Agenda

What have we done?

- Probabilistic thinking!
- Balls and Bins
- Probabilistic Method
- Foundations of DTMC
- Random Walks on Graphs and Expanders

Next

- Approximate Counting and Sampling

Example 1: Number of Spanning Trees

Problem

Given G connected, count the number of spanning trees.

- A: adjacency matrix of G
- D: diagonal matrix of vertex degrees
- $L = D - A$: Laplacian of G
- L_{ij}: submatrix of L obtained by removing column i, row j
- $(-1)^{i+j}\det(L_{ij})$: ij-cofactor of L
- $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n$ the Laplacian spectrum

Theorem (Matrix-Tree, also Kirchhoff's Theorem)

Number of spanning trees of G is $(-1)^{i+j}\det(L_{ij})$ for all i, j, which is equal to $\frac{1}{n}\mu_1\cdots\mu_n$.

Example 2: Number of Perfect Matchings

"Dimer Covers"

Given a graph G, count the number of perfect matchings.

- A Pfaffian orientation of G is an orientation $\overrightarrow{G}$ such that: for any two perfect matchings M_1 and M_2 of G, every cycle of $M_1 \cup M_2$ has an odd number of same-direction edges.
- In particular, if $\overrightarrow{G}$ is an orientation in which every even cycle is *oddly oriented*, then $\overrightarrow{G}$ is a Pfaffian orientation.
- Skew adjacency matrix $A_s(\overrightarrow{G}) = (a_{uv})$:

$$a_{uv} = \begin{cases} +1 & (u,v) \in E(\overrightarrow{G}) \\ -1 & (v,u) \in E(\overrightarrow{G}) \\ 0 & \text{otherwise} \end{cases}$$

Kasteleyn's Theorem

Theorem (Kasteleyn)

For any Pfaffian orientation $\overrightarrow{G}$ of G,

$$\text{number of perfect matchings} = \sqrt{\det(A_s(\overrightarrow{G})}$$

Theorem

Every planar graph has a Pfaffian orientation which can be found in polynomial time. In particular, Dimer Covers is solvable for planar graphs!

Open Question

Complexity of deciding if a graph G has a Pfaffian orientation. (Known to be in $\mathbf{P}$ if G is bipartite.)

Example 1: Routing in Intermittently Connected Networks

- G: ad hoc network of mobile users
- For every $(u, v) \in E$, p_{uv} is the probability that u and v are "in contact"
- For simplicity, say $p_{uv} = 1/2$
- Want: send a message from s to d
- If routed through a length-k s, t-path, delivery probability is $(1/2)^k$
- To increase delivery probability, send messages along edges of a subgraph $H \subseteq G$ such that Prob$[s$ and t connected in $H]$ is maximized
- If $H = G$, we are just broadcasting $\Rightarrow$ broadcast storm problem
- If H is a path, delivery prob. is too low

The Problem is Hard

Routing on a Probabilistic Graph

Given G (and p_{uv}), and a parameter k, find a subgraph $H \subseteq G$ with at most k edges so that Prob$[s$ and t connected in $H]$ is maximized

- Given H, how to compute Prob$[s$ and t connected in $H]$? (let alone finding an optimal H)
- (Ghosh, Ngo, Yoon, Qiao – INFOCOM'07) The optimization problem is $\#\mathbf{P}$-Hard, if solvable then $\mathbf{P} = \mathbf{NP}$
- Subtle: $\mathbf{P} = \mathbf{NP}$ does not necessarily imply problem solvable

Probability Estimation $\approx$ Counting

Network Reliability Problem

Given H (and p_{uv}), compute $\mathbf{P} = \mathbf{NP}$ and t connected in $H]$.

