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THE COMPLEXITY OF DETERMINACY PROBLEM ON GROUP TESTING

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The complexity of group testing is a long-standing open problem. Recently, Du and Ko studied some related problems which can explain the hardness of group testing undirectly. One of such problems is called the determinacy problem on which they left open questions for some models. In this paper, we answer all of them.

Keywords. Group testing, NP-completeness, polynomial-time algorithm.

1. Introduction

Group testing was first discovered by Dorfman [1] when he did blood testing during World War II. Since then, one has found many applications in statistics, combinatorics and computer science. The general form of the problem is as follows: Let N be a set of n items denoted by $1,2,\dots,n$; some items are defective. We need to find out all defective items by a given kind of tests. What is the best way in some sense to do so? All defective items form a set that is called a sample. The sample space consists of all possible samples. For every set S, let $|S|$ denote the number of elements of S, we will study the following sample spaces: The space φ_n consists of all subsets S of N, the space $\varphi_{n,d}$ consists of all subsets S of N with $|S|=d$ and the space $\varphi_{n,\le d}$ consists of all subsets S of N with $|S|\le d$. Each test in group testing problems is on a subset T of N. For simplicity, we denote by the same symbol T the test and the subset on which one tests. For a sample S, $ANS_S(T)$ is the result of the test T and is called the answer function. There are different definitions in various models. In this paper, we consider the following definition:

$$ANS_S(T)=\begin{cases} i, & \text{if } |S\cap T|=i<k,\\ k, & \text{if } |S\cap T|\ge k,\end{cases}$$

where k is a fixed natural number.

Let $A_k(A_k', A_k'')$ denote the model with the above answer function and the sample space $\varphi_n(\varphi_{n,d}, \varphi_{n,\le d})$. The determinacy problem for model $M\in\{A_k, A_k', A_k''\,|\,k\ge 1\}$ is as follows.

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DM. Given a natural number n (or two natural number n and d) and a collection of tests $Q=\{T_j \mid j=1,2,\ldots,m\}$, determine whether Q is determinant for model M, i.e., for any two different samples S_1 and S_2, there is $T\in Q$ such that $ANS_{S_1}(T)\neq ANS_{S_2}(T)$.

Du and Ko [2] showed that DA_1 is polynomial-time solvable, and DA_k, $k\geq 4$, and DA'_k, $k\geq 4$, are co-NP-complete. But DA_2, DA_3 and DA'_k, $k=1,2,3$, remain open. We will show that DA_3 and DA'_k, $k=1,2,3$ are co-NP-complete, and DA_2 is polynomial-time solvable. Meanwhile, we will also show that DA_3 is polynomial-time solvable when there exists no test $T\in Q$ with $|T|=3$. This is interesting because in the same special case, DA_k, $k\geq 4$, is co-NP-complete. (In [2], Du and Ko did not use tests of size 3 to prove the completeness of DA_k, $k\geq 4$.) we will also show DA''_k, $k\geq 1$, to be co-NP-complete.

2. DA'_k and DA''_k

We now use the following NP-complete problem [4] to prove our results.

VERTEX-COVER. Given a graph $G=(V,E)$ without isolated vertex and a positive integer $h<|V|-1$, determine whether there is a set $Y\subset V$ with $|Y|\leq h$ such that each edge $e\in E$ is incident with some $v\in Y$.

The above statement is a little different from the usual one on the restrictions of G and h. However, it is easy to see that the VERTEX-COVER problem remains NP-complete with the current statement.

Theorem 2.1. *DA''_k $(k\geq 1)$ is co-NP-complete.*

Proof. It is easy to see that DA''_k $(k\geq 1)$ is in co-NP. Now we show that VERTEX-COVER is polynomial-time reducible to the complement of DA''_k, and hence DA''_k is co-NP-complete.

Let $G=(V,E)$ and integer h $(0<h<|V|-1)$ form an instance of VERTEX-COVER. Assume $V=\{1,2,\ldots,n\}$. Every edge e is represented by a subset of two elements of V. Define an instance of DA''_k as follows:

$$X_i=\{i\},\quad \text{for } i=1,2,\ldots,n+k-1,$$

$$T_e=e\cup\{n+1,n+2,\ldots,n+k\},\quad \text{for } e\in E,$$

and

$$Q=\{X_i \mid i=1,2,\ldots,n+k-1\}\cup\{T_e \mid e\in E\}.$$

In the following, we will show that G has a vertex-cover Y with $|Y|\leq h$ if and only if $\{n+k,d,Q\}$ is not determinant, where $d=h+k$.

