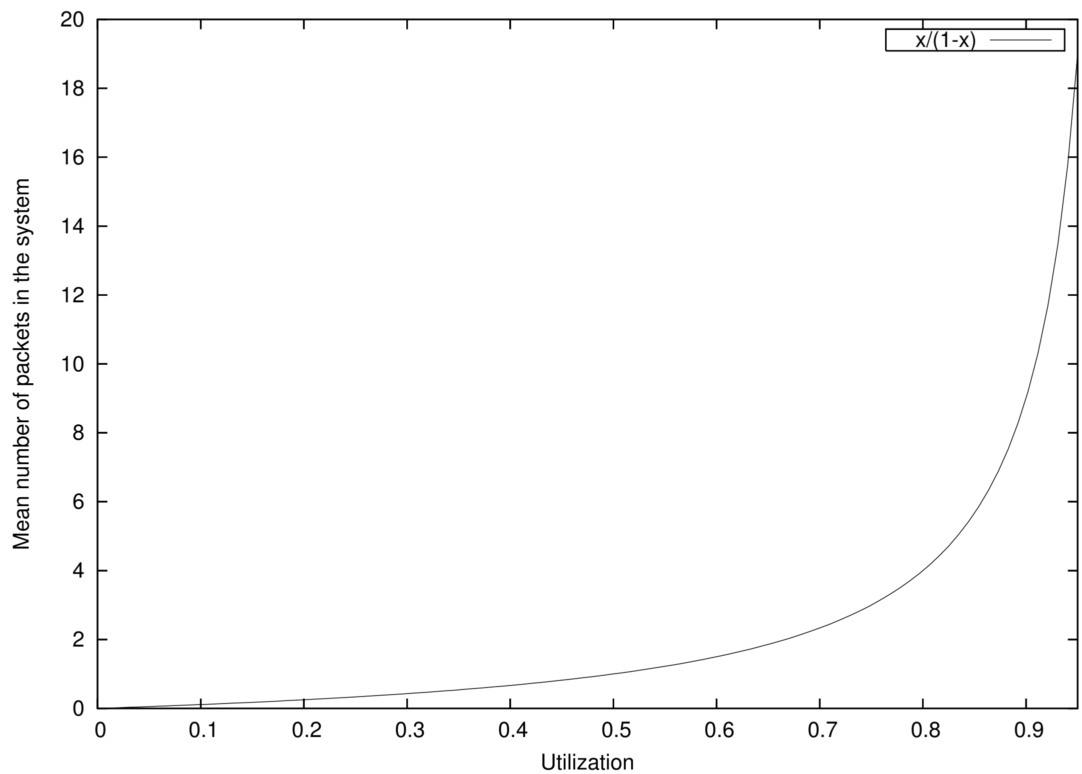
- Steady state probabilities

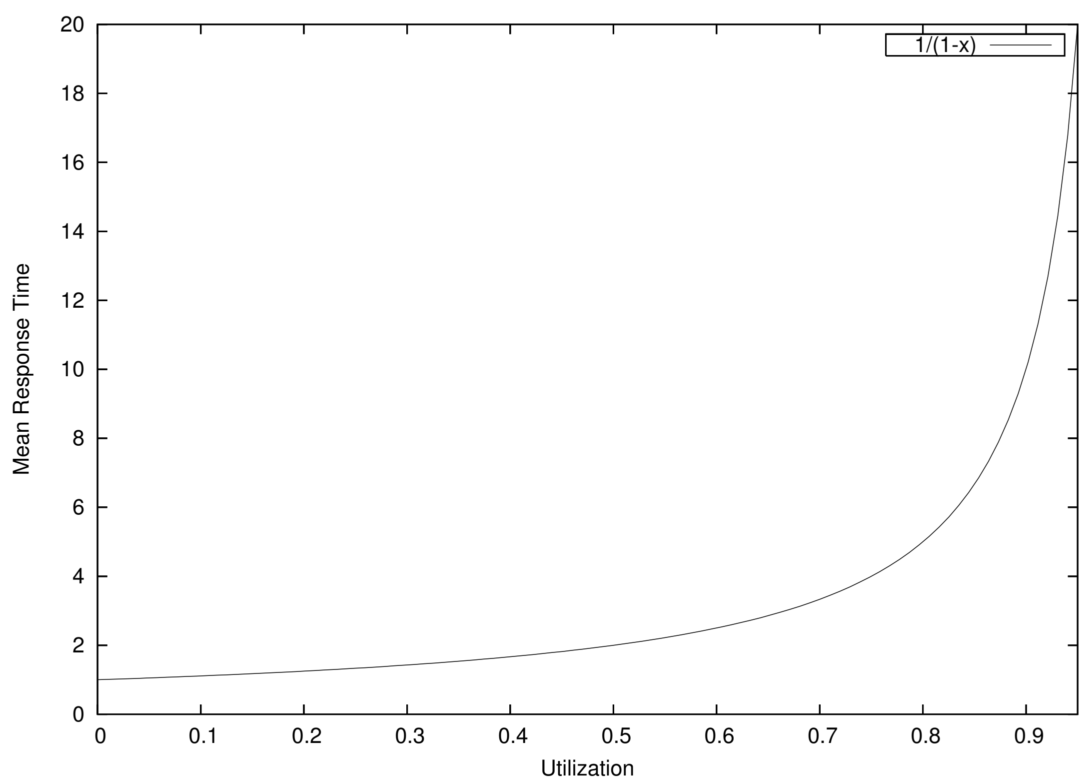
$$\tau_n = \frac{1}{C}\rho^n = (1-\rho)\rho^n \quad n \geq 0$$

