A dual frequency-selective bounded real lemma and its applications to IIR filter design*

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***Abstract*— Given a transfer function $H(s)$ of order n, the celebrated bounded real lemma characterises the untractable semi-infinite programming (SIP) condition $|H(\jmath\omega)|^2 \leq \gamma^2 \forall \omega \in R$ of function bounded realness (BR) by a tractable semi-definite programming (SDP). Some recent results generalise this result for the SIP condition $|H(\jmath\omega)|^2 \leq \gamma^2 \forall |\omega| \leq \bar{\omega}$ of frequency-selective bounded realness (FSBR). The SDP characterisations are given at the expense of an introduced Lyapunov matrix variable of dimension $n \times n$. As a result, the dimension of the resultant SDPs grows so quickly in respect to the function order, making them much less computationally tractable and practicable. Moreover, they do not allow to formulate synthesis problems as SDPs.**

In this paper, a completely new SDP characterizations for general FSBR for all-pole transfer functions is proposed. Our motivation is the design of infinite-impulse-response(IIR) filters involving a few of simutaneous FS-BRs. Our SDP characterizations are of moderate size and free from Lyapunov variables and thus allow to address problems involving transfer functions of arbitrary order. Examples are also provided to validate the effectiveness of the resulting SDP design formulation.

Finallly we also raise some issues arising with practicability of SDP for multi-dimensional filter design problems. In particular, any bilinear matrix inequality (BMI) optimization is shown to be solved by a SDP with any prescribed tolerance but the issue is dimensionality of this SDP.

I. Introduction

The positive real lemma and its variations such as Kalman-Popov-Yakubovich lemma, bounded real lemmas (see e.g. [1], [14]) are certainly among most important results of modern control. They allow to express a computationally untractable SIP constraint of a transfer function $H(s)$ by a computationally tractable SDPs for its state-space realization. For instance, the bounded real (BR) condition $|H(\jmath\omega)|^2 \leq \gamma^2\ \forall \omega \in R$ of $n-$order transfer function $H(s)$ is characterised by a SDP involving its state-space realization (A, B, C, D) and the Lyapunov matrix function variable of dimension $n \times n$ [1], [14]. As it can be seen, the variable dimension $n \times n$ of the Lyapunov variable, which is equivalent to $n(n+1)/2$ scalar variables, increases very quickly as the function order n increases moderately. As a consequence, the resultant SDPs are large dimensional and hardly solved by the presently available SDP solvers such as [16]. For instance, a function order $n = 100$ already requires the Lyapunov variable of dimension 100×100, which is equivalent to 5000 scalar variables. The generalised results (see e.g. [3], [7]) for the FSBR $|H(\jmath\omega)|^2 \leq \gamma^2\ \forall\ |\omega| \leq \bar{\omega}$ also experience a similar drawback. The SDP formulation of [7] also does not allow to formulate a synthesis problem as a SDP. More exactly, it leads to a bilinear matrix inequality (BMI) formulation for the problem. In [17], we have obtained a new SDP characterisation for the FRPR of discrete-time function $H(e^{\jmath\omega})$ of finite impulse response (FIR). Our SDP formulation is of substantially reduced order and its dual formulation does not involve any additional variables and thus open a new way for effective solution of large dimensional digital (discrete) systems. We have implemented in [17] computational examples involving the design of 1200-order system arising from digital filter design.

In this paper, we obtain a new FSBR lemma for all-pole functions. They are among the most popular classes in IIR analog filter design, which is the most fundamental problem of signal processing [12]. Like our previous results for discrete-time case, our new SDP characterization is moderate side and its dual formulation is free from slack variables and leads to a very robustly numerical algorithm.

The paper is organised as follows. A new optimisation-based design formulation for the class of all-pole filter is presented next in Section 2. FSBR lemmas and effi-

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cient SDP based solution to the proposed formulation is given in Section 3. An extension to multi-dimensional filters is provided in Section 4 and conclusions are drawn in Section 5.

Notations. For symmetric matrix X, $X \geq 0$ or $X \leq 0$ means X is semi-positive definite or semi-negative definite, respectively. Also, as usual ^{T} symbolizes the matrix transposition operator and if matrices Y and Z are of appropriate dimensions then $\langle Y, Z\rangle = \text{Trace}(YZ)$. The convex hull conv$(C)$ (conic hull cone(C), respectively) of a set $C \subset R^n$ is the smallest convex set (smallest convex cone, respectively) containing C. The polar cone C^* of C is defined as the set $\{x \in R^n : \langle x, c\rangle \geq 0,\ \forall c \in C\}$. We refer the reader to [15] for the background of convex analysis. Also $[n/2]$ is the largest integer not exceeding $n/2$ for an integer n.