- Suppose H has m edges. Then, Prob$[s$ and t connected in $H]$ is

$$\frac{1}{2^m}\,(\#\text{subgraphs of } H \text{ which contains an } s,t\text{-paths})$$

Network Reliability, Counting Version

Given H, find the number of subgraphs of H in which there is a path from s to t

Example 2: #CNF, #DNF, 01-PERM, #BIPARTITE-PM

#CNF

Given a CNF formula φ, count number of satisfying assignments

#DNF

Given a DNF formula φ, count number of satisfying assignments

#BIPARTITE-PM

Given a bipartite graph G, count number of perfect matchings

01-PERM

Given a 01-square matrix $\mathbf{A}$, compute per A, defined by

$$\operatorname{per} A = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i\pi(i)}$$

Rough Classification of Counting Problems

"Easy" Counting Problems

- # Subsets of a Set
- # Spanning trees of G
- # Perfect matchings in planar graphs

"Hard" Counting Problems (At least, no solution is known)

- Network reliability
- #CNF
- #DNF
- 01-PERM, #BIPARTITE-PM
- #CYCLES, #HAMILTONIAN CYCLES, #CLIQUES, #k-CLIQUES, etc.

How to Show a Counting Problem is Hard?

Suppose we want to prove any problem Π is "hard" to solve

Try This First

Prove that if Π can be solved in polynomial time, then some $\mathbf{NP}$-complete problem can be solved in polynomial time.

- Typically Done with Optimization Problem.
- $\#$CNF, $\#$HAM-CYCLES, ... are certainly $\mathbf{NP}$-hard
- We'll show $\#$DNF and $\#$CYCLES are $\mathbf{NP}$-hard to illustrate.

Try This Next

Define a new complexity class $\mathcal{C}$ for Π, and show Π is complete in that class. Provide evidence that $\mathcal{C}$ is not complete as a whole.

This was what Valiant did in 1978 for 01-PERM and NETWORK RELIABILITY. The new class $\mathcal{C}$ is $\#\mathbf{P}$

$\#$DNF is $\mathbf{NP}$-hard

Theorem

If we can count the number of satisfying assignments of a DNF formula, then we can decide if a CNF formula is satisfiable.

Given φ in CNF:

$$\varphi = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \bar{x}_4)$$

φ is satisfiable iff $\overline{\varphi}$ has $< 2^n$ satisfying assignments.

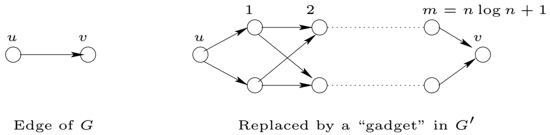
$$\overline{\varphi} = (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge \bar{x}_3 \wedge x_4)$$

$\#$CYCLES is NP-hard

Theorem

If we can count the number of cycles of a given graph in polynomial time, then we can decide if a graph has a Hamiltonian cycle in polynomial time.

- To decide if G has a Hamiltonian cycle, construct G' as shown



- Each length-l cycle in G becomes $(2^m)^l$ cycles in G'
- If G has a Hamiltonian cycle, G' has at least $(2^m)^n > n^{n^2}$ cycles
- If all cycles of G have lengths $\leq n-1$, there can be at most n^{n-1} cycles in G, implying $\leq (2^m)^{n-1} n^{n-1} < n^{n^2}$ cycles in G'

P, NP, FP, #P Intuitively

Sample Problems (each have a #-version)

1. SPANNING TREE: does G have a spanning tree?
2. BIPARTITE-PM: does bipartite G have a perfect matching?
3. CNF: does φ in CNF have a satisfying assignment?
4. DNF: does φ in DNF have a satisfying assignment?

- **P**: problems whose solutions can be **found efficiently**: SPANNING TREE, DNF, BIPARTITE-PM
- **NP**: problems whose solutions can be **verified efficiently**: all four
- **FP**: problems whose solutions can be **counted efficiently**: #SPANNING TREE
- **#P**: problems of **counting efficiently verifiable** solutions: all four.

$\#\mathbf{P}$-Complete, Intuitively

A counting problem $\#\Pi$ is $\#\mathbf{P}$-complete iff it is in $\#\mathbf{P}$ and, if $\#\Pi$ can be solved efficiently, then we can solve all $\#\mathbf{P}$ problems efficiently.

Lemma

$\#\text{CNF}$ is $\#\mathbf{P}$-complete (for the same reason SAT is $\mathbf{NP}$-complete)

This implies $\#\text{DNF}$ is $\#\mathbf{P}$-complete. (Why?)