First, assume that G has a vertex-cover Y with $|Y| \le h$. Define two sets $S_1 = Y \cup \{n+1, n+2, \ldots, n+k-1\}$ and $S_2 = S_1 \cup \{n+k\}$. Obviously, $S_1 \ne S_2$, $|S_1| < d$ and $|S_2| \le d$. It is easy to see that $ANS_{S_1}(X_i) = ANS_{S_2}(X_i)$, $i = 1, 2, \ldots, n+k-1$ and $ANS_{S_1}(T_e) = k = ANS_{S_2}(T_e)$ $\forall e \in E$. Hence $\{n+k, d, Q\}$ is not determinant.

Conversely, assume that $\{n+k, d, Q\}$ is not determinant, then there exists $S_1, S_2 \in \wp_{n+k, \le d}$, $S_1 \ne S_2$, such that for all $T \in Q$, $ANS_{S_1}(T) = ANS_{S_2}(T)$. Without loss of generality, we assume $S_1 \setminus S_2 \ne \emptyset$. From $ANS_{S_1}(X_i) = ANS_{S_2}(X_i)$, we see that $i \notin S_1 \setminus S_2$ and $i \notin S_2 \setminus S_1$ for all $i = 1, 2, \ldots, n+k-1$. Hence, $S_1 \setminus S_2 = \{n+k\}$ and $S_2 \setminus S_1 = \emptyset$. It follows that for any $e \in E$, $|S_1 \cap T_e| = |S_2 \cap T_e| + 1$. Since $ANS_{S_1}(T_e) = ANS_{S_2}(T_e)$, we have

$$|S_2 \cap T_e| \ge k \quad \forall e \in E, \tag{2.1}$$

$$S_2 \cap e \ne \emptyset \quad \forall e \in E. \tag{2.2}$$

Next, we show that

$$\{n+1, n+2, \ldots, n+k-1\} \subset S_2. \tag{2.3}$$

Assume, for otherwise, that (2.3) is not true, then, $|S_2 \cap \{n+1, n+2, \ldots, n+k-1\}| = k-2$, and $|S_2 \cap e| = 2$ $\forall e \in E$. Because G has no isolated vertex, $V \sqsubseteq S_2$, we have $n+k-2 = |S_2| = |S_1| - 1 \le d-1 = k+h-1$, which implies that $n-1 \le h$. By assumption, $h < n-1$, a contradiction. Hence $\{n+1, n+2, \ldots, n+k-1\} \subset S_2$. Define $Y = S_2 \setminus \{n+1, n+2, \ldots, n+k-1\}$, then $|Y| \le d-k = h$. By (2.2), $Y \cap e \ne \emptyset$ $\forall e \in E$, so Y is a vertex-cover of G with $|Y| \le h$. $\square$

Theorem 2.2. *DA'_k $(k \ge 1)$ is co-*NP*-complete*.

Proof. It is easy to see that DA'_k $(k \ge 1)$ is in co-NP. Now, we show that VERTEX-COVER is polynomial-time reducible to the complement of DA'_k, and hence DA'_k is co-NP-complete.

Let $G = (V, E)$ and integer h $(0 < h < |V| - 1)$ form a given instance of VERTEX-COVER. Assume $V = \{1, 2, \ldots, n\}$. Every edge e is represented by a subset of two elements of V. Define an instance of DA'_k as follows:

$$X_i = \{i\} \quad \forall i = 1, 2, \ldots, n+k-1,$$

$$T_e = e \cup \{n+1, n+2, \ldots, n+k\} \quad \forall e \in E,$$

and

$$Q = \{X_i \mid i = 1, 2, \ldots, n+k-1\} \cup \{T_e \mid e \in E\}.$$

It is only routine checking that G has a vertex-cover Y with $|Y| \le h$ if and only if $\{n+k+1, k+h, Q\}$ is not determinant, in particular, it cannot distinguish the following two sets:

$$S_1 = Y \cup \{n+1, n+2, \ldots, n+k\},$$

$$S_2 = Y \cup \{n+1, n+2, \ldots, n+k-1, n+k+1\}. \quad \square$$

3. DA_2 and DA_3

To prove that DA_2 is polynomial-time solvable, we first show two lemmas.

