Agenda

We've discussed

- C++ basics
- Some built-in data structures and their applications: stack, map, vector, array
- The Fibonacci example showing the importance of good algorithms and asymptotic analysis

Now

- Growth of functions
- Asymptotic notations
- Scare some people off

Next

- Recurrence relations & a way to solve them
- Binary search and some sorting algorithms to illustrate

Some conventions

$$
\begin{aligned}
\lg n &= \log_2 n \\
\log n &= \log_{10} n \\
\ln n &= \log_e n
\end{aligned}
$$

Growth of functions

Consider an Intel Core i7 Extreme Edition 980X (Hex core), 3.33GHz, top speed $< 150 \cdot 10^9$ instructions per second (IPS). For simplicity, say it's 10^9 IPS.

	10	20	30	40	50	1000
$\lg\lg n$	1.7 ns	2.17 ns	2.29 ns	2.4 ns	2.49 ns	3.3 ns
$\lg n$	3.3 ns	4.3 ns	4.9 ns	5.3 ns	5.6 ns	9.9 ns
n	10 ns	20 ns	3 ns	4 ns	5 ns	1 μs
n^2	0.1 μs	0.4 μs	0.9 μs	1.6 μs	2.5 μs	1 ms
n^3	1 μs	8 μs	27 μs	64 μs	125 μs	1 sec
n^5	0.1 ms	3.2 ms	24.3 ms	0.1 sec	0.3 sec	277 h
2^n	1 μs	1 ms	1 s	18.3 m	312 h	$3.4 \cdot 10^{282}$ Cent.
3^n	59 μs	3.5 s	57.2 h	386 y	227644 c	$4.2 \cdot 10^{458}$ Cent.

1.6^{100} ns is approx. 82 centuries (Recall `fib1()`).

$$\lg 10^{10} = 33, \quad \lg\lg 10^{10} = 4.9$$

Some other large numbers

- The age of the universe $\leq$ 13 G-Years $= 13 \cdot 10^7$ centuries.

$\Rightarrow$ Number of seconds since big-bang $\approx 10^{18}$.

- $4 * 10^{78} \leq$ Number of atoms is the universe $\leq 6 * 10^{79}$.
- The probability that a monkey can compose Hamlet is $\approx \frac{1}{10^{60}}$

so what's the philosophical implication of this?

Robert Wilensky, speech at a 1996 conference

We've heard that a million monkeys at a million keyboards could produce the complete works of Shakespeare; now, thanks to the Internet, we know that is not true.

Dominating Terms

When n is sufficiently large, order the following functions:

$$\begin{aligned}
f_1(n) &= 2000n^2 + 1{,}000{,}000n + 3 \\
f_2(n) &= 100n^2 \\
f_3(n) &= n^5 + 10^7 n \\
f_4(n) &= 2^n + n^{10{,}000} \\
f_5(n) &= 2^n \\
f_6(n) &= \frac{3^n}{10^6}
\end{aligned}$$

(Only need to look at the dominating term)

Our functions

We will only look at functions of the type

$$f : \mathbb{N} \to \mathbb{R}^+$$

as they are used for time and space complexity estimation.

Behind comparing functions

- Mathematically, $f(n) \gg g(n)$ for "sufficiently large" n means

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

  We also say $f(n)$ is asymptotically larger than $g(n)$.
- They are comparable (or of the same order) if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0$$

- and $f(n)$ is asymptotically smaller than $g(n)$ when

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Question

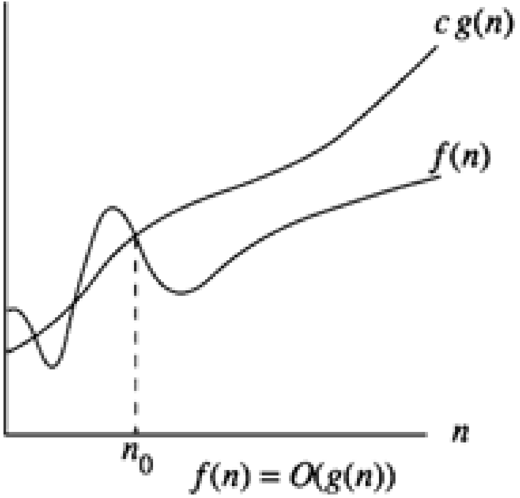
Are there $f(n)$ and $g(n)$ not falling into one of the above cases?

