Solution to CSE 250 Homework Assignment 4

Solution for problem 1

$$(\log n)^3 n^{1.2} \ll n^2\sqrt{n} \ll n^3 \ll n^3(\log n)^{2.1} \ll \frac{n^4}{(\log n)^3} \ll n^4 \ll \frac{2^n}{n^2} \ll \frac{(\sqrt{3})^{2n}}{n^4}$$

$$\lim_{n\to\infty} \frac{(\log n)^3 n^{1.2}}{n^2\sqrt{n}} = \lim_{n\to\infty} \frac{(\log n)^3}{n^{1.2}} = 0$$

because the numerator is in the log-class, and the denominator is in the polynomial class.

Similarly,

$$\lim_{n\to\infty} \frac{n^2\sqrt{n}}{n^3} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\lim_{n\to\infty} \frac{n^3}{n^3(\log n)^{2.1}} = \lim_{n\to\infty} \frac{1}{(\log n)^{2.1}} = 0$$

$$\lim_{n\to\infty} \frac{n^3(\log n)^{2.1}}{\frac{n^4}{(\log n)^3}} = \lim_{n\to\infty} \frac{(\log n)^{5.1}}{n} = 0$$

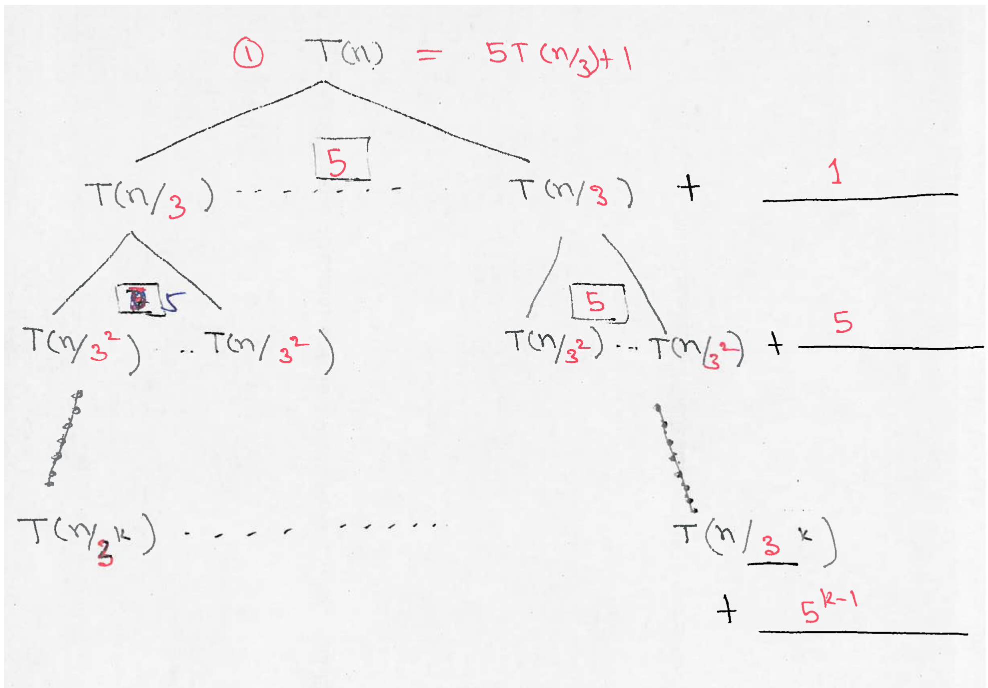
$$\lim_{n\to\infty} \frac{\frac{n^4}{(\log n)^3}}{n^4} = \lim_{n\to\infty} \frac{1}{(\log n)^3} = 0$$

$$\lim_{n\to\infty} \frac{n^4}{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{n^6}{2^n} = 0$$

because the numerator is in the polynomial-class, and the denominator is in the exponential class.

$$\lim_{n\to\infty} \frac{\frac{2^n}{n^2}}{\frac{(\sqrt{3})^{2n}}{n^4}} = \lim_{n\to\infty} \frac{\frac{2^n}{n^2}}{\frac{3^n}{n^4}} = \lim_{n\to\infty} \frac{n^2}{(1.5)^n} = 0$$