II. Motivation: all-pole analog filter design

The reader is referred to [5] for a brief introduction for the state of art of analog filter design.

In its plain form, the problem is to design a filter that matches as smoothly as possible an ideal non-implementable filter of the frequency response 1 at a given passband and zero at a given stopband. Thus the ripple constraints on the passband and stopband, which are in fact particular FSBR, are introduced to attain the smooth filter performance and also high signal-to-noise (SNR) performance in its implementation at the noisy environment.

Thus, the following criteria are set for designing a normalized low pass filter of order n with transfer function $F(s)$:

(i) Least aggregate squared error over the pass band

$$\Sigma_E = \int_0^1 \left(|F(j\omega)|^2 - 1\right)^2 d\omega \to \min \tag{1}$$

(ii) Given a peak error $0 < \delta_P < 1$, FSBR constraint of magnitude error or ripple constraint over the pass band

$$-\delta_P \leq |F(j\omega)|^2 - 1 \leq \delta_P,\ \forall\omega \in [0,1]. \tag{2}$$

(iii) Given the stopband bound ω_S, and a peak error δ_S, FSBR constraints of magnitude error constraint or ripple constraint over the stop band

$$0 \leq |F(j\omega)|^2 \leq \delta_S,\ \forall\omega \geq \omega_S \tag{3}$$

For an all-pole analog filter of order n, its transfer function $F(s)$ is

$$|F(j\omega)|^2 = F(j\omega)F(-j\omega) = \frac{1}{1+P(\omega^2)} \tag{4}$$

with

$$P(\omega^2) = \sum_{i=0}^n p_i\omega^{2i},\ p_n > 0, \tag{5}$$

where $\mathbf{p} = (p_0,\ p_1,\ p_2,\ \cdots,\ p_n)^T \in R^n$ and of course, for any ω, the inequality $1 + P(\omega^2) > 0$ must hold.

The filter design problem for all-pole analog filter is now to find the vector $\mathbf{p}$ to minimise the objective (1) subject to constraints (2) and (3).

To begin with, the FSBR constraints (2) and (3) for the all-pole transfer function $F(j\omega)$ are equivalent to the following FSBR for the polynomial $P(\omega^2)$

$$\Delta_{PL} \leq P(\omega^2) \leq \Delta_{PU},\ \forall\omega \in [0,1] \tag{6}$$

$$\Delta_S \leq P(\omega^2),\ \forall\omega \geq \omega_S, \tag{7}$$

where

$$\Delta_{PL} = -\frac{\delta_P}{1+\delta_P},\ \Delta_{PU} = \frac{\delta_P}{1-\delta_P},\ \Delta_S = \frac{1-\delta_S}{\delta_S}. \tag{8}$$

In view of (6),

$$\frac{1}{(1+\Delta_{PU})^2}\int_0^1 P^2(\omega^2)d\omega \leq \int_0^1 \frac{P^2(\omega^2)}{(1+P(\omega^2))^2}d\omega \leq \frac{1}{(1+\Delta_{PL})^2}\int_0^1 P^2(\omega^2)d\omega, \tag{9}$$

where Δ_{PL} and Δ_{PU} are small as defined by (6). Hence, an adequate approximation of Σ_E in (1) is the convex quadratic function

$$\int_0^1 P^2(\omega^2)d\omega = \mathbf{p^T}Q\mathbf{p}, \tag{10}$$

with Q defined as

$$\int_0^1 \begin{bmatrix} 1 \\ \omega^2 \\ \vdots \\ \omega^{2n} \end{bmatrix} \begin{bmatrix} 1 \\ \omega^2 \\ \vdots \\ \omega^{2n} \end{bmatrix}^T d\omega = \begin{bmatrix} 1 & \frac{1}{3} & \cdots & \frac{1}{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2n+3} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \cdots & \frac{1}{4n+1} \end{bmatrix} > 0.$$

To sum up, our filter design problem is reduced to the following quadratic objective minimisation under FSBR constraints

$$\min_{\mathbf{p}\in R^{n+1}} \mathbf{p}^T Q\mathbf{p} : (6), (7). \tag{11}$$

In what follows we define the sets of FSBR constraints

$$\mathcal{P}(a,b) = \{\mathbf{p} \in R^{n+1} :\ \sum_{i=0}^n p_i t^i \geq 0,\ \forall t \in [a,b]\},$$
$$\mathcal{P}(a) = \{\mathbf{p} \in R^{n+1} :\ \sum_{i=0}^n p_i t^i \geq 0,\ \forall t \in [a,+\infty]\}. \tag{12}$$