Lemma 3.1. *Let Q be a collection of tests. If there exist $S_1, S_2 \in \varphi_n$, $S_1 \neq S_2$ such that for all $T \in Q$, $ANS_{S_1}(T) = ANS_{S_2}(T)$ in the model A_k, then there exist $S_1', S_2' \in \varphi_n$, $S_1' \neq S_2'$ and $S_1' \cup S_2' = N$ such that for all $T \in Q$, $ANS_{S_1'}(T) = ANS_{S_2'}(T)$ in the model A_k.*

Proof. Let $S_i' = S_i \cup (N \setminus (S_1 \cup S_2))$, $i = 1, 2$, then

$$ANS_{S_i'}(T) = \min\{k, ANS_{S_i}(T) + |T \cap (N \setminus (S_1 \cup S_2))|\}.$$

From $ANS_{S_1}(T) = ANS_{S_2}(T)$, we see that $ANS_{S_1'}(T) = ANS_{S_2'}(T)$. □

Lemma 3.2. *Let $\{n, Q\}$ be an instance of DA_2. Then, $\{n, Q\}$ is not determinant if and only if there exist Y_1, Y_2 in φ_n, $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 \neq \emptyset$, such that for any $T \in Q$ the following conditions hold:*

(1) *If $|T| \leq 2$, then $|Y_1 \cap T| = |Y_2 \cap T|$.*
(2) *If $|T| \geq 3$, then $|Y_1 \cap T| \leq |T| - 2$ and $|T \cap Y_2| \leq |T| - 2$.*

Proof. Assume that $\{n, Q\}$ is not determinant. By Lemma 3.1, there exist $S_1', S_2' \in \varphi_n$, $S_1' \neq S_2'$ and $S_1' \cup S_2' = N$ such that $ANS_{S_1'}(T) = ANS_{S_2'}(T)$ for all $T \in Q$. Define $Y_1 = S_1' \setminus S_2' = N \setminus S_2'$, $Y_2 = S_2' \setminus S_1' = N \setminus S_1'$. If $|T| \leq 2$, then $|S_1' \cap T| = |S_2' \cap T|$, so $|Y_1 \cap T| = |S_1' \cap T| - |T \cap S_1' \cap S_2'| = |S_2' \cap T| - |T_2 \cap S_1' \cap S_2'| = |Y_2 \cap T|$. If $|T| \geq 3$, from $S_1' \cup S_2' = N$, we see that $|S_1' \cap T| \geq 2$ or $|S_2' \cap T| \geq 2$. Assume, without loss of generality, that $|S_1' \cap T| \geq 2$. Then $ANS_{S_2'}(T) = ANS_{S_1'}(T) = 2$, so $|S_2' \cap T| \geq 2$. Therefore, $|Y_1 \cap T| \leq |T| - 2$ and $|Y_2 \cap T| \leq |T| - 2$.

Conversely, assume that there exist $Y_1, Y_2 \in \varphi_n$, $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 \neq \emptyset$, satisfying conditions (1) and (2). Define $S_1 = N \setminus Y_2$ and $S_2 = N \setminus Y_1$, then $S_1 \neq S_2$. It is easy to see that $ANS_{S_1}(T) = ANS_{S_2}(T)$ for all $T \in Q$. □

Let $\{n, Q\}$ be an instance of DA_2. Define $GQ = (N, E)$ to be the graph with vertex set N and edge set $E = \{T \in Q \,||\, |T| = 2\}$. A graph $G = (V, E)$ is said to be bicolourable if its vertex set V can be partitioned into two disjoint parts V_1 and V_2 such that every edge of G is between V_1 and V_2. For a connected graph G, if such partition exists, then it is unique. In this case V_1 and V_2 are called monochromatic subsets of G.

Theorem 3.3. *Let $\{n, q\}$ be an instance of DA_2. Then, $\{n, Q\}$ is not determinant if and only if GQ has a connected component that is bicolourable and its monochromatic vertex subsets Y_1 and Y_2 satisfy the following conditions:*

(1) *If $T \in Q$, $|T| = 1$, then $T \cap (Y_1 \cup Y_2) = \emptyset$.*
(2) *If $T \in Q$, $|T| \geq 3$, then $|T \cap Y_1| \leq |T| - 2$, and $|T \cap Y_2| \leq |T| - 2$.*

Proof. Assume that GQ has a connected component that is bicolourable, the monochromatic vertex subsets Y_1 and Y_2 satisfy the conditions (1) and (2), then by Lemma 3.2, $\{n, Q\}$ is not determinant. Conversely, assume that $\{n, Q\}$ is not determinant, then there exist Y_1 and Y_2 satisfying Lemma 3.2. For any $T \in E$, since $|Y_1 \cap T| = |Y_2 \cap T| = 0$ or 1, we have either $T \cap (Y_1 \cup Y_2) = \emptyset$ or $T \subset Y_1 \cup Y_2$. Hence the subgraph $GQ|_{Y_1 \cup Y_2}$ induced by $Y_1 \cup Y_2$ is the union of some connected components of GQ. Moreover, for each edge T of $GQ|_{Y_1 \cup Y_2}$, we must have that $|T \cap Y_1| = |T \cap Y_2| = 1$. Thus $GQ|_{Y_1 \cup Y_2}$ is bicolourable. Consider a connected component of $GQ|_{Y_1 \cup Y_2}$. Its two monochromatic vertex subsets must be subsets of Y_1 and Y_2 and hence satisfy the conditions (1) and (2). □