Solution for problem 2

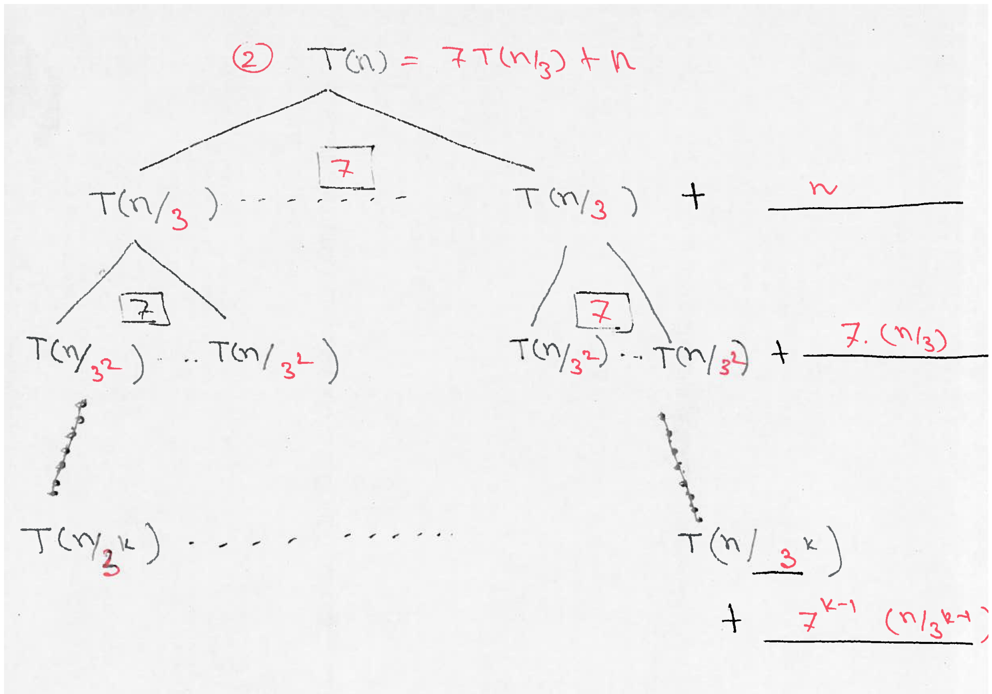


1. $T(n) = 5T(n/3) + 1$

$$T(n) = 5T(n/3) + 1 = 5^k T(n/3^k) + \sum_{i=0}^{k-1} 5^i = 5^k T(n/3^k) + \frac{5^k - 1}{4}$$

Hence, by setting $k = \log_3 n$, we have

$$T(n) = 5^k T(1) + \Theta(5^k) = \Theta(5^k) = \Theta(5^{\log_3 n}) = \Theta(n^{\log_3 5}) \approx \Theta(n^{1.465})$$