Then the optimisation problem (11) is just

$$\min_{\mathbf{p}\in R^{n+1}} \mathbf{p}^T Q\mathbf{p} :\ \mathbf{p} - \Delta_{PL}e_1 \in \mathcal{P}(0,1), \tag{13}$$

$$\Delta_{PU}e_1 - \mathbf{p} \in \mathcal{P}(0,1),\ \ \mathbf{p} - \Delta_S e_1 \in \mathcal{P}(\omega_S^2), \tag{14}$$

where $e_1 = [\,1 \quad 0 \quad \cdots \quad 0\,]^T \in R^{n+1}$.

Note that the stable transfer function $F(s)$ is found from the optimal solution of the above problem (13)-(14) through spectral factorisation for the function [11]

$$F(s)F(-s) = \frac{1}{1+P(-s^2)} = \frac{1}{1+\sum_{i=0}^{n} p_i(-s^2)^i}. \tag{15}$$

III. FSBR LEMMA AND POLYNOMIAL CURVES

We have seen that FSBR constraints (2) and (3) for the all-pole transfer function $F(s)$ are in fact equivalent to FSBR ones (6) and (7) for the polynomial $P(\omega^2) = 1/|F(\jmath\omega)|^2 - 1$. As both (6) and (7) are particular cases of FSBRs (12), we now derive SDP characterisation for (12).

There are two interpretations for (12). The first most natural one is its appeared form (12) saying merely that the polynomial $\sum_{i=0}^{n} p_i t^i$ is positive on the segment $[a, b]$. As firstly shown in [10], it is equivalent to a SDP constraint through the Markov-Lukacs theorem on positive polynomials. However, as we will see in Subsection 3.1, the resulting SDP is of potentially high dimension. The second less visible one is that $\mathbf{p}$ belongs to the polar cone of the polynomial curve $\{(1, t, ..., t^n)^T :\ t \in [a, b]\}$. As it will be seen in the Subsection 3.2, it will lead to a SDP description of substantially reduced dimension and result in a very robust numerical algorithm. Let us describe them in order.

A. Markov-Lukacs theorem as a primal FSBR lemma

The $k-$th order moment matrix is $(k+1)\times(k+1)$-positive semi-definite and defined as [8]

$$\mathcal{M}_k(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^k \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^k \end{bmatrix}^T = \begin{bmatrix} 1 & t & \cdots & t^k \\ t & t^2 & \cdots & t^{k+1} \\ \vdots & \vdots & \cdots & \vdots \\ t^k & t^{k+1} & \cdots & t^{2k} \end{bmatrix}, \tag{16}$$

and accordingly, the matrix $\mathcal{M}_k(y)$ is created from $\mathcal{M}_k(t)$ by the variable change

$$t^h \leftarrow y_h,\ \ h = 0,\ 1,\ 2,\ \cdots, 2k \tag{17}$$

i.e. $\mathcal{M}_k(y)$ is

$$\mathcal{M}_k(y_0, y_1, ..., y_{2k}) = \begin{bmatrix} y_0 & y_1 & \cdots & y_k \\ y_1 & y_2 & \cdots & y_{k+1} \\ \vdots & \vdots & \cdots & \vdots \\ y_k & y_{k+1} & \cdots & y_{2k} \end{bmatrix}. \tag{18}$$

Define also

$$\mathcal{M}_{1k}(t) = t\mathcal{M}_k(t) \tag{19}$$

and accordingly

$$\mathcal{M}_{1k}(y) \equiv \mathcal{M}_{\ell k}(y_0, y_1, ..., y_{2k+1})$$

is created from $\mathcal{M}_{\ell k}(t)$ by the variable change (17) for $h = 0, 1, ..., 2k+1$.

For instance, it can be easily checked that

$$\mathcal{M}_{1(k-1)}(y) = [\,I_{k\times k} \quad 0_{k\times 1}\,]\,\mathcal{M}_k(y) \begin{bmatrix} 0_{1\times k} \\ I_{k\times k} \end{bmatrix} \in R^{k\times k}$$

The role of moment matrices is shortly clarified in the following theorem.