Corollary 3.4. *DA_2 is polynomial-time solvable.*

Proof. A graph is bicolourable if and only if it contains no odd cycle, the latter holds if and only if there exists no odd cycle in a basis of cycles. Hence, the bicolouring of a graph is polynomial-time solvable. If a connected graph is bicolourable, then its vertex set can be uniquely partitioned into two disjoined monochromatic subsets. By Theorem 3.3, it is easy to see that the corollary is true. □

Next, we consider DA_3.

Lemma 3.5. *Let $\{n, Q\}$ be an instance of DA_3, then $\{n, Q\}$ is not determinant if and only if there exist $Y_1, Y_2 \in \varphi_n$, $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 \neq \emptyset$, such that for any $T \in Q$ the following conditions hold:*

(1) *If $|T| \leq 3$, then $|Y_1 \cap T| = |Y_2 \cap T|$.*
(2) *If $|T| = 4$, then $|Y_1 \cap T| = |Y_2 \cap T|$, or $|T \cap (Y_1 \cup Y_2)| = 1$.*
(3) *If $|T| \geq 5$, then $|Y_1 \cap T| \leq |T| - 3$ and $|Y_2 \cap T| \leq |T| - 3$.*

Proof. Assume that $\{n, Q\}$ is not determinant. By Lemma 3.1, there exist $S_1', S_2' \in \varphi_n$, $S_2' \neq S_1'$, $S_1' \cup S_2' = N$ such that $ANS_{S_1'}(T) = ANS_{S_2'}(T)$ for all $T \in Q$. Define $Y_1 = S_1' \setminus S_2' = N \setminus S_2'$ and $Y_2 = S_2' \setminus S_1' = N \setminus S_1'$. If $|T| = 4$ and $|Y_1 \cap T| \neq |Y_2 \cap T|$, then $|Y_i \cap T| = |T| - |S_{3-i}' \cap T|$ implies $|S_1' \cap T| \neq |S_2' \cap T|$. But $ANS_{S_1'}(T) = ANS_{S_2'}(T)$, so $|S_1' \cap T| \geq 3$ and $|S_2' \cap T| \geq 3$, hence $|Y_1 \cap T| = 0$, (or 1) if and only if $|Y_2 \cap T| = 1$ (or 0), i.e., $|(Y_1 \cup Y_2) \cap T| = 1$. The proof of other cases is similar to that of Lemma 3.2.

Conversely, assume that there exist Y_1 and Y_2 satisfying the conditions of the lemma, then define $S_1' = N \setminus Y_2$, $S_2' = N \setminus Y_1$. It is easy to verify that S_1' and S_2' satisfy Lemma 3.1. Hence, $\{n, Q\}$ is not determinant. □

Let $\bar{i}DA_3$ denote the special case of DA_3 that there exists no test $T \in Q$ with $|T| = i$.

Theorem 3.6. *$\bar{3}DA_3$ is polynomial-time solvable.*

Proof. The first of all, we state an algorithm, then show its correctness. Let $\{n, Q\}$ be an instance of $\bar{3}DA_3$, define $E=\{T\in Q \mid |T|=2\}$ and the graph $GQ=(N,E)$. Assume that the bicolourable connected components of GQ are $G_1, G_2, \ldots, G_m$ and the monochromatic vertex subsets of G_i are X_i and Z_i.

Step 0. Let

$$R:=N\setminus \bigcup_{i=1}^{m}(X_i\cup Z_i).$$

Step i $(1\le i\le m)$. Let $Y:=X_i$ and $Y':=Z_i$.

(A) If Y and Y' satisfy the following conditions:

(1) If $T\in Q$, $|T|=1$, then $T\cap(Y\cup Y')=\emptyset$.

(2) If $T\in Q$, $|T|=4$, then $|Y\cap T|=|Y'\cap T|$ or $|(Y\cup Y')\cap T|=1$.

(3) If $T\in Q$, $|T|\ge 5$, then $|Y\cap T|\le |T|-3$ and $|Y'\cap T|\le |T|-3$.

Then stop, and conclude that $\{n, Q\}$ is not determinant.