Theorem

If any $\#\mathbf{P}$-complete problem can be solved in poly-time, then $\mathbf{P} = \mathbf{NP}$.

The converse is not known to hold (open problem!)

Theorem (Valiant)

$\#\text{BIPARTITE-PM}$ and 01-PERM are $\#\mathbf{P}$-complete

Approximate Counting: What and Why

- Suppose we want to estimate some function f on input x
  - $x = G$, $f(G) =$ number of perfect matchings
  - $x = \varphi$ in DNF, $f(\varphi) =$ number of satisfying assignments
- For many problems, computing $f(x)$ efficiently is (extremely likely to be) difficult
- The next best hope is: given ϵ, δ, efficiently compute $\tilde{f}(x)$ such that

$$\text{Prob}[|\tilde{f}(x) - f(x)| > \epsilon f(x)] < \delta$$

Definition (FPRAS)

A randomized algorithm producing such $\tilde{f}$ is called a fully polynomial time randomized approximation scheme (**FPRAS**) if its running time is polynomial in $|x|, 1/\epsilon, \log(1/\delta)$

An Alternative Definition of FPRAS

Definition (FPRAS)

A fully polynomial time randomized approximation scheme (**FPRAS**) for computing f is a randomized algorithm satisfying the following:

- on inputs x and ϵ
- A outputs $\tilde{f}(x)$, such that

$$\text{Prob}[|\tilde{f}(x) - f(x)| > \epsilon f(x)] < 1/4$$

- A's running time is polynomial in $|x|$ and $1/\epsilon$

The *median trick* shows the equivalence between the two definitions.

The Essence of the Monte Carlo Method

Basic idea: to estimate μ

- Design an **efficient** process to generate t i.i.d. variables $X_1, \dots, X_t$ such that $\mathsf{E}[X_i] = \mu$, $\mathsf{Var}[X_i] = \sigma^2$, for all i
  (X_i is called an unbiased estimator for μ)
- Output the sample mean $\tilde{\mu} = \frac{1}{t}\sum_{i=1}^{t} X_i$
- Chebyshev gives the following theorem

Theorem (Unbiased Estimator Theorem)

If $t \geq \frac{4\sigma^2}{\epsilon^2\mu^2}$*, then*

$$\text{Prob}[|\tilde{\mu} - \mu| > \epsilon\mu] < 1/4.$$

In particular, if X_i *are all indicators, then* $\sigma^2 = \mu(1-\mu) \leq \mu$*; we only need* $t \geq \frac{4}{\epsilon^2\mu}$.

Potential Bottlenecks of the Monte Carlo Method

- Each single sample value X_i must be generated efficiently
- The number of samples t needs to be a polynomial in $|x|$ (and $1/\epsilon$)
- So, if μ is really small then we're in trouble!

#DNF with Naive Monte Carlo Algorithm

Line of thought

- $f = f(\varphi)$ is the number of satisfying assignments
- Probability that a random truth assignment satisfies φ is $\mu = f/2^n$
- Let X_i indicates if the ith truth assignment satisfies φ
- $\text{Prob}[X_i = 1] = \text{E}[X_i] = \mu$
- After taking t samples, output

$$\tilde{f} = 2^n \tilde{\mu} = 2^n \cdot \frac{1}{t} \sum_{i=1}^{t} X_i$$

- Then, by the unbiased estimator theorem, when $t \geq \frac{4}{\epsilon^2 \mu}$ we have

$$\text{Prob}[|\tilde{f} - f| > \epsilon f] = \text{Prob}[|\tilde{\mu} - \mu| > \epsilon \mu] < 1/4$$

- If $f \ll 2^n$, say $f = n^2$, then $\mu = n^2/2^n$ and $t = \Omega(2^n/n^2)$

What is the Main Problem with the Naive Method?