Markov-Lucacs theorem. *One has* $\mathbf{p} \in \mathcal{P}(a, b)$ *if and only if there are matrices* $X \geq 0$ *and* $Z \geq 0$ *of size* $([n/2]+1)\times([n/2]+1)$ *such that*

$$\sum_{i=0}^{n} p_i t^i = (t-a)\langle X, M_{[n/2]}(t)\rangle + (b-t)\langle Z, M_{[n/2]}(t)\rangle \tag{20}$$

Also, $\mathbf{p} \in \mathcal{P}(a)$ *if and only if*

$$\sum_{i=0}^{n} p_i t^i = (t-a)\langle X, M_{[n/2]}(t)\rangle + \langle Z, M_{[n/2]}(t)\rangle, \tag{21}$$

for some $X \geq 0,\ Z \geq 0$ *of size* $([n/2]+1)\times([n/2]+1)$.

The above theorem follows from the so called snake theorem [8] with a quite complex and long proof involving many mathematical constructs. Actually, the whole chapter 7 of [8] is devoted to its proof! A short and elementary proof for the above version of this famous theorem is provided in the appendix I.

Based on (20) and (21), **the primal FSBR lemma for the transfer function** $F(s)$ can be easily stated by obtaining the linear constraints in $\mathbf{p}$ and X, Z derived by comparison of terms with the same power t^i of t in both sides of (20) and (21). As indicated in the statement of the Markov-Lucacs Theorem, X and Z must also satisfy the SDP constraints $X \geq 0$ and $Z \geq 0$, so each constraint in (14) in $\mathbf{p}$ is in fact equivalently expressed by a LMI constraint in $\mathbf{p}$ and additional variables $X,\ Z$. As a result, for instance (13)-(14) is reduced to a SDP in $\mathbf{p}$ and 6 additional matrix variables of dimensions $([n/2]+1)\times([n/2]+1)$, i.e. the number of variables increases substantially in such a SDP formulation. In the next subsection, we use the polar cone interpretation for the constraint (12) to provide a novel technique, which allows us to solve (13)-(14) by a SDP with a much smaller number of variables.

B. The convex hull of polynomial curves and dual FSBR lemma

The dual FSBR lemma is based on the following LMI characterizations of the convex hull of (nonconvex) polynomial curve (in R^{n+1}).

Theorem 1: The conic hull of the polynomial curve $C_{a,b}$ defined as

$$C_{a,b} = \{[1 \quad t \quad \cdots \quad t^n]^T : t \in [a,b]\} \qquad (22)$$

is fully characterised by LMIs: $y \in \text{cone}(C_{a,b})$ if and only and it satisfies the LMIs

$$b\mathcal{M}_{[n/2]}(y) \geq \mathcal{M}_{1[n/2]}(y) \geq a\mathcal{M}_{[n/2]}(y) \qquad (23)$$

The conic hull of the polynomial curve C_a defined as

$$C_a = \{[1 \quad t \quad \cdots \quad t^n]^T : t \in [a,+\infty]\} \qquad (24)$$

is fully characterised by LMIs: $y \in \text{cone}(C_a)$ if and only if it satisfies the LMIs

$$\mathcal{M}_{[n/2]}(y) \geq 0, \ \mathcal{M}_{1[n/2]}(y) \geq a\mathcal{M}_k(y) \qquad (25)$$

Proof: See the appendix II. □

Remark. For n even, by the definition, $\mathcal{M}_{1[n/2]}$ is a matrix function of $(y_0, y_1, ..., y_n, y_{n+1})$ and accordingly LMIs (23), (25) are understood for some y_{n+1}.

As a further step, from the definition of $\mathcal{P}(a,b)$ and $\mathcal{P}(a)$, it is clear that

$$\mathcal{P}(a,b) = (C_{a,b})^* = (\text{conv}(C_{a,b}))^*$$
$$\mathcal{P}(a) = (C_a)^* = (\text{conv}(C_a))^*$$

so (13)-(14) can be written as

$$\min_{\mathbf{p}} \mathbf{p}^T Q\mathbf{p} \ : \ \ \mathbf{p} - \Delta_{PL}e_1 \in C^*_{0,1}, \ \Delta_{PU} - \mathbf{p} \in C^*_{0,1}, \ \mathbf{p} - \Delta_s e_1 \in C^*_{\omega_s^2} \qquad (26)$$

where for simplicity of notations we use $C^*_{0,1}$ and $C^*_{\omega_s^2}$ to refer $(\text{cone}C_{0,1})^*$ and $(\text{cone}C_{\omega_s^2})^*$, respectively.
The dual problem of (26) is the SDP problem

$$\max_{y^{(1)},y^{(2)},y^{(3)},\nu} \ \Delta_{PL}e_1^T y^{(1)} - \Delta_{PU}e_1^T y^{(2)} + \Delta_S e_1^T y^{(3)} - \nu : \qquad (27)$$
$$\begin{bmatrix} \nu & (-y^{(1)} + y^{(2)} - y^{(3)})^T \\ -y^{(1)} + y^{(2)} - y^{(3)} & 4Q \end{bmatrix} \geq 0,$$
$$(23) \text{ with } y^{(1)}, y^{(2)} \text{ for } 0 \to a, \ 1 \to b,$$
$$(25) \text{ with } y^{(3)} \text{ for } \omega_S^2 \to a,$$