- Fraction of time system has n packets is $\tau_n(\lambda + \mu)$
- It then follows that

$$\begin{aligned}
\Pr[N \geq m] &= 1 - \sum_{n=0}^{m-1} \tau_n = \rho^m \\
\bar{N} &= \sum n\tau_n = \frac{\rho}{1-\rho} \\
\bar{W} &= \frac{\bar{N}}{\lambda} = \frac{1}{\mu - \lambda} \\
\bar{W}_q &= \bar{W} - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda} \\
r &= (1-\tau_0)\mu = \lambda \\
u &= 1 - \tau_0 = \rho
\end{aligned}$$

$M/M/1$: Some Performance Graphing





The $M/M/k$ Queue

- This is a birth and death process with constant birth rate λ and death rates

$$\mu_i = \begin{cases} i\mu & 1 \leq i \leq k \\ k\mu & i > k \end{cases}.$$

- The stability condition is

$$C = \sum_{n=0}^{k-1} \frac{\rho^n}{n!} + \frac{\rho^k}{k!} \sum_{i=0}^{\infty} \left(\frac{\rho}{k}\right)^i < \infty,$$

which holds iff $\rho < k$.

- At equilibrium,

$$\begin{aligned} \tau_0 &= \left[\sum_{n=0}^{k-1} \frac{\rho^n}{n!} + \frac{\rho^k}{k!} \frac{1}{1-\rho/k}\right]^{-1} \\ \tau_n &= \begin{cases} \tau_0 \frac{\rho^n}{n!} & 1 \leq n \leq k \\ \tau_0 \frac{\rho^n}{k!k^{n-k}} & n > k \end{cases} \end{aligned}$$

The $M/M/1/c$ Queue

- This is a birth and death process with constant birth rate λ and constant death rate μ, state space $\{0, 1, \dots, c\}$
- If $\rho \neq 1$,

$$\tau_n = \begin{cases} \rho^n \frac{(1-\rho)}{1-\rho^{c+1}} & n \leq c \\ 0 & n > c \end{cases}$$