- To find a few needles in a haystack, we need **many** samples
- More concretely, the sample space is too large, while the "good set" is too small.
- Karp-Luby (STOC 1973) designed a much smaller sample space from which we can still sample efficiently

The Karp-Luby Algorithm for #DNF

- Suppose φ has m terms

$$\varphi = T_1 \vee T_2 \vee \cdots \vee T_m = (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_2 \wedge x_4) \vee \cdots$$

- Let S_j be the set of assignments satisfying T_j which has v_j variables
- Then, $|S_j| = 2^{n-v_j}$; and we want $f = \left|\bigcup_{j=1}^{n} S_j\right|$
- The haystack

$$\begin{aligned} \Omega &= \{(a,j) \mid a \in S_j\} \\ |\Omega| &= \sum_{j=1}^{m} 2^{n-v_j} \leq m2^n \end{aligned}$$

- The needles (represent each satisfying a by the minimum j for which $a \in S_j$)

$$N = \left\{(a,j) \mid j = \min(j', (a,j') \in \Omega)\right\}, \implies f = |N|$$

The Karp-Luby Algorithm for $\#\mathrm{DNF}$

The Algorithm

for $i = 1$ to t **do**

Choose (a, j) uniformly from Ω

$$X_i = \begin{cases} 1 & (a,j) \in N \\ 0 & \text{otherwise} \end{cases}$$

end for

Output $|\Omega| \cdot \frac{1}{t} \sum_{i=1}^{t} X_i$

The Analysis

- $\text{Prob}[X_i = 1] = \mathsf{E}[X_i] = \frac{|N|}{|\Omega|}$
- To chose (a, j) uniformly from Ω, pick j with probability $\frac{|S_j|}{\sum |S_j|}$, then choose $a \in S_j$ uniformly
- Checking if $(a, j) \in N$ is the same as checking if a satisfies $T_{j'}$ for some $j' < j$.

Concluding Remarks

The algorithm can be used to estimate

$$\left|\bigcup_{j=1}^{m} S_j\right|$$

for any collection of sets S_j for which similar operations can be done efficiently.

Almost Uniform Sampling

Definition (FPAUS)

A fully polynomial time almost uniform sampler is a randomized algorithm A:

- A's input is an instance x of the problem (like a graph G)
- A internally chooses a random string R
- A outputs $A(x, R) \in \Omega$, Ω is the set of solutions to x
- the total variation distance between A's output distribution and the uniform distribution is at most ϵ

$$\max_{S \subseteq \Omega} \left| \text{Prob}_{R}[A(x, R) \in S] - \frac{|S|}{|\Omega|} \right| \leq \epsilon$$

- A's running time is polynomial in $|x|$ and $\log(1/\epsilon)$

(Approximate) Sampling and Counting

$$\begin{array}{ccc} \text{Exact Counting} & \stackrel{*}{\Longrightarrow} & \text{Exact Sampling} \\ \Downarrow & & \Downarrow \\ \text{Approximate Counting} & \stackrel{*}{\Longleftrightarrow} & \text{Approximate Sampling} \\ \text{(FPRAS)} & & \text{(FPAUS)} \end{array}$$

(* means "true for a class of problems," which is fairly large)

Approximate Sampling $\Longrightarrow$ Approximate Counting

Counting number of matchings (#MATCHINGS): given a graph G

- $\mathcal{M}(G)$ = set of matchings (not necessarily perfect)
- $f(G) = |\mathcal{M}(G)|$
- Compute $f(G)$

Theorem

If there is a FPAUS for #MATCHINGS then there is a FPRAS for it too

Making Use of "Self-Reducibility"

- Suppose $G = (V, \{e_1, e_2, \dots, e_m\}$
- Let $G_k = (V, \{e_1, \dots, e_k\})$, $0 \leq k \leq m$
- Key idea:

$$\begin{aligned} f(G) &= f(G_m) \\ &= \frac{f(G_m)}{f(G_{m-1})} \cdot \frac{f(G_{m-1})}{f(G_{m-2})} \cdots \frac{f(G_1)}{f(G_0)} \cdot f(G_0) \\ &= \frac{1}{r_m} \cdot \frac{1}{r_{m-1}} \cdots \frac{1}{r_1} \cdot 1 \end{aligned}$$

We will approximate all the

$$r_k = \frac{f(G_{k-1})}{f(G_k)}, \quad 1 \leq k \leq m$$

then take the reciprocal of their product as an estimate for $f(G)$

How Well Must We Approximate the r_k?

