Lecture 10: Introduction to Algebraic Graph Theory

Standard texts on linear algebra and algebra are [2, 14]. Two standard texts on algebraic graph theory are [3, 6]. The monograph by Fan Chung [5] and the book by Godsil [7] are also related references.

1 The characteristic polynomial and the spectrum

Let $A(G)$ denote the adjacency matrix of the graph G. The polynomial $p_{A(G)}(x)$ is usually referred to as the *characteristic polynomial* of G. For convenience, we use $p(G,x)$ to denote $p_{A(G)}(x)$. The *spectrum* of a graph G is the set of eigenvalues of $A(G)$ together with their multiplicities. Since A (short for $A(G)$) is a real symmetric matrix, basic linear algebra tells us a few thing about A and its eigenvalues (the roots of $p(G,x)$). Firstly, A is diagonalizable and has real eigenvalues. Secondly, if λ is an eigenvalue of A, then the λ-eigenspace has dimension equal to the multiplicity of λ as a root of $p(G,x)$. Thirdly, if $n = |V(G)|$, then $\mathbb{C}^n$ is the direct sum of all eigenspaces of A. Last but not least,

$$rank(A) = n - m[0],$$

where $m[0]$ is the multiplicity of the 0-eigenvalue.

Suppose $A(G)$ has s distinct eigenvalues $\lambda_1 > \cdots > \lambda_s$, with multiplicities $m[\lambda_1], \dots, m[\lambda_s]$ respectively, then we shall write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m[\lambda_1] & m[\lambda_2] & \dots & m[\lambda_s] \end{pmatrix}$$

We also use $\lambda_{max}(G)$ and $\lambda_{min}(G)$ to denote λ_1 and λ_s, respectively.

Example 1.1 (The Spectrum of The Complete Graph).

$$\begin{aligned} p(K_n,\lambda) &= \lambda I - J \\ &= \det \begin{bmatrix} \lambda & -1 & -1 & \dots & -1 \\ 0 & \frac{(\lambda+1)(\lambda-1)}{\lambda} & \frac{-(\lambda+1)}{\lambda} & \dots & \frac{-(\lambda+1)}{\lambda} \\ 0 & 0 & \frac{(\lambda+1)(\lambda-2)}{(\lambda-1)} & \dots & \frac{-(\lambda+1)}{\lambda} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{(\lambda+1)(\lambda-(n-1))}{(\lambda-(n-2))} \end{bmatrix} \\ &= (\lambda+1)^{n-1}(\lambda-n+1) \end{aligned}$$

So,

$$Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

Remark 1.2. Two graphs are *co-spectral* if they have the same spectrum. There are many examples of co-spectral graphs which are not isomorphic. There are also examples all the graphs with a particular spectral must be isomorphic. I don't know of an intuitive example of co-spectral graphs (yet). Many examples can be found in the "bible" of graph spectra [15].

A *principal minor* of a square matrix A is the determinant of a square submatrix of A obtained by taking a subset of rows and the same subset of columns. The principal minor is of *order* k if it has k rows and k columns.

Proposition 1.3. *Suppose* $p(G,x) = x^n + c_1x^{n-1} + \cdots + c_n$, *then*

(i) $c_1 = 0$.

(ii) $-c_2 = |E(G)|$.

(iii) $-c_3$ *is twice the number of triangles in* G.

Proof. It is not difficult to see that $(-1)^i c_i$ is the sum of the principal minors of $A(G)$ of order i. Given this observation, we can see that

(i) $c_1 = 0$ since $trA(G) = 0$.

(ii) $-c_2 = |E(G)|$ since each non-zero principal minor of order 2 of $A(G)$ corresponds to $\det\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and there is one such minor for each pair of adjacent vertices in G.

(iii) Of all possible order-3 principal minors of $A(G)$, the only non-zero minor is

$$\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2$$

which corresponds to a triangle in G.