Furthermore, the optimal solution $\mathbf{p}$ of (26) can be directly retrieved from the optimal solution $y^{(i)}$ of (27) by the unique solution of the linear equation system

$$Q\mathbf{p} = -\frac{1}{2}(-y^{(1)} + y^{(2)} - y^{(3)}). \qquad (28)$$

TABLE I
DESIGN SPECIFICATIONS AND AGGREGATE SQUARED ERROR PERFORMANCES OF THE FREQUENCY NORMALIZED FILTERS

Filters	Specification details			
	ω_S	δ_S	δ_P	Σ_E
4^{th} order filter	1.6	0.015	0.03	$\mathbf{1.09 \times 10^{-4}}$
5^{th} order filter	1.5	0.001	0.02	$\mathbf{4.75 \times 10^{-5}}$
6^{th} order filter	1.4	0.004	0.014	$\mathbf{6.53 \times 10^{-6}}$
7^{th} order filter	1.35	0.003	0.010	$\mathbf{6.09 \times 10^{-7}}$
8^{th} order filter	1.3	0.0022	0.008	$\mathbf{8.06 \times 10^{-7}}$
9^{th} order filter	1.25	0.0017	0.006	$\mathbf{1.06 \times 10^{-6}}$
10^{th} order filter	1.22	0.0012	0.005	$\mathbf{2.87 \times 10^{-7}}$

Let us emphasise the advantage of the analytical dual SDP formulation (27). Unlike the primal SDP (26), the dual SDP (27) involves variables of dimension $n+1$ and of moderate dimension so it can be effectively solved by any existing SDP software no matter how n can be large.
The below theorem is now at hand.

Theorem 2: Given the design specifications: stop frequency ω_S, pass band and stop band ripple constraints δ_P and δ_S, the optimal filter of order n in the sense of least aggregate squared error over the pass band Σ_E in (1) has the transfer function $F(s)$ resulting from spectral factorization for the function (15), where the coefficients $p_0, \ p_1, \ \cdots, \ p_n$ are derived from the optimal solution $y^{(i)}, i = 1, \ 2, \ 3$ of SDP (27) from the linear equations (28).

C. Numerical Illustration

We examine our design formulation via the designs of a number of analog filters using SDP (27). Design parameters are given in Table 1 where the resultant design objectives are written in boldface. Magnitude responses of designed 10^{th} order filters is depicted in Figures 1. By the table 2, we can see that reduction in the number of scalar variables achieved by (27) gets better fast as the order of the desired filter increases.

Table 3 (with the objective performances of filters stressed in boldface) gives performances of the proposed filters and the Chebysev ones. In the case of the 4^{th} order filter, gain of $0.004/0.030 = 13.3\%$ in ripple constraint made by the Chebyshev filter results in $0.002/0.015 = 13.3\%$ loss in aggregate error. For the 5^{th} and 6^{th} filters, that is 15% versus 36.5% and 36.4% versus 51.4%. Thus, in all the three cases our proposed formulation offers considerable aggregate error reduction over the pass band. The sacrifice of the error peak trades off well with the improvement of pass band aggregate error. This subsection together with the

TABLE II
NUMBER OF SCALAR VARIABLES OF DIFFERENT DESIGN FORMULATIONS

Filters	Number of scalar variables	
	(27)	(13)-(14)
5^{th} order filter	19	43
6^{th} order filter	22	56
7^{th} order filter	25	69
8^{th} order filter	28	85
9^{th} order filter	31	101
10^{th} order filter	34	120

Designed Filters	Specification details			
	ω_S	δ_S	δ_P	Σ_e
Prop. 4-order filter	1.6	0.03	0.034	**0.013**
Cheb. 4-order filter	1.6	0.03	**0.030**	0.015
Prop. 5-order filter	1.5	0.015	0.020	**0.0063**
Cheb. 5-order filter	1.5	0.015	**0.0174**	0.0086
Prop. 6-order filter	1.41	0.01	0.015	**0.0035**
Cheb. 6-order filter	1.41	0.01	**0.011**	0.0053

TABLE III
AGGREGATE ERROR OVER THE PASS BAND VERSUS PASS BAND RIPPLE PERFORMANCES

last one consolidate our motivation and validate of our design formulations.