If Y and Y' do not satisfy (1) or (3), then let $R:=R\cup X_i\cup Z_i$, and go to Step $i+1$.

If Y and Y' satisfy (1) and (3), but do not satisfy (2), then there exists $T\in Q$, $|T|=4$ such that either

$$|T\cap Y|\ge 2 \quad\text{and}\quad |T\cap Y'|\le 1 \tag{3.1}$$

or

$$|T\cap Y|\le 1 \quad\text{and}\quad |T\cap Y'|\ge 2. \tag{3.2}$$

If $T\sqsubseteq Y\cup Y'$, then let $R:=R\cup X_i\cup Z_i$ and go to Step $i+1$, else choose $x\in T\setminus (Y\cup Y')$. If $x\in R$, then let $R:=R\cup X_i\cup Z_i$, go to Step $i+1$. If $x\notin R$, then x must be a vertex of G_j for some $j=1,2,\ldots,m$. When (3.1) occurs, let

$$\left.\begin{array}{l} Y:=\text{the union of } Y \text{ and the monochromatic vertex subset that} \\ \qquad \text{does not contain } x, \\ Y':=\text{the union of } Y' \text{ and the monochromatic vertex subset that} \\ \qquad \text{contains } x; \end{array}\right. \tag{3.3}$$

when (3.2) occurs, let

$$\left.\begin{array}{l} Y:=\text{the union of } Y \text{ and the monochromatic vertex subset that} \\ \qquad \text{contains } x, \\ Y':=\text{the union of } Y' \text{ and the monochromatic vertex subset that} \\ \qquad \text{does not contain } x, \end{array}\right. \tag{3.4}$$

and go to (A).

Step $m+1$. Stop and conclude that $\{n, Q\}$ is determinant.

Now, we show its correctness as follows.

Assume that $\{n, Q\}$ is not determinant, then there exist Y_1 and Y_2 satisfying Lemma 3.5. If $|T|=2$, $T\in Q$, then $|T\cap Y_1|=|T\cap Y_2|$, so, $GQ|_{Y_1\cup Y_2}$ is the union of some bicolourable connected components of GQ. Let G_i be the connected component of $GQ|_{Y_1 \cup Y_2}$ with the smallest index. We claim that the algorithm stops not later than Step i and conclude that $\{n, Q\}$ is not determinant. We first see two facts.

Claim 1. *At step i, $Y \sqsubseteq Y_1$, $Y' \sqsubseteq Y_2$ (or $Y \sqsubseteq Y_2$, $Y' \sqsubseteq Y_1$).*

Proof. If $Y = X_i$, $Y' = Z_i$, then, obviously $Y \sqsubseteq Y_1$, $Y' \sqsubseteq Y_2$. In the following, we show that the claim is still true when Y and Y' are redefined by (3.3) or (3.4). For convenience, let $\tilde{Y}$ and $\tilde{Y}'$ denote the redefined Y and Y'. We must show that $Y \sqsubseteq Y_1$ and $Y' \sqsubseteq Y_2$ imply $\tilde{Y} \sqsubseteq Y_1$ and $\tilde{Y}' \sqsubseteq Y_2$. If (3.1) occurs, then $Y \sqsubseteq Y_1$, $|T| = 4$, $|Y_1 \cap T| \geq 2$. By Lemma 3.5, $|Y_1 \cap T| = |Y_2 \cap T| = 2$. So, $T \setminus (Y \cup Y') \sqsubseteq Y_2$, $X \in Y_2$. Hence, the $\tilde{Y}$ and $\tilde{Y}'$ obtained from (3.3) satisfy $\tilde{Y} \sqsubseteq Y_1$, $\tilde{Y}' \sqsubseteq Y_2$. If (3.2) occurs, we can show the claim by a similar argument. □

Claim 2. *The algorithm cannot go to Step $i+1$ from Step i.*

Proof. For otherwise, assume that the algorithm goes to Step $i+1$ from Step i, then one of the following occurs.

(a) Y and Y' do not satisfy (1) or (3).

(b) Y and Y' do not satisfy (2) with T and $T \sqsubseteq Y \cup Y'$.

(c) $(T \setminus (Y \cup Y')) \cap R \neq \emptyset$ holds.

