CSE 694 Homework Assignment 2

Due in class on Monday, Sep 26.

September 16, 2011

There are totally 6 problems, 10 points each. You are encouraged to discuss problems with your classmates. Again, tell me who you talk to on which problem. The final writeup must still be your own, in your own words!

Problem 1. Prove that: if there is a $p \in [0, 1]$ where

$$\binom{n}{a} p^{\binom{a}{2}} + \binom{n}{b}(1-p)^{\binom{b}{2}} < 1,$$

then the Ramsey number $R(a, b)$ is more than n. Using this, show that

$$R(4, b) = \Omega\left(\frac{b^{3/2}}{(\ln b)^{3/2}}\right).$$

(If get stuck: go back to my first lecture and try to understand the Ramsey number example.)

Problem 2. Suppose $n \geq 2$ and let $H = (V, E)$ be an n-uniform hypergraph. Show that if $|E| \leq 4^{n-1}$ then there exists a coloring of vertices in V with 4 colors such that no edge $e \in E$ is monochromatic. (Note that each "edge" $e \in E$ is a subset of size n of V.)

> For the next two problems, look again at the proof of the Sperner Theorem in the first Lecture.
> However, they are only similar in spirit, so please don't over-analyze the similarity.

Problem 3. Let F be a finite set of binary strings of finite lengths so that no string in F is a prefix of another string in F. Let n_i be the number of strings of length i in F. Show that

$$\sum_i \frac{n_i}{2^i} \leq 1.$$

Problem 4. Let k, l be two fixed positive integers. Let $\{(A_i, B_i), 1 \leq i \leq h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ for all i and $|B_i| = l$ for all i, $A_i \cap B_i = \emptyset$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that $h \leq \frac{(k+l)^{k+l}}{k^k l^l}$.

Problem 5. For all n sufficiently large and divisible by 8, show that there exists a family $\mathcal{F}$ of subsets of $[n]$ satisfying the following conditions

- For each $F \in \mathcal{F}$, $|F| = n/4$

- For every two different members F_1, F_2 of $\mathcal{F}$, $|F_1 \cap F_2| \leq n/8$

- $|\mathcal{F}| = 2^{\Omega(n)}$

(**Hint:** fix n, choose "randomly" members of the family, prove that the probability that the family is "bad" is < 1 for n sufficiently large.)

Problem 6. Recall that a $t \times n$ binary matrix $\mathbf{A} = (a_{ij})$ is a d-*disjunct* matrix iff no column is contained in the union of d other columns. (We think of each column j as the set of rows i for which $a_{ij} = 1$. We will use set terminologies and operations on columns.) In Lecture note 1, we have seen a probabilistic method proof of the existence of $t \times n$ d-disjunct matrix with $t \approx 2e(d+1)^2 \ln(en/(d+1)) = O(d^2 \log(n/d))$. In this problem, we will prove a similar result using a different method of generating a random matrix.

1. Let n, m, b, d be any positive integers. Show that if

$$(d+1)\binom{n}{d+1}\left(1 - (1-1/b)^d\right)^m < 1$$

   then there exists a $t \times n$ d-disjunct matrix $\mathbf{A}$ with $t = mb$.

   (**Hint:** generate a random $mb \times n$ matrix as follows. Partition the $t = mb$ rows into m blocks, each block has exactly b rows. Each column of $\mathbf{A}$ is chosen randomly by having exactly one 1 in each block. The 1 is chosen uniformly out of b choices. Thus, each column of $\mathbf{A}$ has precisely m 1s.)

2. Given n and d, we would like to know a small value of t for which a $t \times n$ d-disjunct matrix exists. To simplify some calculations, assume $d \geq 2$ and $(d+1)^2 \leq n$. From the previous question, show that a $t \times n$ d-disjunct matrix exists when $t \geq ed(2d+3)\log(n/(d+1))$.

   (With more careful analysis you might get a better bound than that!)