On writing proofs about asymptotic relations

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In this document, I formally write a few things discussed in previous lectures. Most problems are first analyzed in a *draft* form, indicating how I think about the solution to the problem. Then, formal proofs are presented. When writing homework solutions, only formal proofs are required.

Problem 1. Given two functions $f, g : \mathbb{N}^+ \to \mathbb{R}^+$ such that both $f(n)$ and $g(n)$ tend to ∞ as $n \to \infty$, is it true that $\lg(f(n)) = O(\lg(g(n))$ implies $f(n) = O(g(n))$?

Informal analysis. The value $\lg(f(n))$ roughly is the "power part" of the function $f(n)$. If $f(n) = n^3$, then $\lg(f(n)) = 3\lg n$. The relation $\lg(f(n)) = O(\lg(g(n))$ says that the power-part of $f(n)$ is upper bounded by some constant times the power-part of $g(n)$. Hence, it is possible that $\lg(f(n))$ is greater than $\lg(g(n)$ by a constant factor, yet the relation $\lg(f(n)) = O(\lg(g(n))$ still holds. For instance, $f(n) = n^{100}$, $g(n) = n^1$, i.e. $\lg(f(n)) = 100\lg(g(n))$, yet $\lg(f(n)) = O(\lg(g(n))$. However, clearly $n^{100} \neq O(n)$. This is a perfect counter example to the claim! $\square$

Formal proof. NOT TRUE. Take, for instance, $f(n) = n^{100}$, $g(n) = n$. Then, $\lg(f(n)) = 100\lg n = O(\lg(n))$, yet $n^{100} \neq O(n)$. $\square$

Note again: a formal proof is all we need for homework problems. Do not go at length explaining your thinking!

Problem 2. Given two functions $f, g : \mathbb{N}^+ \to \mathbb{R}^+$ such that both $f(n)$ and $g(n)$ tend to ∞ as $n \to \infty$, is it true that $\lg(f(n)) = o(\lg(g(n))$ implies $f(n) = o(g(n))$?

Informal analysis. In Problem 1, the assertion was not true because the O relation is not very strong: f could be $O(g)$ even though f is a constant factor greater than g. The o relation, however, indicates that the power-part of g grows infinitely faster than the power-part of f. It only makes sense then, that g grows infinitely faster than f.

How are we going to prove something like this? Let's start from the definitions.

What we know is: $\lg(f(n)) = o(\lg(g(n))$, which, by definition, means that for all $c > 0$, $\lg(f(n)) \leq c\lg(g(n))$ for large enough n (say $n \geq n_0$, for some n_0).

What we want is: for all $\bar{c} > 0$, $f(n) \leq \bar{c}g(n)$ when $n \geq n_1$, for some n_1.

Let's start from what we want to prove, to see what it is equivalent to, at the same time try to make a connection to what we know. Consider any constant $\bar{c} > 0$.

$$f(n) \leq \bar{c}g(n) \quad \Leftrightarrow \quad \lg(f(n)) \leq \lg(g(n)) + \lg(\bar{c}).$$

The reason we want to take $\lg$ is clear: what we know involves the $\lg$ of the two functions!

Now, for **any** constant c,

$$\lg(f(n)) \leq c\lg(g(n)), \quad \text{for } n \geq n_c. \tag{1}$$

*Please let me know of any mistakes/typos as soon as you find them

How do we use this to show

$$\lg(f(n)) \le \lg(g(n)) + \lg(\bar{c}), \quad \text{for large enough } n. \tag{2}$$

It is only natural to pick $c > 0$, so that

$$c\lg(g(n)) \le \lg(g(n)) + \lg(\bar{c}), \tag{3}$$

in which case (3) and (1) imply (2)!

When $\lg(\bar{c}) \ge 0$, we can pick $c = 1$ and (3) would definitely hold.

When $\lg(\bar{c}) < 0$, (3) is equivalent to

$$\begin{array}{rcl} c\lg(g(n)) & \le & \lg(g(n)) + \lg(\bar{c}) \\ -\lg(\bar{c}) & \le & (1-c)\lg(g(n)) \end{array}$$

We thus have to choose c so that $1 - c > 0$, in which case the last inequality is the same as

$$\frac{-\lg(\bar{c})}{1-c} \le \lg(g(n)),$$

or

$$2^{\frac{-\lg(\bar{c})}{1-c}} \le g(n).$$

This is definitely true since the left hand side is a constant (for a fixed $\bar{c}$ and a constant $c < 1$ we have chosen), while $g(n)$ was assumed to go to ∞. (This is true for n is large enough!) $\square$

Formal proof. We want to show that, for any $\bar{c} > 0$, there is some constant n_0 such that $f(n) \le \bar{c}g(n)$ when $n \ge n_0$.

Consider any $\bar{c} > 0$.

Case 1: $\bar{c} \ge 1$, or $\lg(\bar{c}) \ge 0$.

Since $\lg(f(n)) = o(\lg(g(n))$, by definition there is some n_1 such that

$$\lg(f(n)) \le 1 \cdot \lg(g(n)) \ \text{ for all } n \ge n_1.$$

Thus,

$$\lg(f(n)) \le \lg(g(n)) + \lg(\bar{c}), \ \ \forall n \ge n_1,$$

which is equivalent to

$$f(n) \le \bar{c}g(n), \ \ \forall n \ge n_1.$$

Hence, when $\bar{c} \ge 1$, we can pick $n_0 = n_1$, and our assertion is proved.

Case 2: $0 < \bar{c} < 1$, or $\lg(\bar{c}) < 0$.

Since $\lg(f(n)) = o(\lg(g(n))$, by definition there is some n_1 such that

$$\lg(f(n)) \le \frac{1}{2} \cdot \lg(g(n)) \ \ \forall n \ge n_1.$$

Since $\lim_{n\to\infty} g(n) = \infty$, there is some n_2 such that

$$2^{\frac{-\lg(\bar{c})}{1/2}} \le g(n), \ \ \forall n \ge n_2.$$

Now, pick $n_0 = \max\{n_1, n_2\}$, we have, for all $n \ge n_0$,

$$\begin{array}{rl} & 2^{\frac{-\lg(\bar{c})}{1/2}} \le g(n) \\ \Leftrightarrow & -\lg(\bar{c}) \le \dfrac{1}{2}\lg(g(n)) \\ \Leftrightarrow & \dfrac{1}{2}\lg(g(n)) \le \lg(\bar{c}) + \lg(g(n)) = \lg(\bar{c} \cdot g(n)). \end{array}$$

Consequently, for all $n \geq n_0$, we have

$$\lg(f(n)) \leq \frac{1}{2} \cdot \lg(g(n)) \leq \lg(\bar{c} \cdot g(n)),$$

which is the same as

$$f(n) \leq \bar{c} \cdot g(n), \forall n \geq n_0,$$

as desired. $\square$