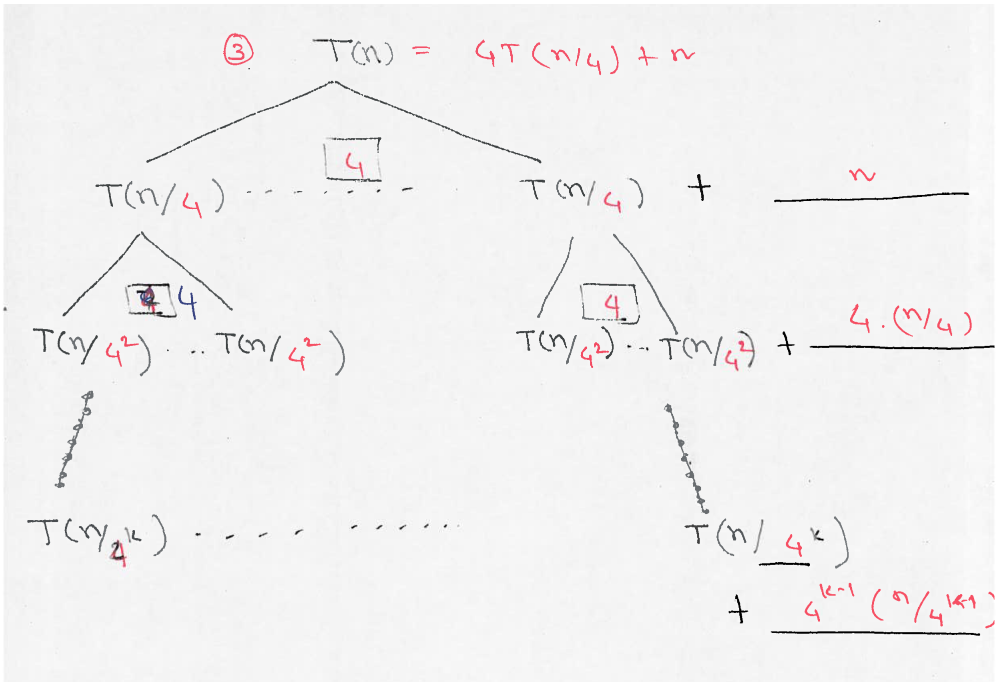


2. $T(n) = 7T(n/3) + n$

$$T(n) = 7T(n/3) + n = 7^k T(n/3^k) + \sum_{i=0}^{k-1} 7^i (n/3^i) = 7^k T(n/3^k) + n \sum_{i=0}^{k-1} (7/3)^i)$$
$$= 7^k T(n/3^k) + n \frac{(7/3)^k - 1}{(4/3)}$$

Hence, by setting $k = \log_3 n$, we have

$$T(n) = 7^k T(1) + \Theta(n \cdot (7/3)^k) = \Theta(7^k) + \Theta(7^k) = \Theta(7^{\log_3 n}) = \Theta(n^{\log_3 7}) \approx \Theta(n^{1.771})$$

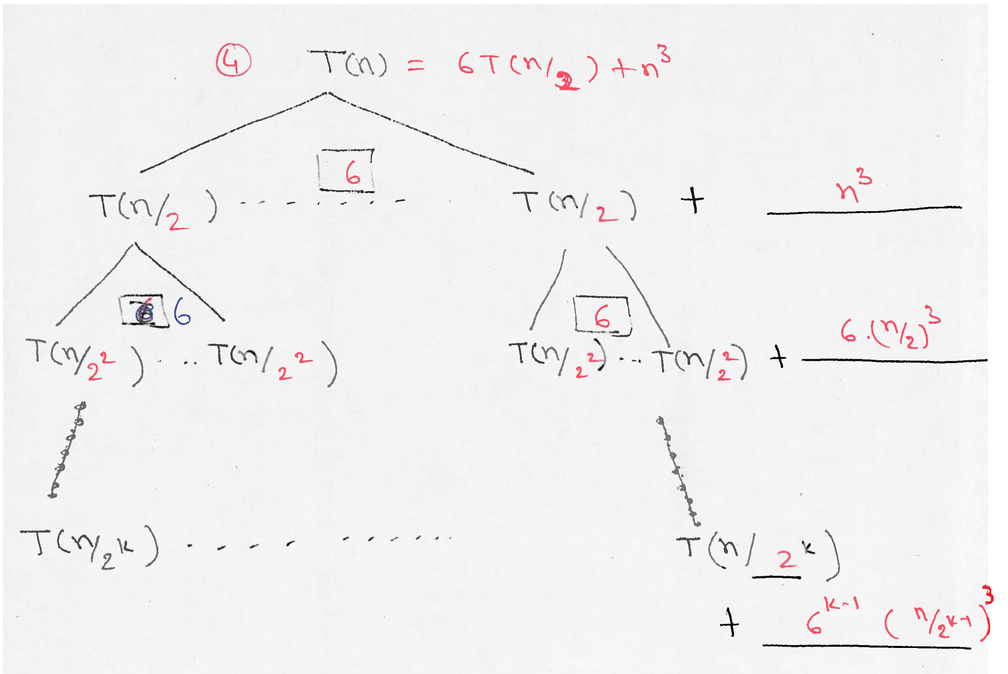


3. $T(n) = 4T(n/4) + n$

$$T(n) = 4T(n/4) + n = 4^k T(n/4^k) + \sum_{i=0}^{4-1} 4^i (n/4^i) = 4^k T(n/4^k) + n \sum_{i=0}^{k-1} (4/4)^i)$$
$$= 4^k T(n/4^k) + n \cdot k$$