Asymptotic notations

$$\begin{array}{rcl}
f(n) = O(g(n)) & \text{iff} & \exists c > 0, n_0 > 0 : f(n) \le cg(n), \text{ for } n \ge n_0 \\
f(n) = \Omega(g(n)) & \text{iff} & \exists c > 0, n_0 > 0 : f(n) \ge cg(n), \text{ for } n \ge n_0 \\
f(n) = \Theta(g(n)) & \text{iff} & f(n) = O(g(n)) \ \& \ f(n) = \Omega(g(n)) \\
f(n) = o(g(n)) & \text{iff} & \forall c > 0, \exists n_0 > 0 : f(n) \le cg(n), \text{ for } n \ge n_0 \\
f(n) = \omega(g(n)) & \text{iff} & \forall c > 0, \exists n_0 > 0 : f(n) \ge cg(n), \text{ for } n \ge n_0
\end{array}$$

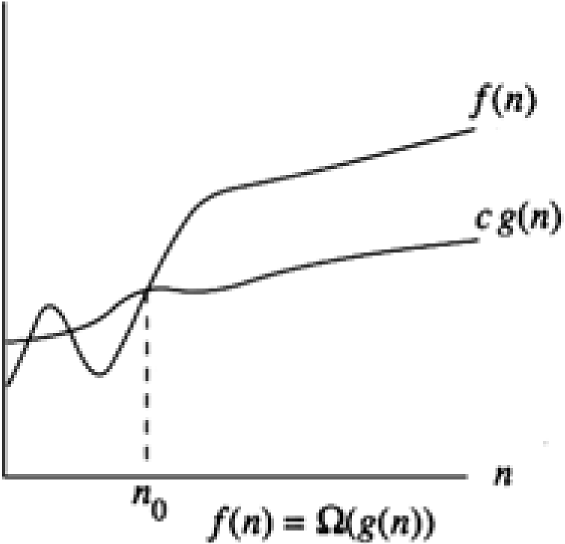
Note:

- we shall be concerned only with functions f of the form $f : \mathbb{N}^+ \to \mathbb{R}^+$, unless specified otherwise.
- $f(n) = \mathbf{x}(g(n))$ isn't mathematically correct; $f(n) \in \mathbf{x}(g(n))$ is, but not commonly used.

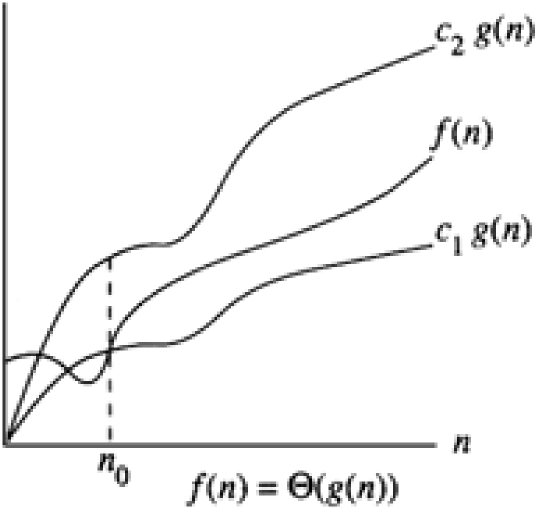


c g(n)
f(n)
n
n0
f(n) = O(g(n))

An illustration of big-Ω



An illustration of Θ



Some examples

$$
\begin{aligned}
a(n) &= \sqrt{n} \\
b(n) &= n^5 + 10^7 n \\
c(n) &= (1.3)^n \\
d(n) &= (\lg n)^{100} \\
e(n) &= \frac{3^n}{10^6} \\
f(n) &= 3180 \\
g(n) &= n^{0.0000001} \\
h(n) &= (\lg n)^{\lg n}
\end{aligned}
$$