□

Example 1.4. All principal minors of $A(K_{m,n})$ of order $k \neq 2$ are 0. Hence, $p(K_{m,n}, x) = x^{m+n} + c_2x^{m+n-2}$. By previous proposition, $c_2 = -mn$. Thus,

$$Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$$

Notice that $Spec(K_{m,n})$ is symmetric above the eigenvalue 0. This beautiful property turns out to be true for all bipartite graphs, as the following lemma shows.

Lemma 1.5 (The Spectrum of a Bipartite Graph). *The following are equivalent statements about a graph* G

(a) G *is bipartite.*

(b) The non-zero eigenvalues of G *occurs in pairs* λ_i, λ_j *such that* $\lambda_i + \lambda_j = 0$ *(with the same multiplicity).*

(c) $p(G,x)$ *is a polynomial in* x^2 *after factoring out the largest common power of* x.

(d) $\sum_{i=1}^{n} \lambda_i^{2t+1} = 0$ *for all* $t \in \mathbb{N}$.

Proof. $(a \Rightarrow b)$. First of all, we could assume that the bipartitions of G have the same size, otherwise adding more isolated vertices into one of the bipartitions only give us more 0 eigenvalues. We can permute the vertices of G so that $A = A(G) = \left[\begin{smallmatrix} 0 & B \\ B^T & 0 \end{smallmatrix}\right]$. Let $v = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ be a λ-eigenvector. We have $\lambda v = Av = \left[\begin{smallmatrix} 0 & B \\ B^T & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} By \\ B^Tx \end{smallmatrix}\right]$. So, $By = \lambda x$ and $B^Tx = \lambda y$. Let $v' = \left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$ then $Av' = \left[\begin{smallmatrix} -By \\ B^Tx \end{smallmatrix}\right] = -\lambda\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$. Hence, v' is a $(-\lambda)$-eigenvector of A. The multiplicity of λ is the dimension of its eigenspace. The mapping $v \to v'$ just described is clearly an invertible linear transformation, so the λ-eigenspace and the $(-\lambda)$-eigenspace have the same dimension.

$(b \Rightarrow c)$. Easy as $(x-\lambda_i)(x+\lambda_i) = x^2 - \lambda_i^2$.

$(c \Rightarrow d)$. When $p(G,x)$ is a polynomial in x^2, its roots come in pairs $\lambda_i + \lambda_j = 0$, so that $\lambda_i^{2t+1} + \lambda_j^{2t+1} = 0$ for each pair.

$(d \Rightarrow a)$. $= \sum_{i=1}^{n} \lambda_i^{2t+1} = trA^{2t+1}$ by Proposition **??**. Also, trA^{2t+1} is at least the total number of closed walks of length $2t+1$ in G. So G does not have any cycle of odd length. It must be bipartite. □

Proposition 1.6 (A Reduction Formula for $p(G,x)$). *Suppose v_i is a vertex of degree 1 of G, and v_j is v_1's neighbor. Let $G_1 = G - v_i$, and $G_2 = G - \{v_i, v_j\}$, then*

$$p(G,x) = (xp(G_1,x) - p(G_2,x)).$$

Proof. Expanding the determinant of $(xI - A)$ along row i and then column j yields the result. □

Example 1.7 (The Characteristic Polynomial of a Path). Let P_n be the path with n vertices $\{v_1, \ldots, v_n\}$, then

$$p(P_n,x) = xp(P_{n-1},x) - p(P_{n-2},x), n \geq 3;$$

which is a straightforward application of the previous proposition. Note that this implies $p(P_n,x) = U_n(x/2)$ where U_n is the Chebyshev polynomial of the second kind.