- Suppose $\tilde{r}_k$ is an $(\bar{\epsilon}, \bar{\delta})$-approximation for r_k, $1 \leq k \leq m$
- Want: $\tilde{f} = \frac{1}{\tilde{r}_1 \cdots \tilde{r}_m}$ to be an (ϵ, δ)-approximation for $f = \frac{1}{r_1 \cdots r_m}$:

$$\text{Prob}\left[\left|\frac{1}{\tilde{r}_1 \cdots \tilde{r}_m} - \frac{1}{r_1 \cdots r_m}\right| < \epsilon \frac{1}{r_1 \cdots r_m}\right] > 1 - \delta$$

which is the same as

$$\text{Prob}\left[1 - \epsilon < \frac{r_1 \cdots r_m}{\tilde{r}_1 \cdots \tilde{r}_m} < 1 + \epsilon\right] > 1 - \delta$$

- What we have is:

$$\text{Prob}\left[|\tilde{r}_k - r_k| < \bar{\epsilon} r_k\right] > 1 - \bar{\delta}$$

which is equivalent to

$$\text{Prob}\left[(1 + \bar{\epsilon})^{-1} < \frac{r_k}{\tilde{r}_k} < (1 - \bar{\epsilon})^{-1}\right] > 1 - \bar{\delta}$$

How Well Must We Approximate the r_k?

- Choose $\bar{\delta} = \delta/m$, then

$$\text{Prob}\left[(1+\bar{\epsilon})^{-1} < \frac{r_k}{\tilde{r}_k} < (1-\bar{\epsilon})^{-1}, \quad \text{for all } k\right] > 1-\delta$$

- Hence,

$$\text{Prob}\left[(1+\bar{\epsilon})^{-m} < \prod_{k=1}^{m} \frac{r_k}{\tilde{r}_k} < (1-\bar{\epsilon})^{-m}\right] > 1-\delta$$

- Now, setting $\bar{\epsilon} = \frac{\epsilon}{4m}$ we get

$$\begin{aligned} (1+\bar{\epsilon})^{-m} &\geq 1-\epsilon \\ (1-\bar{\epsilon})^{-m} &\leq 1+\epsilon \end{aligned}$$

In Case You're Wondering

We made use of a subset of the following inequalities:

$$\begin{aligned}
1-x &\leq e^{-x} && \forall x \in [0,1] \\
1-x &> e^{-x-x^2} && \forall x < 1 \\
1-x &> e^{-x-\frac{1}{2}x^2-\frac{1}{2}x^3} && \forall x < 1 \\
1+x &\leq e^{x} && \forall x \in [-1,1] \\
1+x &> e^{x-\frac{1}{2}x^2} && \forall x > -1 \\
1+x &> e^{x-\frac{1}{2}x^2+\frac{1}{4}x^3} && \forall x > -1
\end{aligned}$$

Estimating r_k: Which Needles? In Which Haystack?

To estimate $r_k = \frac{f(G_{k-1})}{f(G_k)}$:

- The haystack: $\Omega_k = \mathcal{M}(G_k)$
- The needles: $\Omega_{k-1} = \mathcal{M}(G_{k-1})$
- Are there enough needles to reduce number of samples? **yes!**

$$r_k \geq \frac{1}{2}$$

- Thus, if we had an *exact* uniform sampler we only need $t \geq \frac{4}{\bar{\epsilon}^2 r_k}$ samples to get an $(\bar{\epsilon}, 1/4)$-approximation for r_k

Main Question Now

How many samples does an $(\bar{\epsilon}, 1/4)$-approximator for r_k need if it only has access to a FPAUS, i.e. it can only sample approximately uniformly from Ω_k?