- If $\rho = 1$,

$$\tau_n = \begin{cases} \frac{1}{c+1} & n \leq c \\ 0 & n > c \end{cases}$$

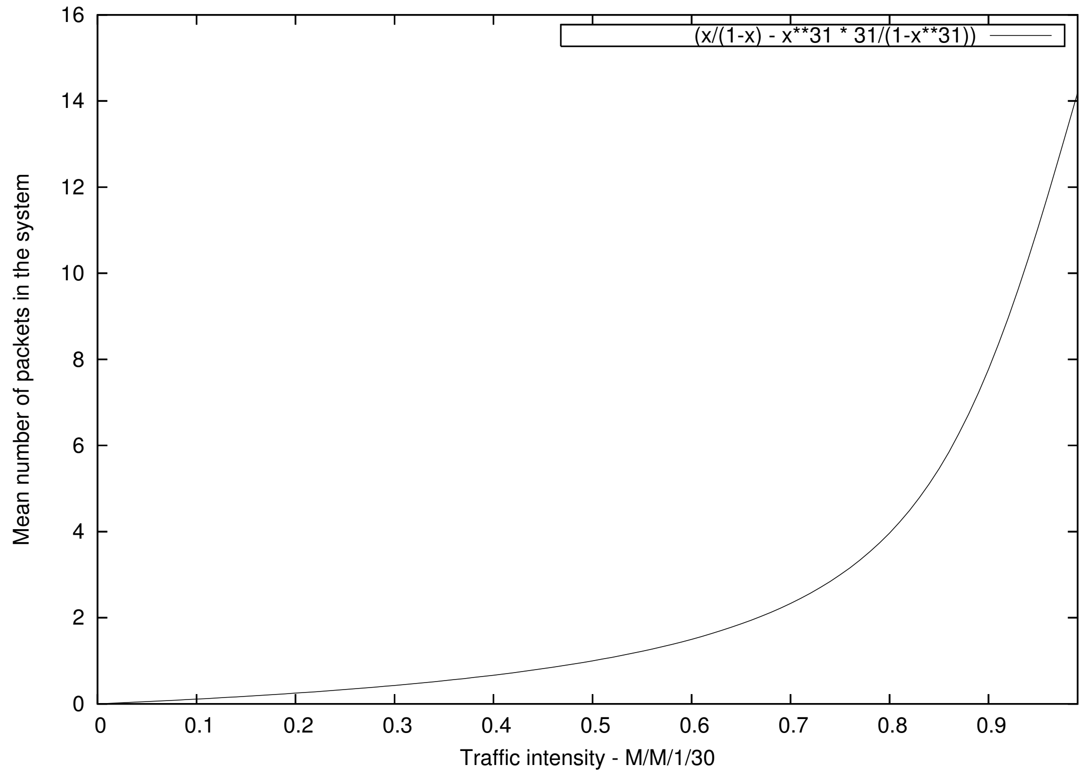
- Thus,

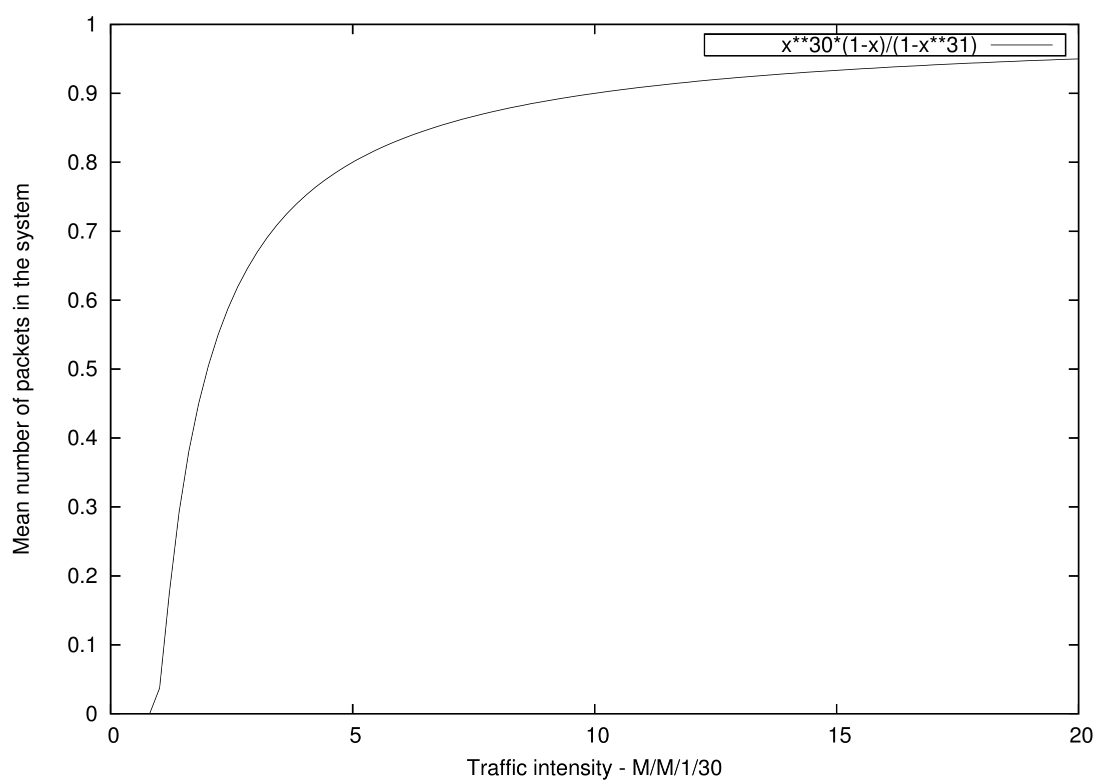
$$\bar{N} = \begin{cases} \frac{\rho}{1-\rho} - \frac{c+1}{1-\rho^{c+1}} \rho^{c+1} & \rho \neq 1 \\ c/2 & \rho = 1 \end{cases}$$