For the sake of completeness, recall that the Chebyshev polynomial of the second kind has generating function

$$u(t,x) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n,$$

for $|x| < 1$ and $|t| < 1$; which gives the three-term recurrence

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).\text{(why?)}$$

Proposition 1.8 (The Derivative of $p(G,x)$). *For $i = 1, \ldots, n$, let G_i be $G - v_i$ where $V(G) = \{v_1, \ldots, v_n\}$. Then,*

$$p'(G,x) = \sum_i p(G_i,x).$$

Proof. Write

$$\begin{aligned} p'(G,x) &= (x^n + c_1x^{n-1} + \cdots + c_ix^{n-i} + \cdots + c_n)' \\ &= nx^{n-1} + \sum_{j=1}^{n-j}(n-j)c_jx^{n-j-1}. \end{aligned}$$

Now, nx^{n-1} distributes to n leading terms of $p(G_i,x)$. We show that the terms $(n-j)c_jx^{n-j-1}$ also distribute to the corresponding terms of $p(G_i,x)$.

We know c_j is $(-1)^j$ times the sum of all order-j principle minors of A. We want to show that $(n-j)c_j(-1)^j$ is the sum of all order-j principle minors of all $A_i = A(G_i)$. An order-j principle minor of any A_i is an order-j principle minor of A. An order-j principle minor of A is an order-j principle minor of precisely $(n-j)$ of the A_i. The j exceptions are the A_i obtained from A by removing one of the j rows (and columns) corresponding to the minor under consideration. □

Example 1.9. Suppose $A(G)$ has r identical columns indexed $\{i_1, \ldots, i_r\}$, i.e. those r vertices share the same set of neighbors. Let x be a vector all of whose components are 0 except at two components i_s and i_t where $x_{i_s} = -x_{i_t} \neq 0$. Then x is a 0-eigenvector of A. The vector space spanned by all these x has dimension $r-1$ (why?), so the 0-eigenspace of A has dimension at least $r-1$.

This fact could be obtained by seeing that $rank(A) \leq n - r + 1$ due to the r identical columns, then apply $rank(A) = n - m[0]$.

Example 1.10. It's easy to see that the number of closed walks of length k of G is $trA^k = \sum \lambda_i^k$. Hence, if G has n vertices and m edges then $\sum \lambda_i = 0$ and $\sum \lambda_i^2 = 2m$. (Here we let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of G.) It follows trivially that

$$\begin{aligned} \lambda_1^2 &= (\lambda_2 + \cdots + \lambda_n)^2 \\ &\leq (n-1)(2m - \lambda_1^2). \end{aligned}$$

So,

$$\frac{2m}{n} \leq \lambda_1 \leq \sqrt{\frac{2m(n-1)}{n}},$$

where the lower bound is shown in the next section.

2 Eigenvalues and some basic parameters of a graph

The eigenvalues of a graph gives pretty good bounds on certain parameters of a graph. I include here several representative results. More relationships of this kind shall be presented later (e.g. the chromatic number in section 5).

Lemma 2.1. *If G' is an induced subgraph of G, then*

$$\lambda_{min}(G) \leq \lambda_{min}(G') \leq \lambda_{max}(G') \leq \lambda_{max}(G)$$

Proof. Follows directly from the theorem about interlacing of eigenvalues □

Lemma 2.2. *For every graph G, $\delta(G) \leq \lambda_{max}(G) \leq \Delta(G)$.*

Proof. Let x be a λ-eigenvector for some eigenvalue λ of G. Let $|x_j| = \max_i |x_i|$ be the largest absolute coordinate value in x, then

$$|\lambda||x_j| = |(Ax)_j| = \sum_{i \mid ij \in E(G)} |x_i| \leq deg(j)|x_j| \leq \Delta(G)|x_j|$$

For the lower bound, let $\mathbf{1}$ be the all-1 vector. Applying Rayleigh's principle yields

$$\lambda_{max} \geq \frac{\mathbf{1}^T A \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{1}{n} \sum_{i,j} a_{ij} = \frac{2|E(G)|}{n}$$

Thus, actually λ_{max} is at least the average degree. □

Proposition 2.3 (Largest eigenvalue of regular graphs). *If G is a k-regular graph, then*

(i) k is an eigenvalue of G.

(ii) if G is connected, then $m[k] = 1$.

(iii) for any other eigenvalue λ of G, $\lambda \leq k$.