A few properties

$$
\begin{aligned}
f(n) = o(g(n)) &\Rightarrow f(n) = O(g(n)) \text{ \& } f(n) \neq \Theta(g(n)) && (1)\\
f(n) = \omega(g(n)) &\Rightarrow f(n) = \Omega(g(n)) \text{ \& } f(n) \neq \Theta(g(n)) && (2)\\
f(n) = O(g(n)) &\Leftrightarrow g(n) = \Omega(f(n)) && (3)\\
f(n) = \Theta(g(n)) &\Leftrightarrow g(n) = \Theta(f(n)) && (4)\\
\lim_{n\to\infty} \frac{f(n)}{g(n)} = +\infty &\Leftrightarrow f(n) = \omega(g(n)) && (5)\\
\lim_{n\to\infty} \frac{f(n)}{g(n)} = c > 0 &\Rightarrow f(n) = \Theta(g(n)) && (6)\\
\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 &\Leftrightarrow f(n) = o(g(n)) && (7)
\end{aligned}
$$

Remember: we only consider functions from $\mathbb{N}^+ \to \mathbb{R}^+$.

A reminder: L'Hôpital's rule

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

if

$$\lim_{n\to\infty} f(n) \;\text{ and }\; \lim_{n\to\infty} g(n) \;\text{ are both 0 or both }\; \pm\infty$$

Examples:

$$\lim_{n\to\infty} \frac{\lg n}{\sqrt{n}} = 0 \tag{8}$$

$$\lim_{n\to\infty} \frac{(\lg n)^{\lg n}}{\sqrt{n}} = ? \tag{9}$$

Stirling's approximation

For all $n \geq 1$,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n},$$

where

$$\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}.$$

It then follows that

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

The last formula is often referred to as the Stirling's approximation

More examples

$$
\begin{aligned}
a(n) &= \lfloor \lg n \rfloor ! \\
b(n) &= n^5 + 10^7 n \\
c(n) &= 2^{\sqrt{\lg n}} \\
d(n) &= (\lg n)^{100} \\
e(n) &= 3^n \\
f(n) &= (\lg n)^{\lg \lg n} \\
g(n) &= 2^{n^{0.001}} \\
h(n) &= (\lg n)^{\lg n} \\
i(n) &= n!
\end{aligned}
$$

Special functions

Some functions cannot be compared, e.g. $n^{1+\sin(n\frac{\pi}{2})}$ and n.

$$\lg^* n = \min\{i \geq 0 : \lg^{(i)} n \leq 1\},$$

where for any function $f : \mathbb{N}^+ \to \mathbb{R}^+$,

$$f^{(i)}(n) = \begin{cases} n & \text{if } \; i = 0 \\ f(f^{(i-1)}(n)) & \text{if } \; i > 0 \end{cases}$$

Intuitively, compare

$$\begin{aligned} \lg^* n &\quad \text{vs} \quad \lg n \\ \lg^* n &\quad \text{vs} \quad (\lg n)^\epsilon, \; \epsilon > 0 \\ 2^n &\quad \text{vs} \quad n^n \\ \lg^*(\lg n) &\quad \text{vs} \quad \lg(\lg^* n) \end{aligned}$$

How about rigorously?

Asymptotic notations in equations

$$5n^3 + 6n^2 + 3 = 5n^3 + \Theta(n^2)$$

means "the LHS is equal to $5n^3$ plus some function which is $\Theta(n^2)$."

$$o(n^6) + O(n^5) = o(n^6)$$

means "for any $f(n) = o(n^6)$, $g(n) = O(n^5)$, the function $h(n) = f(n) + g(n)$ is equal to some function which is $o(n^6)$."

Be very careful!!

$$O(n^5) + \Omega(n^2) \stackrel{?}{=} \Omega(n^2)$$
$$O(n^5) + \Omega(n^2) \stackrel{?}{=} O(n^5)$$

Tight and not tight

$n \log n = O(n^2)$ is not tight

$n^2 = O(n^2)$ is tight

When comparing functions asymptotically

- Determine the dominating term
- Use intuition first
- Transform intuition into rigorous proof.