CSE 694 Homework Assignment 3

Due in class on Monday, Oct 17

October 7, 2011

There are totally 6 problems, 10 points each. You are encouraged to discuss problems with your classmates. Again, tell me who you talk to on which problem. The final writeup must still be your own, in your own words! The probabilistic method problems are meant for the union bound and expectation arguments only.

Problem 1. Let k be an arbitrary positive integer. Find an example of

(a) a discrete random variable X with finite jth moment $\mathrm{E}[X^j]$ for $1 \leq j \leq k$ but unbounded $(k+1)$th moment

(b) a discrete non-negative random variable X such that

$$\mathrm{Prob}[X \geq k\mathrm{E}[X]] = \frac{1}{k}$$

(This shows that Markov's inequality is tight!)

Problem 2. The CSP problem is defined as follows. We are given a positive integer n, and k "constraints" of the form (x, y, z), where $x, y, z \in [n]$ are distinct integers. The problem is to output a permutation $\pi = \pi_1, \dots, \pi_n$ of $[n]$ satisfying as many constraints as possible. The permutation π satisfies a constraint (x, y, z) if y lies *between* x and z in the listing $\pi_1, \dots, \pi_n$. (It does not matter if x lies before or after y.)

It is known that this problem is **NP**-hard. Let OPT denote the maximum number of constraints which can be satisfied by any permutation for this instance of the problem. Design a randomized algorithm whose output permutation satisfies at least αOPT constraints, on average, where $\alpha > 0$ is a constant. Specify the best α you can prove, i.e. as close to 1 as possible.

Problem 3. Let $G = (V, E)$ be an undirected graph. Let π be a permutation of the vertices of G. Let $S(\pi)$ be a subset of vertices defined as follows: for each vertex $v \in V$, $v \in S$ iff no neighbor of v precedes v in the permutation π. Let d_v denote the degree of $v \in V$.

(i) Show that $S(\pi)$ is an independent set of G.

(ii) Find a (natural) way to generate random permutations such that $\mathrm{E}[|S(\pi)|] = \sum_{v \in V} \frac{1}{d_v+1}$. Conclude that there is an independent set of G of size at least $\sum_{v \in V} \frac{1}{d_v+1}$.

Problem 4. Problem 3 from lecture notes 5 on tail inequalities.

Problem 5. Problem 5 from lecture notes 5 on tail inequalities.

Problem 6. Problem 8 from lecture notes 5 on tail inequalities.