Number of Samples from a FPAUS

The Algorithm

- Let A be an ϵ'-FPAUS for Ω_k (ϵ' to be determined)
- Take t samples using A, let X_i indicate if the ith sample $\in \Omega_{k-1}$
- Output $\tilde{r}_k = \frac{1}{t}\sum_{i=1}^{t} X_i$ as an estimate for r_k

The Analysis

- **Want** $\text{Prob}[|\tilde{r}_k - r_k| > \bar{\epsilon} r_k] < 1/4$, in other words,

$$\text{Prob}[r_k - \epsilon r_k \leq \tilde{r}_k \leq r_k + \epsilon r_k] \geq 3/4$$

- **What do we know?**
  - From definition of A, $\text{Prob}[X_i = 1]$ is near r_k
  - Thus, $\text{E}[\tilde{r}_k]$ is near r_k (within ϵ')
  - $\tilde{r}_k$ is near $\text{E}[\tilde{r}_k]$ with high probability if t is sufficiently large (why?)
  - Should be able to get what we want from here

Number of Samples from a FPAUS

The analysis, more precisely:

- By definition of A,

$$r_k - \epsilon' \leq \text{Prob}[X_i = 1] = \mathsf{E}[X_i] \leq r_k + \epsilon'$$

  Thus,

$$r_k - \epsilon' \leq \mathsf{E}[\tilde{r}_k] \leq r_k + \epsilon'$$

- To apply Chebyshev, need

$$\text{Var}\left[\tilde{r}_k\right] = \frac{1}{t^2}\sum_{i=1}^{t} \text{Var}\left[X_i\right] \leq \frac{1}{t}\mathsf{E}[\tilde{r}_k]$$

- Thus, by Chebyshev

$$\text{Prob}\left[|\tilde{r}_k - \mathsf{E}[\tilde{r}_k]| > a\mathsf{E}[\tilde{r}_k]\right] < \frac{\text{Var}\left[\tilde{r}_k\right]}{a^2(\mathsf{E}[\tilde{r}_k])^2} \leq \frac{1}{ta^2\mathsf{E}[\tilde{r}_k]}$$

Number of Samples from a FPAUS

- Since $\mathsf{E}[\tilde{r}_k] \geq r_k - \epsilon' \geq 1/3$

  $$\text{Prob}\left[(1-a)\mathsf{E}[\tilde{r}_k] \leq \tilde{r}_k \leq (1+a)\mathsf{E}[\tilde{r}_k]\right] \geq 1 - \frac{1}{ta^2\mathsf{E}[\tilde{r}_k]} \geq 1 - \frac{3}{ta^2} \geq 3/4$$

  if we take $t \geq \frac{12}{a^2}$ samples.
- Putting things together

  $$\text{Prob}\left[(1-a)(r_k - \epsilon') \leq \tilde{r}_k \leq (1+a)(r_k + \epsilon')\right] \geq 3/4$$

- Now, just need to choose a and ϵ' so that

  $$\begin{aligned}(1-a)(r_k - \epsilon') &\geq (r_k - \bar{\epsilon} r_k)\\ (1+a)(r_k + \epsilon') &\leq (r_k + \bar{\epsilon} r_k)\end{aligned}$$

- $a = \bar{\epsilon}/4$ and $\epsilon' = \bar{\epsilon}/8$ work!

To Summarize

To get (ϵ, δ)-approximation for f, need

- $(\bar{\epsilon}, \bar{\delta})$-approximation for each r_k, where $\bar{\epsilon} = \epsilon/4m$ and $\bar{\delta} = \delta/m$

To get $(\bar{\epsilon}, \bar{\delta})$-approximation for r_k, need

- ϵ'-FPAUS for Ω_k, with $\epsilon' = \bar{\epsilon}/8 = \epsilon/(64m)$
- this many samples:

$$\frac{12}{a^2} O\left(\log(1/\bar{\delta})\right) = \frac{192}{\bar{\epsilon}^2} O\left(\log(m/\delta)\right) = \frac{3072m^2}{\epsilon^2} O\left(\log(m/\delta)\right)$$

In total, we invoke the FPAUS $\frac{3072m^3}{\epsilon^2} O\left(\log(m/\delta)\right)$ times.
(Number of invocations can be reduced to $\tilde{O}(m^2)$ with a cleverer application of Chebyshev)