If (a) occurs, then by Claim 1, Y_1 and Y_2 do not satisfy the condition (1) or (3) in Lemma 3.5. If (b) occurs, then Y and Y' do not satisfy the condition (2) in Lemma 3.5. Therefore, (a) and (b) cannot occur. Next, suppose that (c) occurs. Note that $|T| = 4$. Since Y_1 and Y_2 satisfy the conditions in Lemma 3.5, we have $|T \cap Y_1| = |T \cap Y_2| = 2$. Thus, $T \sqsubseteq Y_1 \cup Y_2$. (c) implies $R \cap (Y_1 \cup Y_2) \neq \emptyset$. However, during the computation of Step i, we have

$$R = N \setminus \bigcup_{h=i}^{m} (X_h \cup Z_h)$$

and by the assumption on G_i, we have

$$Y_1 \cup Y_2 \sqsubseteq \bigcup_{h=i}^{m} (X_h \cup Z_h)$$

contradicting $R \cap (Y_1 \cup Y_2) \neq \emptyset$. □

By Claim 2, the algorithm must stop before or at Step i. Note that the loop at each step cannot go for infinitely many times since each time when the computation goes to (A) from the last instruction of the step, the number of vertices in $Y \cup Y'$ will increase. Therefore, the algorithm must stop at the place where $\{n, Q\}$ is pointed out not to be determinant.

Now, assume that $\{n, Q\}$ is determinant, we need to show that the computation must enter Step m. For otherwise, suppose that the computation stops before Step m. Then, there must exist Y and Y' satisfying (1), (2) and (3). It follows that Y and Y' satisfy Lemma 3.5 and hence $\{n, Q\}$ is not determinant, a contradiction. □

Next, we show the co-NP-completeness of DA_3 by reducing the following problem to it.

One-in-three-SAT. Given a set U of variables and a set ε of clauses, with each $C \in \varepsilon$ containing exactly three variables from U, determine whether there is a truth assignment t on U such that each clause C in ε contains exactly one TRUE variable.

Theorem 3.7. *DA_3 is co-NP-complete.*

Proof. It is easy to see that DA_3 is in co-NP. Now we show that one-in-three-SAT is polynomial-time reducible to the complement of DA_3, and hence DA_3 is co-NP-complete.

Let (U, ε) be a given instance of one-in-three-SAT such that $U = \{X_1, \ldots, X_p\}$, $\varepsilon = \{C_1, \ldots, C_q\}$, $|C_j| = 3$ for all j. Without loss of generality, assume that every X_i in U occurs in some C_j in ε.

Define an instance $\{n, Q\}$ of DA_3 as follows:

$$\begin{aligned} &n := p + 15q + 1, \qquad m := 22q, \\ &N := \{1, \ldots, p, y, u(j,k), v(j,k), w(j,k), x(j,k), z(j,k) \mid \\ &\qquad k = 1,2,3,\ j = 1, \ldots, q\},\ |N| = n. \end{aligned}$$

For convenience, assume that

$$\begin{aligned} u(j,4) &= u(j,1), & u(j,0) &= u(j,3), \\ v(j,4) &= v(j,1), & v(j,0) &= v(j,3), \\ w(j,4) &= w(j,1), & w(j,0) &= w(j,3), \\ x(j,4) &= x(j,1), & x(j,0) &= x(j,3). \end{aligned}$$

For each $j = 1, \ldots, q$, assume that $C_j = \{X_{j_1}, X_{j_2}, X_{j_3}\}$ (with $j_1 < j_2 < j_3$) and define Q as follows:

$$\begin{aligned} Q = \{&T_{jh}, U_{jk}, M_{jk}, N_{jk}, L_{jk}, H_{jk}, P_{jk} \mid \\ &j = 1, \ldots, q,\ h = 0,1,2,3,\ k = 1,2,3\}, \quad |Q| = m, \end{aligned}$$

where

$$\begin{aligned} T_{j_0} &= \{j_1, j_2, j_3, y\}, \\ T_{jk} &= \{j_k, u(j,k), v(j,k)\}, \\ U_{j_1} &= \{j_2, j_3, u(j,2), z(j,1)\}, \\ U_{j_2} &= \{j_1, j_3, u(j,3), z(j,2)\}, \\ U_{j_3} &= \{j_1, j_2, u(j,1), z(j,3)\}, \\ M_{jk} &= \{u(j,k+1), x(j,k)\}, \\ N_{jk} &= \{z(j,k), x(j,k)\}, \\ L_{jk} &= \{x(j,k), w(j,k), y\}, \\ H_{jk} &= \{z(j,k), w(j,k+1), w(j,k-1), y\}, \\ P_{jk} &= \{z(j,k), v(j,k), x(j,k+1)\}. \end{aligned}$$