Hence, by setting $k = \log_4 n$, we have

$$T(n) = 4^k T(1) + \Theta(n \cdot k) = nT(1) + n \cdot \log_4 n = \Theta(n \log n)$$



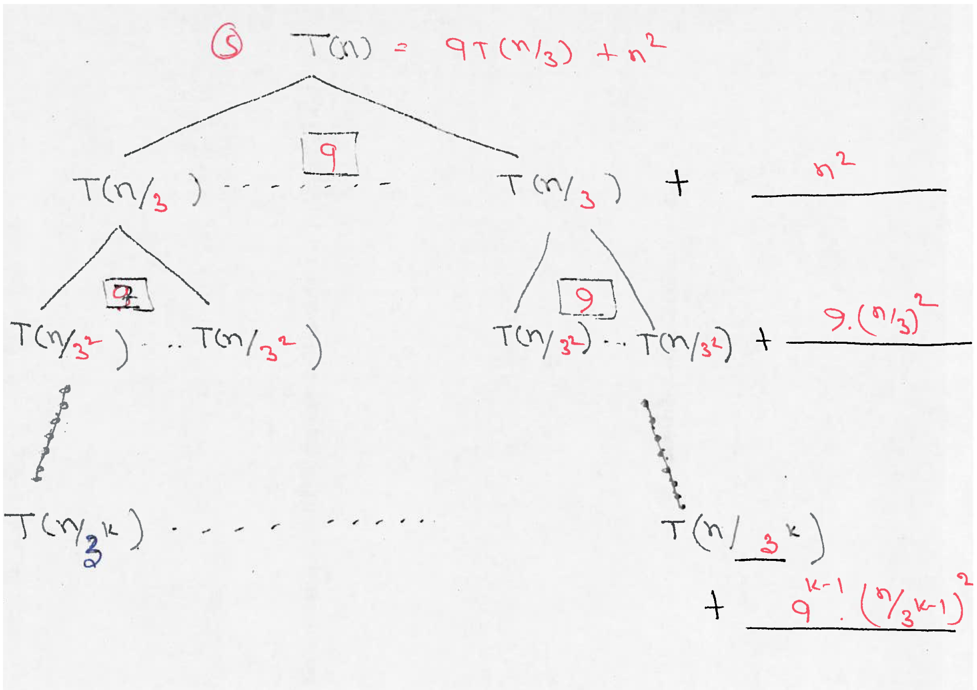
4. $T(n) = 6T(n/2) + n^3$

$$T(n) = 6T(n/2) + n^3 = 6^k T(n/2^k) + \sum_{i=0}^{k-1} 6^i (n/2^i)^3 = 6^k T(n/2^k) + n^3 \sum_{i=0}^{k-1} (6/8)^i)$$

$$= 6^k T(n/2^k) + n^3 \frac{1-(6/8)^k}{(2/8)}$$

Hence, by setting $k = \log_2 n$, we have

$$T(n) = 6^k T(1) + \Theta(n^3 \cdot (1-(6/8)^k)) = \Theta(6^k) + \Theta(n^3) = \Theta(n^{\log_2 6}) + \Theta(n^3) = \Theta(n^3)$$



5. $T(n) = 9T(n/3) + n^2$

$$T(n) = 9T(n/3) + n^2 = 9^k T(n/3^k) + \sum_{i=0}^{k-1} 9^i (n/3^i)^2 = 9^k T(n/3^k) + n^2 \sum_{i=0}^{k-1} (1)^i)$$
$$= 9^k T(n/3^k) + n^2 \cdot k$$

Hence, by setting $k = \log_3 n$, we have

$$T(n) = 9^k T(1) + \Theta(n^2 \cdot k) = \Theta(9^k) + n^2 \Theta(k)$$
$$= \Theta(n^{\log_3 9}) + \Theta(n^2 k) = \Theta(n^2) + \Theta(n^2 \log n) = \Theta(n^2 \log n)$$