Proof. Let $\vec{1}$ denote the all 1 vector, then $A\vec{1} = k\vec{1}$, showing (i). Now, let $x = [x_1, \ldots, x_n]^t$ be any k-eigenvector of G, then $(Ax)_i$ is the sum of k of the x_j for which j is a neighbor of i. Moreover, $(kx)_i$ is kx_i. If x_i was the largest among all components of x, then it follows that all k neighboring x_j must have the same value as x_i. Tracing this neighboring relation we conclude that all of x's components are the same. In fact, if G is a union of m k-regular graphs, then the multiplicity of the eigenvalue k of G is m.

The fact that $\lambda \leq k$ can be shown by a similar argument, we just have to pick a component with largest absolute value. □

Theorem 2.4 (Alon, Milman (1985, [1])). *Suppose G is a k-regular connected graph with diameter d, then*

$$d \leq 2\left\lceil \sqrt{\frac{2k}{k-\lambda_2}}\log_2 n \right\rceil.$$

Proof. □

An improvement was given by Mohar:

Theorem 2.5 (Mohar (1991, [11])). *Suppose G is a k-regular connected graph with diameter d, then*

$$d \leq 2\left\lceil \frac{2k-\lambda_2}{4(k-\lambda_2)}\ln(n-1) \right\rceil.$$

Proof. □

3 The Coefficients of the Characteristic Polynomial

Theorem 3.1 (Harary, 1962 [8]). *Let $\mathcal{H}$ be the collection of spanning subgraphs of a simple graph G such that for all $H \in \mathcal{H}$, every component of H is either an edge or a cycle. Let $c(H)$ and $y(H)$ be the number of components and the number of components that are cycles of H, respectively. Then, $\det A(G) = \sum_{H\in\mathcal{H}}(-1)^{n-c(H)}2^{y(H)}$, where $n = |V(G)|$.*

Proof. We use $\det A = \sum_{\pi\in S_n} sgn(\pi)\prod_{i=1}^n a_{i\pi(i)}$. A term corresponding to π of this product is not zero iff $a_{i\pi(i)} = 1$ for all i, namely π is a permutation such that $(i, \pi(i)) \in E(G)$. In other words, if $H(\pi)$ is the functional digraph of π with edges undirected, then $H(\pi) \in \mathcal{H}$. Hence, there is a one-to-many mapping between $\mathcal{H}$ and the set of π which contribute 1 to $\det A$. We can group the indices of the sum according to H instead, and count how many π with $H(\pi) = H$. Given $H \in \mathcal{H}$, each cycle of length ≥ 3 has 2 choices of direction to construct the corresponding π, this gives the factor $2^{y(H)}$. The sign is readily verified. As we have noticed in the proof of the Matrix Tree theorem, $sgn(\pi) = (-1)^{n-c(\pi)}$ where $c(\pi)$ is the number of cycles of π, which is the number of components of its functional digraph. □

Corollary 3.2 (Sachs, 1967 [13]). *Let $\mathcal{H}_i$ denotes the collection of i-vertex subgraphs of G whose components are edges or cycles. If $p(G,\lambda) = \sum_i c_i\lambda^{n-i}$ is the characteristic polynomial of G, then $c_i = \sum_{H\in\mathcal{H}_i}(-1)^{c(H)}2^{y(H)}$.*

Proof. We already noticed that $(-1)^i c_i$ is the sum of all order i principal minors of $A(G)$. Each principal minor correspond uniquely to an induced subgraph of G on some i vertices. Applying Harary's theorem completes our proof. □

4 The Adjacency Algebra

Recall that an *algebra* is a vector space with an associative multiplication of vectors (thus also imposing a *ring* structure on the space). The *adjacency algebra* $\mathcal{A}(G)$ of G is the algebra of all polynomials in $A(G)$. In other words, $\mathcal{A}(G)$ is the set of all linear combination of powers of A. $\mathcal{A}(G)$ is the basic tool to study a class of graphs called *distance-regular graphs* (see, e.g. [4] for a comprehensive treatment). The theory of distance-regular graphs, in turn, has deep relations to *Coding Theory* (see [10], [**?**]) and *Design Theory* (see [**?**]). We found yet another great reason to study algebraic graph theory. Obviously, it makes sense to first study powers of A.