Case 1. Assume that t is a truth assignment on U such that for each $C_j \in \varepsilon$, t assigns exactly one TRUE value to the variables in C_j. Define two sets S_1, S_2 as follows:

$$\begin{aligned} S_1 = \{i \mid 1 \le i \le p,\ t(X_i) = TRUE\} & \\ \cup \{u(j,k+1), v(j,k-1), z(j,k) \mid & \text{the } k\text{th variable } X_{j_k} \text{ in } C_j \text{ has } t(X_{j_k}) = TRUE\} \\ \cup \{y\}. & \end{aligned}$$

$$\begin{aligned} S_2 = \{i \mid 1 \le i \le p,\ t(X_i) = FALSE\} & \\ \cup \{v(j,k), x(j,k), w(j,k+1), w(j,k-1) \mid & \text{the } k\text{th variable } X_{j_k} \text{ in } C_j \text{ has } t(X_{j_k}) = TRUE\}. \end{aligned}$$

Obviously, $S_1 \ne S_2$, we claim that for all $T \in Q$, $ANS_{S_1}(T) = ANS_{S_2}(T)$, hence $\{n, Q\}$ is not determinant. To show so, we check the following:

Claim 1.1. *For all* j, k, $|S_1 \cap T_{j_0}| = |S_2 \cap T_{j_0}| = 2$, $|S_1 \cap T_{jk}| = |S_2 \cap T_{jk}| = 1$, $|S_1 \cap U_{jk}| = |S_2 \cap U_{jk}| = 1$ *or* 2, $|S_1 \cap M_{jk}| = |S_2 \cap M_{jk}| = 0$ *or* 1, $|S_2 \cap N_{jk}| = |S_1 \cap N_{jk}| = 0$ *or* 1.

Proof. From the definitions of S_1 and S_2, we have that for any j and k, $X(j,k) \in S_2 \Leftrightarrow Z(j,k) \in S_1$. If $t(X_{j_1}) = TRUE$ and $t(X_{j_2}) = t(X_{j_3}) = FALSE$, then

$$\begin{array}{ll} S_1 \cap T_{j_0} = \{j_1, y\}, & S_2 \cap T_{j_0} = \{j_2, j_3\}; \\ S_1 \cap T_{j_1} = \{j_1\}, & S_2 \cap T_{j_1} = \{v(j,1)\}; \\ S_1 \cap T_{j_2} = \{u(j,2)\}, & S_2 \cap T_{j_2} = \{j_2\}; \\ S_1 \cap T_{j_3} = \{v(j,3)\} & S_2 \cap T_{j_3} = \{j_3\}; \\ S_1 \cap U_{j_1} = \{u(j,2), z(j,1)\}, & S_2 \cap U_{j_1} = \{j_2, j_3\}; \\ S_1 \cap U_{j_2} = \{u(j,1)\}, & S_2 \cap U_{j_2} = \{j_3\}; \\ S_1 \cap U_{j_3} = \{j_1\}, & S_2 \cap U_{j_3} = \{j_2\}. \end{array}$$

The other two cases are similar. □

Claim 1.2. *For all* j *and* k, $|S_1 \cap H_{jk}| = 1$ *or* 2, $|S_2 \cap H_{jk}| = 1$ *or* 2, $|S_1 \cap L_{j_k}| = |S_2 \cap L_{j_k}| = 1$, $|S \cap P_{jk}| = |S_2 \cap P_{jk}| = 0$ *or* 1.

Proof. If $t(X_{j_1}) = TRUE$, $t(X_{j_2}) = t(X_{j_3}) = FALSE$, then,

$$\begin{array}{ll} S_1 \cap L_{jk} = \{y\}, \quad k = 1,2,3, & S_2 \cap L_{j_1} = \{x(j,1)\}, \\ S_2 \cap L_{j_2} = \{w(j,2)\}, & S_2 \cap L_{j_3} = \{w(j,3)\}, \\ S_1 \cap H_{j_1} = \{y, z(j,1)\}, & S_2 \cap H_{j_1} = \{w(j,2), w(j,3)\}, \\ S_1 \cap H_{j_2} = S_1 \cap H_{j_3} = \{y\}, & S_2 \cap H_{j_2} = \{w(j,3)\}, \\ S_2 \cap H_{j_3} = \{w(j,2)\}, & S_1 \cap P_{j_1} = \{z(j,1)\}, \\ S_2 \cap P_{j_1} = \{v(j,1)\}, & S_2 \cap P_{j_2} = S_1 \cap P_{j_2} = \emptyset, \\ S_1 \cap P_{j_3} = \{v(j,3)\}, & S_2 \cap P_{j_3} = \{x(j,1)\}. \end{array}$$

The other two cases are similar. □

Case 2. Conversely, assume that $\{n, Q\}$ is not determinant. By Lemma 3.5, there exist $Y_1, Y_2 \in \varphi_n$, $Y_1 \cup Y_2 \neq \emptyset$, $Y_1 \cap Y_2 = \emptyset$ such that for any $T \in Q$,

(1) if $|T| \leq 3$, then $|Y_1 \cap T| = |Y_2 \cap T|$,

(2) if $|T| = 4$, then $|Y_1 \cap T| = |Y_2 \cap T|$, or $|(Y_1 \cup Y_2) \cap T| = 1$.

