

# CSE 250 Homework Assignment 4 (Written)

Due in class on Wednesday, Oct 08

## 1 Objectives

- Take a small break. We'll get back to programming soon.
- Practice comparing growth rates of functions
- Practice solving recurrence relations using the recurrence tree method
- These skills are absolutely necessary, at the most basic level, to analyze algorithms and data structures.

## 2 Sample Problems and Solutions

**Sample Problem 1.** Order the following functions in *increasing* order of asymptotic growth rates.

$$\frac{n^2}{\log n}, \quad n\sqrt{n}, \quad (\log n)^3 n^{1.2}, \quad \frac{3^n}{n^5}, \quad n^2 2^n, \quad n^2 (\log n)^3, \quad n^3.$$

By “increasing order of growth rates” I meant: we write  $f_1(n) \ll f_2(n)$  when  $f_2(n)$  grows faster than  $f_1(n)$ . Please briefly explain your ordering. For example, if you say  $f_1(n) \ll f_2(n)$ , please briefly explain why. (There is no need to compute the limit  $\lim_{n \rightarrow \infty} \frac{f_2(n)}{f_1(n)}$ . You can use the rules shown on slide 12 of the following lecture:

<http://www.cse.buffalo.edu/~hungngo/classes/2014/Fall/250/lectures/growth-handout.pdf>

*Solution.* Here's a listing of all functions from the slowest to the fastest growth rates.

$$(\log n)^3 n^{1.2} \ll n\sqrt{n} \ll \frac{n^2}{\log n} \ll n^2 (\log n)^3 \ll n^3 \ll n^2 2^n \ll \frac{3^n}{n^5}.$$

Here's the justification (by rearranging the limits but not computing the limits):

- $(\log n)^3 n^{1.2} \ll n\sqrt{n}$  because

$$\lim_{n \rightarrow \infty} \frac{(\log n)^3 n^{1.2}}{n\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\log^3 n}{n^{0.3}} = 0.$$

The numerator is in the log-class, and the denominator is in the polynomial class. That's why we know the limit is 0 without computing it.

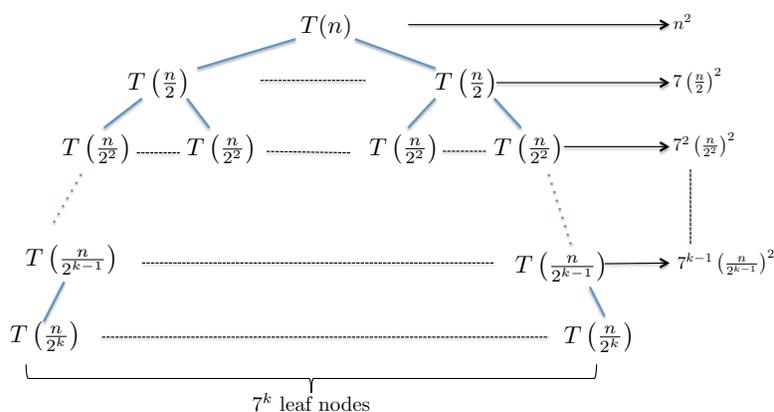
- The rest of the ordering can be justified similarly. (This is a sample problem, so I don't do it – but in your submissions you can't say “the rest is similar.”)

□

**Sample Problem 2.** Use the recurrence tree method to solve the following recurrence. You can assume that  $T(k) \leq C$  for some positive constant  $C$  and for any positive integer  $k \leq 10$ . In particular,  $T(1) = O(1)$ . You are only supposed to “guess” a solution by drawing the recurrence tree and summing up the contributions of all levels as I have done (or will do) several times in class, and as the TA covers in the recitations. You will need to show how you derive the summation and derive the final form  $T(n) = O(f(n))$  for some function  $f(n)$ .

1.  $T(n) = 7T(n/2) + n^2$
2.  $T(n) = 4T(n/3) + 1$
3.  $T(n) = 4T(n/3) + n$
4.  $T(n) = 3T(n/3) + n^2$
5.  $T(n) = 3T(n/3) + 1$
6.  $T(n) = 3T(n/3) + n$
7.  $T(n) = 2T(n/3) + n$

*Sample solution.* Here’s the recurrence tree for the first recurrence relation. The rest of the problems are to practice only.



From the tree, we conclude that, for any  $k$  for which  $n/2^k \geq 1$  we have

$$T(n) = 7^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 7^i \left(\frac{n}{2^i}\right)^2 = 7^k T\left(\frac{n}{2^k}\right) + n^2 \sum_{i=0}^{k-1} \left(\frac{7}{4}\right)^i = 7^k T\left(\frac{n}{2^k}\right) + n^2 \frac{(7/4)^k - 1}{7/4 - 1}.$$

We pick  $k$  so that  $n/2^k = 1$ , or  $k = \log_2 n$ . For that value of  $k$ ,

$$\begin{aligned} 7^k &= 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.81} \\ (7/4)^k &= (7/4)^{\log_2 n} = \frac{7^k}{n^2} \approx \frac{n^{2.81}}{n^2}. \end{aligned}$$

It follows that

$$T(n) = n^{2.81} T(1) + \frac{4}{3} n^2 \left( \frac{n^{2.81}}{n^2} - 1 \right) = n^{2.81} T(1) + \frac{4}{3} (n^{2.81} - n^2) = \Theta(n^{2.81}).$$

□

### 3 Problems to be solved

**Problem 1** (30 points). Order the following functions in *increasing* order of asymptotic growth rates.

$$\frac{2^n}{n^2}, (\log n)^3 n^{1.2}, n^3, \frac{n^4}{(\log n)^3}, n^3 (\log n)^{2.1}, \frac{(\sqrt{3})^{2n}}{n^4}, n^2 \sqrt{n}, n^4.$$

**Problem 2** (70 points). Use the recurrence tree method to solve the following recurrences.

1.  $T(n) = 5T(n/3) + 1$
2.  $T(n) = 7T(n/3) + n$
3.  $T(n) = 4T(n/4) + n$
4.  $T(n) = 6T(n/2) + n^3$
5.  $T(n) = 9T(n/3) + n^2$
6.  $T(n) = T(2n/3) + n^2$
7.  $T(n) = 2T(n/3) + n$