IV. EXTENSIONS TO MULTI-DIMENSIONAL FILTER DESIGN: POTENTIAL DIFFICULTIES

As we can easily see, the key step for converting the optimization problem (11) to the SDP problem (27) is the exact LMI description (23) for the convex hull of the nonconvex set $C_{a,b}$. Particularly, it also implies that the following univariate polynomial optimization problem of nonconvex optimization

$$\min_{t\in[a,b]} \sum_{i=0}^{n} c_i t^i \qquad (29)$$

which is the same as

$$\min_{y\in \mathrm{conv} C_{a,b}} c^T y,\ c=(c_0,c_1,...,c_n)^T \qquad (30)$$

and thus according to Theorem 1 is solved by SDP

$$\min_{y\in R^{n+1}} c^T y:\ \ (23), y_0=1 \qquad (31)$$

This fact has been mentioned in [9] for n odd in a quite different setting.
On the other hand, it can be shown that the problem of multi-dimensional filter design will involve semi-infinite constraints like

$$\mathbf{p}\in R^N: \sum_{\alpha} p_\alpha \prod_i x_i^{\alpha_i} \geq 0\ \forall\ x_i\in[a_i,b_i],\ i=1,2,...,n. \qquad (32)$$

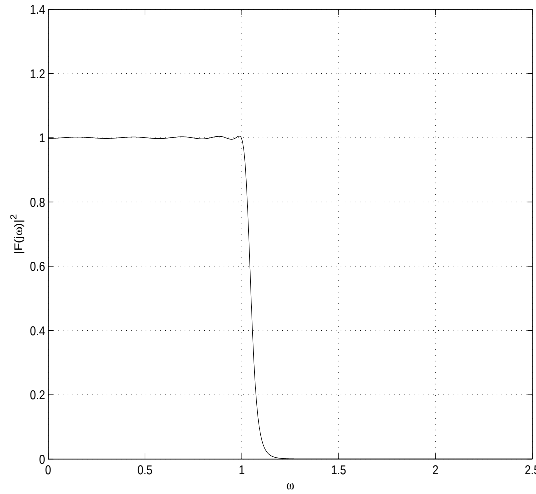

Fig. 1. Magnitude response of the designed 10^{th} order filter

Thus a natural question is whether we can describe the convex hull of the set

$$\mathcal{C}=\{[1\quad x_1\quad ...\quad x_n\quad x_1^2\quad x_1x_2\quad ...\quad x_n^2\quad ...\quad x_n^k]^T: g_\ell(x)\geq 0,\ \ell=1,2,...,L\}\subset R^N \qquad (33)$$

where g_ℓ are polynomials in $x=(x_1,x_2,...,x_n)^T\in R^n$:

$$g_\ell(x)=\sum_\alpha g_{\ell\alpha}x^\alpha,\ x^\alpha=\prod_i x_i^{\alpha_i},\ \sum_i \alpha_i<+\infty. \qquad (34)$$

In what follows we use the following variable changes

$$x^\alpha=\prod_i x_i^{\alpha_i}\leftrightarrow y_\alpha=y_{\alpha_1\alpha_2...\alpha_n}\quad \forall\alpha \qquad (35)$$

The n-dimensional moment matrices are defined as follows

$$\begin{aligned}\mathcal{M}_1(x) &= \begin{bmatrix}1\\ x_1\\ ...\\ x_n\end{bmatrix}\begin{bmatrix}1\\ x_1\\ ...\\ x_n\end{bmatrix}^T\\ &= \begin{bmatrix}1 & x_1 & x_2 & ... & x_n\\ x_1 & x_1^2 & x_1x_2 & ... & x_1x_n\\ x_2 & x_1x_2 & x_2^2 & ... & x_2x_n\\ ... & ... & .... & ... & ....\\ x_n & x_1x_n & x_2x_n & .... & x_n^2\end{bmatrix}\end{aligned} \qquad (36)$$

$$\mathcal{M}_2(x) = \begin{bmatrix} 1 \\ x_1 \\ \ldots \\ x_n \\ x_1^2 \\ x_1x_2 \\ \ldots \\ x_1x_n \\ x_2^2 \\ x_2x_3 \\ \ldots \\ x_n^2 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ \ldots \\ x_n \\ x_1^2 \\ x_1x_2 \\ \ldots \\ x_1x_n \\ x_2^2 \\ x_2x_3 \\ \ldots \\ x_n^2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & x_1 & .... & x_n & x_1^2 & \ldots & x_n^2 \\ x_1 & x_1^2 & .... & x_1x_n & x_1^3 & .... & x_1x_n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n^2 & x_1x_n^2 & \ldots & x_n^3 & x_1^2x_n^2 & \ldots & x_n^4 \end{bmatrix}$$

and so all. Clearly, all moment matrices $\mathcal{M}_i(x)$ are positive semi-definite.

