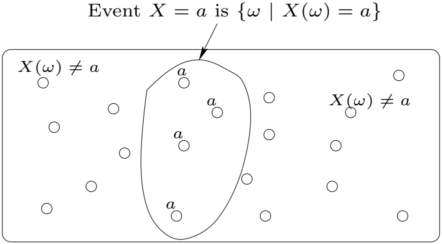
Lecture 2: Random Variables and Expectations

- Random variables
- Expectation
- The argument from expectation



- A random variable is a function $X : \Omega \to \mathbb{R}$
- $p_X(a) = \text{Prob}[X = a]$ is called the probability mass function of X
- $P_X(a) = \text{Prob}[X \leq a]$ is called the (cumulative/probability) distribution function of X

PTCF: Expectation and its Linearity

- The expected value of X is defined as

$$\mathsf{E}[X] := \sum_a a \, \mathsf{Prob}[X = a].$$

- For any set $X_1, \dots, X_n$ of random variables, and any constants $c_1, \dots, c_n$

$$\mathsf{E}[c_1 X_1 + \cdots + c_n X_n] = c_1 \mathsf{E}[X_1] + \cdots + c_n \mathsf{E}[X_n]$$

  This fact is called linearity of expectation

PTCF: Indicator/Bernoulli Random Variable

$$X : \Omega \to \{0, 1\}$$

$$p = \mathsf{Prob}[X = 1]$$

X is called a Bernoulli random variable with parameter p

If $X = 1$ only for outcomes ω belonging to some event A, then X is called an indicator variable for A

$$\begin{aligned} \mathsf{E}[X] &= p \\ \mathsf{Var}[X] &= p(1-p) \end{aligned}$$

The Argument from Expectation: Main Idea

- X a random variable with $\mathrm{E}[X] = \mu$, then
  - There must exist a sample point ω with $X(\omega) \geq \mu$
  - There must exist a sample point ω with $X(\omega) \leq \mu$
- X a random variable with $\mathrm{E}[X] \leq \mu$, then
  - There must exist a sample point ω with $X(\omega) \leq \mu$
- X a random variable with $\mathrm{E}[X] \geq \mu$, then
  - There must exist a sample point ω with $X(\omega) \geq \mu$

Example 1: Large Cuts in Graphs

Intuition & Question

Intuition: every graph must have a "sufficiently large" cut (A, B).
Question: How large?

Line of thought

On average, a *random* cut has size μ, hence there must exist a cut of size $\geq \mu$.

- Put a vertex in either A or B with probability $1/2$
- Expected number of edges X with one end point in each is

$$\mathsf{E}[X] = \mathsf{E}\left[\sum_e X_e\right] = \sum_e \mathsf{Prob}[X_e] = |E|/2$$

Theorem

For every graph $G = (V, E)$, there must be a cut with $\geq |E|/2$ edges

Example 2: ± 1 Linear Combinations of Unit Vectors

Theorem

Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be n unit vectors in $\mathbb{R}^n$.
There exist $\alpha_1, \cdots, \alpha_n \in \{-1, 1\}$ such that

$$|\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n| \leq \sqrt{n}$$

and, there exist $\alpha_1, \cdots, \alpha_n \in \{-1, 1\}$ such that

$$|\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n| \geq \sqrt{n}$$

Specifically, choose $\alpha_i \in \{-1, 1\}$ independently with prob. $1/2$

$$\mathsf{E}\left[|\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n|^2\right] = \sum_{i,j} \mathbf{v}_i \cdot \mathbf{v}_j \mathsf{E}[\alpha_i \alpha_j] = \sum_i \mathbf{v}_i^2 = n.$$

Example 3: Max-E3SAT

- An E3-CNF formula is a CNF formula φ in which each clause has *exactly* 3 literals. E.g.,

$$\varphi = \underbrace{(x_1 \vee \bar{x}_2 \vee x_4)}_{\text{Clause } 1} \wedge \underbrace{(x_1 \vee x_3 \vee \bar{x}_4)}_{\text{Clause } 2} \wedge \underbrace{(\bar{x}_2 \vee \bar{x}_3 \vee x_4)}_{\text{Clause } 3}$$

- Max-E3SAT Problem: given an E3-CNF formula φ, find a truth assignment satisfying as many clauses as possible

A Randomized Approximation Algorithm for Max-E3SAT

- Assign each variable to TRUE/FALSE with probability $1/2$

Analyzing the Randomized Approximation Algorithm

- Let X_C be the random variable indicating if clause C is satisfied
- Then, $\mathsf{Prob}[X_C = 1] = 7/8$
- Let S_φ be the number of satisfied clauses. Then,

$$\mathsf{E}[S_\varphi] = \mathsf{E}\left[\sum_C X_C\right] = \sum_C \mathsf{E}[X_C] = 7m/8 \geq \frac{\text{OPT}}{8/7}$$

  (m is the number of clauses)
- So this is a randomized approximation algorithm with ratio $8/7$

Example 4: Unbalancing Lights

Theorem

For $1 \le i, j \le n$, we are given $a_{ij} \in \{-1, 1\}$. Then, there exist $\alpha_i, \beta_j \in \{-1, 1\}$ such that

$$\sum_i \sum_j a_{ij}\alpha_i\beta_j \ge \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}$$

- Choose $\beta_j \in \{-1, 1\}$ independently with prob. $1/2$.
- $R_i = \sum_j a_{ij}\beta_j$, then

$$\mathsf{E}[|R_i|] = 2\frac{n\binom{n-1}{\lfloor (n-1)/2 \rfloor}}{2^n} \approx \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{1/2}$$

- Choose α_i with the same sign as R_i, for all i