Proposition 4.1. *The number of walks of length l in G, from v_i to v_j, is the (i, j) entry of $A(G)^l$.*

Proof. Easy to see by inspection or by induction □

Lemma 4.2. *If G is a connected graph with diameter d, then $deg(m(A)) = dim(\mathcal{A}(G)) \geq d + 1$.*

Proof. Let $x, y \in V(G)$ with distance d apart. Suppose $x = v_0, v_1, \ldots, v_d = y$ is a path of length d joining x and y. Then, for all $i \in [d]$ the distance from x to v_i is i. Consequently, $(A^i)_{x,v_i} > 0$ but $(A^j)_{x,v_j} = 0$, $\forall j < i$. This implies that for all $i \in [d]$ A^i is independent from $\{I, A, \ldots, A^{i-1}\}$, or $\{I, A, \ldots, A^d\}$ is a set of independent members of $\mathcal{A}(G)$. □

Corollary 4.3. *A graph with diameter d has at least $d + 1$ distinct eigenvalues. In other words, the diameter of a graph is strictly less than the number of its distinct eigenvalues.*

Proof. If $A(G)$ has s distinct eigenvalues, then by Lemma **??**, the minimum polynomial of $A(G)$ has degree s, making $dim(\mathcal{A}(G)) = s$. So, $s \geq d + 1$ by the previous lemma. □

5 The Chromatic Number

The following theorem improves the greedy bound $\chi(G) \leq 1 + \Delta(G)$.

Theorem 5.1 (Wilf, 1967 [16]). *For every graph G, $\chi(G) \leq 1 + \lambda_{max}(G)$, where $\chi(G)$ is the chromatic number of G.*

Proof. If $\chi(G) = k$, successively delete vertices of G until we obtain a k-*critical* subgraph H of G, i.e. $\chi(H - v) = k - 1$, $\forall v \in V(H)$. We claim $\delta(H) \geq k - 1$. Suppose $\delta(H) \leq k - 2$, let v be the vertex in H with $deg(v) \leq k - 2$. $H - v$ is $(k - 1)$-colorable, so H is also $k - 1$ colorable since adding back v wouldn't require a new color. Consequently,

$$k \leq 1 + \delta(H) \leq 1 + \lambda_{max}(H) \leq 1 + \lambda_{max}(G)$$

□

It must be noted that this bound is still a poor estimate for the chromatic number. A parallel result concerning the lower bound is as follows.

Theorem 5.2 (Hoffman, 1970 [9]). *For any graph G with non-empty edge set*

$$\chi(G) \geq 1 + \frac{\lambda_{max}(G)}{-\lambda_{min}(G)}$$

We first need two auxiliary results.

Lemma 5.3. *Let X be a real symmetric matrix, partitioned in the form*

$$X = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$$

where P and R are square symmetric matrices, then

$$\lambda_{max}(X) + \lambda_{min}(X) \leq \lambda_{max}(P) + \lambda_{max}(R)$$

Proof. Let $\lambda = \lambda_{min}(X)$. Let $X' = X - \lambda I$, $P' = P - \lambda I$ and $R' = R - \lambda I$, then clearly

$$X' = \begin{bmatrix} P' & Q \\ Q^T & R' \end{bmatrix}$$

Let A and B be defined as follows

$$\begin{aligned} A &= \begin{bmatrix} P' & 0 \\ Q^T & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 & Q \\ 0 & R' \end{bmatrix} \end{aligned}$$

then, every eigenvalue of A (B) is an eigenvalue of P' (R') since $Ax = \mu x \Rightarrow P'y = \mu y$ where $x = [y\ z]^T$ with y being the part corresponding to P'. Consequently, the eigenvalues of A and B are all real. Theorem **??** implies

$$\begin{aligned} \lambda_{max}(X) - \lambda &= \lambda_{max}(X') = \lambda_1(A + B) \\ &\leq \lambda_1(A) + \lambda_1(B) \\ &\leq \lambda_1(P') + \lambda_1(R') \\ &= \lambda_1(P) - \lambda + \lambda_1(R) - \lambda \end{aligned}$$