First, we show the following facts.

Claim 2.1. *For all* $j = 1, \ldots, q$, *and* $k = 1, 2, 3$,

$$u(j, k+1) \in Y_1 \Leftrightarrow x(j,k) \in Y_2 \Leftrightarrow z(j,k) \in Y_1,$$
$$u(j, k+1) \in Y_2 \Leftrightarrow x(j,k) \in Y_1 \Leftrightarrow z(j,k) \in Y_2.$$

Proof. It follow immediately from $|Y_1 \cap M_{jk}| = |Y_2 \cap M_{jk}|$ and $|Y_1 \cap N_{jk}| = |Y_2 \cap N_{jk}|$. □

Claim 2.2. *For any* j, *if* $|\{j_1, j_2, j_3\} \cap Y_1| \leq 1$ *and* $|\{j_1, j_2, j_3\} \cap Y_2| \leq 1$, *then* $u(j,k)$, $v(j,k)$, $w(j,k)$, $x(j,k)$, $z(j,k) \notin Y_1 \cup Y_2$ *for all* $k = 1, 2, 3$.

Proof. By Claim 2.1, $\{u(j, k+1), z(j,k)\} \cap (Y_1 \cup Y_2) \neq \emptyset$ implies either $\{j_{k+1}, j_{k+2}\} \sqsubseteq Y_1$, or $\{j_{k+1}, j_{k+2}\} \sqsubseteq Y_2$. Thus, we must have $u(j,k)$, $z(j,k) \notin Y_1 \cup Y_2$ for $k = 1, 2, 3$, and hence $x(j,k) \sqsubseteq Y_1 \cup Y_2$ for $k = 1, 2, 3$. Since $|P_{jk} \cap Y_1| = |P_{jk} \cap Y_2|$ we also have $v(j,k) \notin Y_1 \cup Y_2$ for $k = 1, 2, 3$. If $y \notin Y_1 \cup Y_2$, then by considering L_{jk}, we can obtain $w(j,k) \notin Y_1 \cup Y_2$ for $k = 1, 2, 3$. If $y \in Y_1 \cup Y_2$, then by considering L_{jk}, we can obtain $w(j,k) \in Y_1 \cup Y_2$ for $k = 1, 2, 3$. Thus, $|H_{jk} \cap (Y_1 \cap Y_2)| = 3$. This is impossible. □

Claim 2.3. $y \in Y_1 \cup Y_2$.

Proof. Suppose to the contrary that $y \notin Y_1 \cup Y_2$. Then the hypothesis of Claim 2.2 holds for all $j = 1, \ldots, q$, and hence none of $u(j,k)$, $v(j,k)$, $w(j,k)$, $z(j,k)$ is in $Y_1 \cup Y_2$. Furthermore, by considering T'_{jk} we can see that none of j_k is in $Y_1 \cup Y_2$. Thus, $Y_1 \cup Y_2 = \emptyset$, a contradiction. □

Since $y \in Y_1 \cup Y_2$, without loss of generality, we assume $y \in Y_1$.

Claim 2.4. *For all* $j = 1, \ldots, q$, $|\{j_1, j_2, j_3\} \cap Y_1| = 1$.

Proof. Suppose to the contrary that for some j, $|\{j_1, j_2, j_3\} \cap Y_1| \neq 1$. Then, we must have $|\{j_1, j_2, j_3\} \cap Y_2| = 0$. By considering test T_{j0}, we see that $|\{j, j_2, j_3\} \cap Y_2| \leq 1$. Thus, the hypothesis of Claim 2.2 holds for j and hence for $k = 1, 2, 3$, $u(j,k)$, $v(j,k)$, $w(j,k)$, $x(j,k)$ and $z(j,k)$ are not in $Y_1 \cup Y_2$. It follows that $|L_{jk} \cap Y_1| \neq 0 = |L_{jk} \cap Y_2|$, a contradiction. □

Now, we define a truth assignment t on V by

$$t(X_i) = \begin{cases} TRUE, & i \in Y_1, \\ FALSE, & i \notin Y_1, \end{cases}$$

for $i = 1, \ldots, q$. From Claim 2.4, it is easy to see that this is a one-in-three truth assignment. □

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