- Loss probability

$$p_{\text{loss}} = \tau_c = \begin{cases} \rho^c \frac{(1-\rho)}{1-\rho^{c+1}} & \rho \neq 1 \\ \frac{1}{c+1} & \rho = 1 \end{cases}$$

$M/M/1/30$: Some Performance Graphing





Simple Comparisons of Queueing Systems

Question

Intuitively, is it better (in terms of response time) to have

an $M/M/1$ with service rate 10μ or

an $M/M/10$ with service rate μ,

Simple Comparisons of Queueing Systems

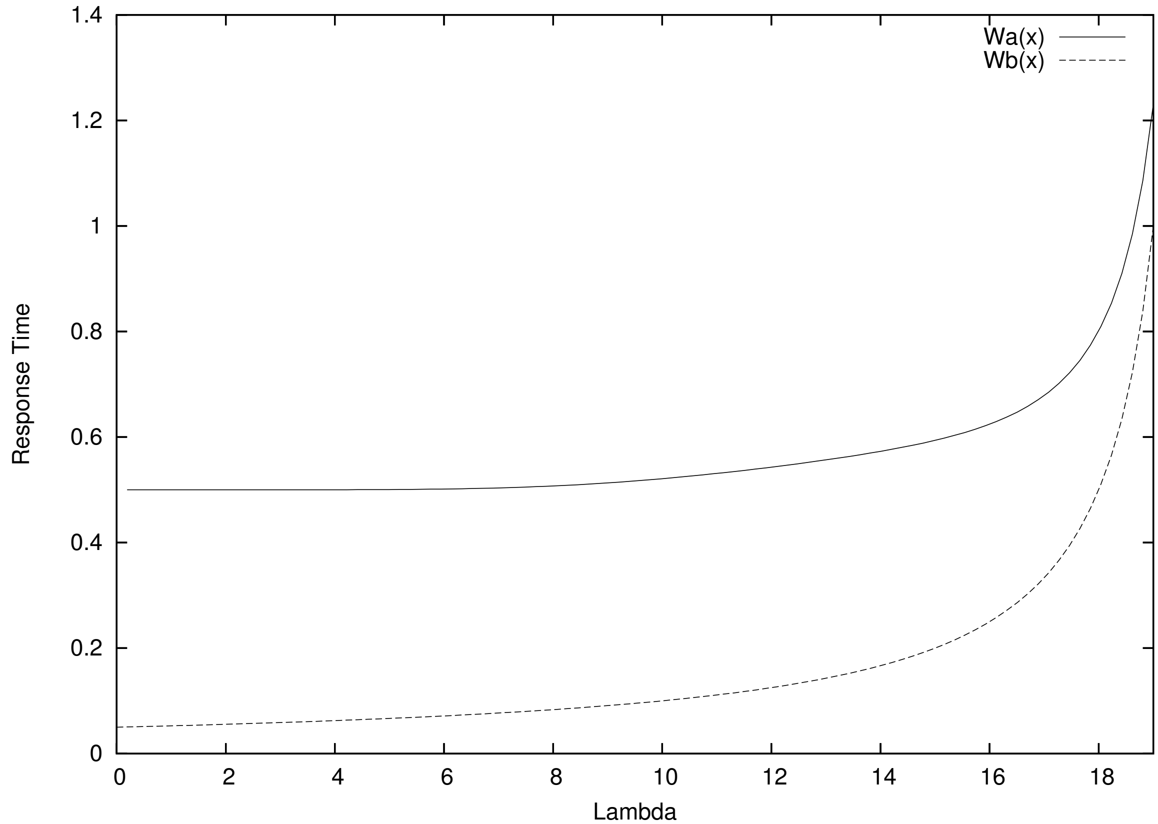


Figure 1: Wa(x): waiting time of the $M/M/10$ queue
Wb(x): waiting time of the $M/M/1$ queue