□

Corollary 5.4. *Let A be a real symmetric matrix, partitioned into t^2 submatrices A_{ij} in such a way that the rows and columns are partitioned in the same way, i.e. the diagonal submatrices A_{ii} are all square matrices. Then*

$$\lambda_{max}(A) + (t-1)\lambda_{min}(A) \leq \sum_{i=1}^{t} \lambda_{max}(A_{ii})$$

Proof. Induction and apply previous lemma □

Proof of Theorem 5.2. Let $c = \chi(G)$ and partition $V(G)$ into c color classes, inducing a partition of $A(G)$ into c^2 submatrices where all diagonal submatrices A_{ii} consist entirely of 0's. Thus,

$$\lambda_{max}(A) + (c-1)\lambda_{min}(A) \leq \sum_{i=1}^{c} \lambda_{max}(A_{ii}) = 0$$

But if G has at least one edge, $p_A(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n \neq \lambda^n$, because $c_2 = -|E(G)|$. Hence, $\lambda_{min}(A) < 0$. This completes the proof. □

6 The Laplacian

This section is built upon the first chapter's outline of Fan Chung's book [5]. See has an entirely different system of notations and definitions (she normalized everything and defined the eigenvalues of a graph to be the eigenvalues of the Laplacian). So, I'll try my best of map them back to our, I believe, more standard notations.

However, the mapping isn't so simple. It will take me some time to link the two definitions. Thus, courtesy Bill Gate : "the best is yet to come."

6.1 The Laplacian and eigenvalues

Definition 6.1. Let G be a simple graph, D the diagonal matrix with $(D)_{ii} = deg(i)$, and A the adjacency matrix of G. Then, the matrix $L := D - A$ is called the *Laplacian* matrix of G. We shall often use $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ to denote the eigenvalues of L.

Definition 6.2. Let N be the incident matrix of any orientation H of $G(V, E)$. Let $L^2(V)$ ($L^2(E)$) be the space of real valued functions on V (E), with the usual inner product $\langle f, g\rangle$ and the usual norm $\|f\| = \sqrt{\langle f, f\rangle}$.

Note that $L^2(V)$ is isomorphic to $\mathbb{R}^n$ and thus we can define the Rayleigh quotient for f similarly: $R_A(f) = \frac{\langle Lf,f\rangle}{\|f\|^2}$. Also note that

$$\begin{aligned}
\langle Lf, f\rangle &= \langle N^T Nf, f\rangle = \langle Nf, Nf\rangle \\
&= \sum_{(u,v)\in E(H)} (f(u) - f(v))^2 \\
&= \sum_{u\sim v} (f(u) - f(v))^2
\end{aligned}$$

So, L is non-negative definite, which implies L has non-negative eigenvalues. We've just proved the first statement of the following proposition.

Proposition 6.3. *We have $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0,\ \forall i$. Moreover, $\mu_{n-1} = 0$ iff G is not connected; and, when G is regular, $m[0]$ is the number of connected components of G.*

Proof. Firstly, $\mu_n = 0$ because $L\mathbf{1} = 0$, i.e. $\mathbf{1}$ is a 0-eigenvalue of L. Secondly, notice that any function y which is non-zero and constant on the connected components of G would make $Ly = \mathbf{0}$, and thus y is a 0-eigenvector of G. Hence, the multiplicity of 0, being the dimension of the 0-eigenspace, is ≥ 2 when G is disconnected. For the converse, we assume $\mu_{n-1} = 0$ so that the 0-eigenspace has dimension ≥ 2. Let f be any μ_{n-1}-eigenvector orthogonal to $\mathbf{1}$ then

$$\mu_{n-1} = \frac{\displaystyle\sum_{u\sim v} (f(u) - f(v))^2}{\displaystyle\sum_{v} f^2(v)}$$

This means that f has to be constant on all connected components of G. If G has only 1 connected component, f has to be identically 0 contradicting the fact that it is an eigenvector.