Accordingly, one can define

$$\mathcal{M}_{ki}(x) = g_k(x)\mathcal{M}_i(x) \tag{37}$$

and $\mathcal{M}_{ki}(y)$ are created from $\mathcal{M}_{ki}(x)$ through the variable change (35).

Theorem 3: Suppose that the set $\mathcal{C}$ defined by (33) is compact.

Then the convex hull conv($\mathcal{C}$) of $\mathcal{C}$ can be analytically described by LMIs with any prescribed tolerance ϵ, i.e. one can show a convex set conv$_\epsilon\mathcal{C}$ defined analytically by LMIs and satisfying

$$\text{conv}(\mathcal{C}) \subset \text{conv}_\epsilon\mathcal{C} \subset \text{conv}(\mathcal{C}) + \epsilon O_N, \tag{38}$$

where O_N is the unit ball in R^N. One of such convex set conv$_\epsilon\mathcal{C}$ is described by

$$\text{conv}_\epsilon\mathcal{C} = \text{Proj}\{y:\ \mathcal{M}_{\bar{N}}(y) \geq 0,\ \mathcal{M}_{\ell(\bar{N}-1)}(y) \geq 0,\ \ell = 1,2,...,L\} \tag{39}$$

for some $\bar{N}$.

Proof: See the appendix III. □

An immediate consequence of the above theorem is the following result on solvability of BMIs, which frequently arise in control synthesis (see e.g. [2]), by SDPs.

Proposition 1: Suppose that the set

$$\mathcal{C} = \{\tilde{x} = \begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} : G_\ell(x) \geq 0,\ \ell = 1,2,...,L\} \subset R^n \tag{40}$$

is compact, where $G_\ell(x) \geq 0$ are BMIs in x, i.e. G_ℓ admits the form

$$G_\ell(x) = G_{\ell 0} + \sum_{i,j} x_i x_j G_{\ell ij}. \tag{41}$$

Then the convex hull conv($\mathcal{C}$) of $\mathcal{C}$ can be analytically described by SDPs with any prescribed tolerance ϵ, i.e. one can show a convex set conv$_\epsilon\mathcal{C}$ defined analytically by SDPs and satisfying (38). Consequently, any BMI optimization problem

$$\min_x \langle c, x\rangle :\ G_\ell(x) \geq 0,\ \ell = 1,2,...,L \tag{42}$$

can be solved by an SDP with any prescribed tolerance.

Proof: By the Sylvester's criterion, the matrix $G_\ell(x)$ is positive-semidefinite if and only if its determinant and principle minors are nonnegative. Obviously, these are polynomials in x and thus the constraint $G_\ell(x) \geq 0$ are equivalent to an analytically described polynomial constraints and the above proposition is a direct consequence of Theorem 3. □

However, in sharp contrast to the one-dimensional case, there is no closed form for predicting of the highest order $\bar{N}$ of the moment matrices. Actually, $\bar{N}$ is very potentially high. Even for moderate $\bar{N}$ (so we would have a relaxed problem) the resultant SDPs are already very high dimensional and unlikely solved by the existing SDP solvers. Some techniques handling this issue has been initialled in [6].

V. Conclusions

The paper has cast a new design of all-pole analog filter into a convex optimization problem, which is based on a new version of FSBR lemmas. Our proposed design is really practical and can be a more appropriate alternative to the classical filters in some actual analog filtering contexts. We have also developed a general framework for SDP applications to multi-dimensional problems and analysed some potential difficulties with their practicability.

Appendix I: An elementary proof of Markov-Lucacs theorem

For convenience, we deal with polynomials that are non-negative on $[0,\infty)$. One can turn $f(t)$ non-negative on $[a,b]$ into $\bar{f}(t)$ non-negative on $[0,\infty)$ with the so-called *Goursat transform*.

Definition 1: Given a polynomial $f(t)$ of degree n and an interval $[a,b]$, the Goursat transform of f on $[a,b]$ is defined as follows.

$$\bar{f}(t) = (1+t)^n f\left(\frac{b+at}{1+t}\right). \tag{43}$$

Lemma 1: (Variation of Goursat's Lemma) $f(t)$ is non-negative on $[a,b]$ if and only if $\bar{f}(t)$ is non-negative on $[0,\infty)$.