$T(n) = T(2n/3) + n^2$

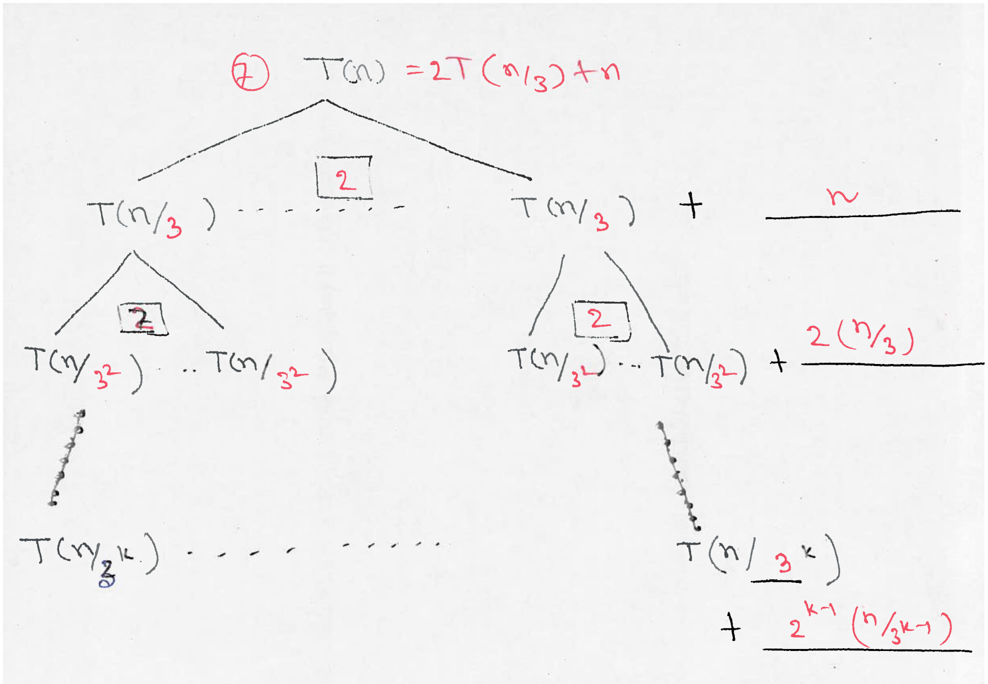
$$
\begin{array}{ll}
T(n) & \\
\Big| & \\
T(2n/3) & + n^2 \\
\Big| & \\
T\left(\frac{2^2 n}{3^2}\right) & + \left(\frac{2n}{3}\right)^2 \\
\Big| & \\
T\left(\frac{2^3 n}{3^3}\right) & + \left(\frac{2^2 n}{3^2}\right)^2 \\
\vdots & \\
T\left(\frac{2^k n}{3^k}\right) & + \left(\frac{2^{k-1} n}{3^{k-1}}\right)^2
\end{array}
$$

6. $T(n) = T(2n/3) + n^2$

$$T(n) = T(2n/3) + n^2 = T(2^k n/3^k) + \sum_{i=0}^{k-1} (2^i n/3^i)^2 = T(2^k n/3^k) + n^2 \sum_{i=0}^{k-1} (4/9)^i)$$
$$= T(2^k n/3^k) + n^2 \frac{1 - (4/9)^k}{(5/9)}$$

Hence, by setting $k = \log_{(3/2)} n$, we have

$$T(n) = T(1) + \Theta(n^2 \cdot (4/9)^k) = \Theta(1) + n^2 \Theta(1) = \Theta(n^2)$$



7. $T(n) = 2T(n/3) + n$

$$T(n) = 2T(n/3) + n = 2^k T(n/3^k) + \sum_{i=0}^{k-1} 2^i (n/3^i) = 2^k T(n/3^k) + n \sum_{i=0}^{k-1} (2/3)^i)$$
$$= 2^k T(n/3^k) + n \frac{(1-2/3)^k}{(1/3)}$$

Hence, by setting $k = \log_3 n$, we have

$$T(n) = 2^k T(1) + \Theta(n \cdot (2/3)^k) = \Theta(2^k) + \Theta(n) = \Theta(n^{\log_3 2}) + \Theta(n) = \Theta(n)$$