Randomized Algorithms

Randomized Rounding

- Brief Introduction to Linear Programming and Its Usage in Combinatorial Optimization
- **Randomized Rounding for Cut Problems**
- Randomized Rounding for Satisfiability Problems
- Randomized Rounding for Covering Problems
- Randomized Rounding and Semi-definite Programming

Approximate Sampling and Counting

- ...

(Randomized) Rounding

- A (minimization) combinatorial problem $\Pi \Leftrightarrow$ an ILP
- Let $\bar{\mathbf{y}}$ be an optimal solution to the ILP
- Relax ILP to get an LP; let $\mathbf{y}^*$ be an optimal solution to the LP
- Then,
$$\textsc{opt}(\Pi) = \text{cost}(\bar{\mathbf{y}}) \geq \text{cost}(\mathbf{y}^*)$$
(If Π is maximization, reverse the inequality!)
- Carefully "round" $\mathbf{y}^*$ (rational) to get a feasible solution $\mathbf{y}^A$ (integral) to the ILP, such that $\mathbf{y}^A$ is *not too bad*, say $\text{cost}(\mathbf{y}^A) \leq \alpha\, \text{cost}(\mathbf{y}^*)$
- Conclude that $\text{cost}(\mathbf{y}^A) \leq \alpha \cdot \textsc{opt}(\Pi)$
- Thus, we get an α-approximation algorithm for Π
- If $\alpha = 1$, then we have solved Π exactly!

An Integer Linear Program for Minimum Cut

Definition (Min-Cut Problem)

Given a (undirect/directed) graph $G = (V, E)$, edge capacities $c : E \to \mathbb{N}$, a source $s \in V$, sink $t \in V$, find a subset C of edges such that removing C disconnect t from s (i.e. there's no path from s to t), such that C has minimum total capacity.

Let $\mathcal{P}$ be the set of all s, t-paths

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e y_e \\ \text{subject to} \quad & \sum_{e \in P} y_e \geq 1, \quad \forall P \in \mathcal{P}, \\ & y_e \in \{0, 1\}, \quad \forall e \in E. \end{aligned} \tag{1}$$

Let $\bar{\mathbf{y}}$ be an optimal solution to this ILP.

Relaxation

To Relax and Integer LP is to relax the integral constraints
The relaxation of the ILP is a linear program:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e y_e \\ \text{subject to} \quad & \sum_{e \in P} y_e \geq 1, \quad \forall P \in \mathcal{P}, \\ & y_e \geq 0, \quad \forall e \in E. \end{aligned} \tag{2}$$

The Randomized Rounding Step

- Let $\mathbf{y}^*$ be an optimal solution to the LP
- Think of y_e^* as the "length" of e. Let $d(s, u)$ be the distance from s to u in terms of $\mathbf{y}^*$-length. Then, $d(s, t) \geq 1$.
- For each $r \in [0, 1]$, let $B(r) := \{u \mid d(s, u) \leq r\}$ and

$$C(r) = [B(r), \overline{B(r)}]$$

- Choose $r \in [0, 1)$ uniformly at random (a continuous distribution now!). Output the cut $C = C(r)$

(Expected) Quality of the Solution

- Expected quality of the solution

$$\begin{aligned}\mathsf{E}[\mathsf{cap}(C)] &= \sum_{e=(u,v)\in E} c_e \,\mathsf{Prob}[e \in C] \\ &\leq \sum_{e=(u,v)\in E} c_e \frac{d(s,v)-d(s,u)}{1-0} \leq \sum_{e\in E} c_e y_e^* = \mathsf{cost}(\mathbf{y}^*).\end{aligned}$$

- And so,

$$\mathsf{E}[\mathsf{cap}(C)] \leq \mathsf{cost}(\mathbf{y}^*) \leq \mathsf{cost}(\bar{\mathbf{y}}) = \text{min-cut capacity of } G$$

- Anything "weird"?
- Conclude that, just **output $C(r)$ for any $r \in [0,1)$ and we have a minimum cut!**

Additional Remarks

- Computers cannot choose $r \in [0, 1)$ uniformly at random! (They can't deal with continuous things.)
- Fortunately, there are only finitely many $B(r)$, even though $r \in [0, 1)$.

There are $0 < r_1 < r_2 < \cdots < r_k < 1$ such that

- For $r \in [0, r_1)$ we get cut C_0 (with prob r_1)
- For $r \in [r_1, r_2)$ we get cut C_1 (with prob $r_2 - r_1$)
- ...

Let $\mathcal{C}$ be this set of k cuts, then,

$$\mathsf{cost}(\mathbf{y}^*) \geq \mathsf{E}[\mathsf{cap}(C)] = \sum_{i=0}^{k-1} \mathsf{cap}(C_i)\,\mathsf{Prob}[C_i]$$

The Dual Linear Program (DLP)

Here's the dual linear program of LP (5)

$$
\begin{aligned}
\max \quad & \sum_{P \in \mathcal{P}} f_P \\
\text{subject to} \quad & \sum_{P : e \in P} f_P \leq c_e, \quad \forall e \in E, \\
& f_P \geq 0, \quad \forall P \in \mathcal{P}.
\end{aligned} \tag{3}
$$

- This is precisely the maximum flow problem!
- Let $\mathbf{f}^*$ be a maximum flow, then by "strong duality"

$$\text{cost}(\mathbf{f}^*) = \text{cost}(\mathbf{y}^*)$$

Maxflow-Mincut Theorem

Lemma (Maxflow-Mincut, Weak Duality)

For every cut C *of* G, $\text{cap}(C) \geq$ *maxflow.*

Theorem (Maxflow-Mincut, Strong Duality)

There exists a cut C *such that* $\text{cap}(C) =$ *maxflow.*

An Integer Linear Program for Multiway Cut

Definition (Multiway-Cut Problem)

Given a graph $G = (V, E)$, edge capacities $c : E \to \mathbb{N}$, and k terminals $t_1, t_2, \dots, t_k \in V$, find a subset C of edges such that removing C disconnect all terminals from each other such that C has minimum total capacity.

Let $\mathcal{P}$ be the set of all t_i, t_j-paths, $i \neq j$, $i, j \in [k]$

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e \\
\text{subject to} \quad & \sum_{e \in P} y_e \geq 1, \quad \forall P \in \mathcal{P}, \\
& y_e \in \{0, 1\}, \quad \forall e \in E.
\end{aligned} \tag{4}
$$

Let $\bar{\mathbf{y}}$ be an optimal solution to this ILP.

Relaxation

The relaxation of the ILP is a linear program:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e y_e \\
\text{subject to} \quad & \sum_{e \in P} y_e \geq 1, \quad \forall P \in \mathcal{P}, \\
& y_e \geq 0, \quad \forall e \in E.
\end{aligned} \tag{5}
$$

The Randomized Rounding Step

- Let $\mathbf{y}^*$ be an optimal solution to the LP
- Think of y_e^* as the "length" of e. Let $d(t_i, u)$ be the distance from t_i to u in terms of $\mathbf{y}^*$-length. Then, $d(t_i, t_j) \geq 1$ for every pair $i, j \in [k], i \neq j$.
- For each $r \in [0, 1]$, let $B_i(r) := \{u \mid d(t_i, u) \leq r\}$ and

$$C_i(r) = [B_i(r), \overline{B_i(r)}]$$

- Choose $r \in [0, 1/2)$ uniformly at random Output the cut

$$C = C_1(r) \cup \cdots \cup C_k(r)$$

The rest is a homework problem! We get a 2-approximation algorithm for multiway cut