Lastly, also note that if each connected component of G is regular, then the multiplicity of 0 is equal to the number of connected components. $\square$

Theorem 6.4. *Let $f \in L^2(V)$ such that $\sum_v f(v) = 0$. Let μ_{n-1} be the second smallest eigenvalue of L then*

$$\mu_{n-1} \leq \frac{\langle Lf, f\rangle}{\|f\|} = \frac{\displaystyle\sum_{u\sim v}(f(u)-f(v))^2}{\displaystyle\sum_v f^2(v)}$$

In fact, a stronger statement holds

$$\mu_{n-1} = \min_{f\neq 0} \frac{\langle Lf, f\rangle}{\|f\|}$$

with the $\min$ *runs over all f satisfying $\sum_v f(v) = 0$.*

Note. $\displaystyle\sum_{u\sim v}(f(u)-f(v))^2$ is sometime called the *Dirichlet sum* of G.

Proof. Let $u_n = \mathbf{1}/\sqrt{n}$ be the unit μ_n-eigenvector, then by Theorem **??** we have

$$\mu_{n-1} = \min_{\substack{0\neq f\in\mathbb{C}^n\\ f\perp u_n}} R_L(f) = \min_{\substack{0\neq f\in\mathbb{C}^n\\ f\perp \mathbf{1}}} \frac{\langle Lf, f\rangle}{\|f\|}$$

The condition $f \perp \mathbf{1}$ is the same as $\sum_u f(u) = 0$. □

Theorem 6.4 gives us a very useful upper bound for μ_{n-1}. However, sometime we need also a lower bound. The following Proposition fills our gap.

Proposition 6.5. *Let G be a connected graph, $\mu = \mu_{n-1}(G)$ and $f \in L^2(V)$ be any μ-eigenvalue. Let $V^+ := \{v \in V \mid f(v) > 0$ and $V^- := V - V^+$, then define $g \in L^2(V)$ as follows.*

$$g(v) = \begin{cases} f(v) & \text{if } v \in V^+ \\ 0 & \text{otherwise.}\end{cases}$$

Then, we have

$$\mu \geq \frac{\displaystyle\sum_{u\sim v}(g(u)-g(v))^2}{\displaystyle\sum_v g^2(v)}$$

Proof. Note that since G is connected, $\mu \neq 0$, making $f \neq 0$. Hence, $V^+ \neq \emptyset$. By definition, we have $(Lf)(v) = \mu f(v), \forall v \in V$. Thus,

$$\mu = \frac{\displaystyle\sum_{v\in V^+}(Lf)(v)f(v)}{\displaystyle\sum_{v\in V^+} f^2(v)}$$

But,

$$\sum_{v\in V^+} f^2(v) = \sum_{v\in V} g^2(v)$$

and,

$$\begin{aligned}
\sum_{v\in V^+}(Lf)(v)f(v) &= \sum_{v\in V^+}\left(d(v)f^2(v)-\sum_{u\in\Gamma(v)}f(v)f(u)\right)\\
&= \sum_{uv\in E(V^+)}(f(u)-f(v))^2+\sum_{uv\in E(V^+,V^-)}f(u)(f(u)-f(v))\\
&\geq \sum_{u\sim v}(g(u)-g(v))^2
\end{aligned}$$

completes our proof. □

6.2 The Laplacian spectrum

6.3 Eigenvalues of weighted graphs

6.4 Eigenvalues and random walks

7 Cycles and cuts

8 More on spanning trees

9 Spectral decomposition and the walk generating function

10 Graph colorings

11 Eigenvalues and combinatorial optimization

This section shall be based on an article with the same title by Bojan Mohar and Svatopluk Poljak [12].

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