Proof. Note that $t \in [0,\infty)$ implies $\frac{b+at}{1+t} \in [a,b]$, hence the non-negativity of f on $[a,b]$ implies the non-negativity of $\bar{f}$ on $[0,\infty)$.

Conversely, suppose $\bar{f}$ is non-negative on $[0,\infty)$. For any $t \in (a,b]$ we can write

$$f(t) = f\left(\frac{b + a\frac{b-t}{t-a}}{1+\frac{b-t}{t-a}}\right) = \frac{\bar{f}\left(\frac{b-t}{t-a}\right)}{\left(1+\frac{b-t}{t-a}\right)^n} \geq 0$$

since $t \in (a,b]$ implies $\frac{b-t}{t-a} \in [0,\infty)$.
Lastly, $f(a) \geq 0$ by continuity. □

Lemma 2: Any $\bar{f}(t)$, which is nonnegative in $[0,\infty)$, can be written in the form

$$\bar{f}(t)(t) = p^2(t) + tq^2(t), \tag{44}$$

where $p(t)$ and $q(t)$ are polynomials such that the degree of each of $p^2(t)$ and $tq^2(t)$ is of the degree of $\bar{f}(t)$ at most.

Proof: Suppose $f_1(t) = p_1^2(t) + tq_1^2(t)$, and $f_2(t) = p_2^2(t) + tq_2^2(t)$, then

$$\begin{array}{rcl} f_1(t)f_2(t) & = & (p_1^2(t) + tq_1^2(t))(p_2^2(t) + tq_2^2(t)) \\ & = & (p_1(t)p_2(t) + tq_1(t)q_2(t))^2 \\ & & +t(q_1(t)p_2(t) - p_1(t)q_2(t))^2. \end{array}$$

Thus, if we can write $\bar{f}(t)$ as a product of non-trivial factors each of which has the desired form, then $\bar{f}$ itself has the desired form by induction.
Since $\bar{f}(t)$ is non-negative on $[0,\infty)$, every positive root of $\bar{f}$ must have even multiplicity. Hence, the factors corresponding to positive roots have the form $k^2(t)$ for some polynomial $k(t)$. Since $k^2(t) = k^2(t) + t \times 0^2$, these factors have the right format.
The factors corresponding to non-positive roots have also the right form: $(t + c) = \sqrt{c}^2 + t \times 1^2$. Also, the factors corresponding to complex roots in conjugate pairs:

$$(t-c-id)(t-c+id) = (-\sqrt{c^2+d^2}+t)^2 + t(\sqrt{2\sqrt{c^2+d^2}-2c})^2$$

have the desired form. □

We are now in position to prove the following version of the original Markov-Lukacs theorem [8].

Lemma 3: Let $f(t)$ be a polynomial of degree n which is non-negative on $[a,b]$. Then, $f(t)$ admits the following representation:

$$f(t) = (t-a)(p_1^2(t) + p_2^2(t)) + (b-t)(q_1^2(t) + q_2^2(t)) \tag{45}$$

where $p_i(t)$ and $q_i(t)$ are polynomials, and each of the four terms $(t-a)p_i^2(t), (b-t)q_i^2(t)$ has degree n at most.

Proof We have

$$\begin{array}{rcl} f(t) & = & \frac{(t-a)^n}{(b-a)^n}\bar{f}\left(\frac{b-t}{t-a}\right) \\ & = & \frac{(t-a)^n}{(b-a)^n}\left[p^2\left(\frac{b-t}{t-a}\right) + \frac{b-t}{t-a}q^2\left(\frac{b-t}{t-a}\right)\right] \end{array}$$

which leads to the format

$$f(t) = \begin{cases} p^2(t) + (b-t)(t-a)q^2(t) & n \text{ is even} \\ (t-a)p^2(t) + (b-t)q^2(t) & n \text{ is odd,} \end{cases} \tag{46}$$

Clearly, for n odd, the format (46) is a particular case of (46).
But for n even, like [4, proof of Lemma 1] it can be immediately checked that (46) gives

$$\begin{array}{rcl} (b-a)f(t) & = & (t-a)f(t) + (b-t)f(t) \\ & = & (t-a)(p^2(t) + (b-t)^2q^2(t)) \\ & & +(b-t)(p^2(t) + (t-a)^2q^2(t)) \end{array} \quad \square$$

i.e. the format (45) again.

Proof of Markov Lukacs theorem: Clearly, (45) is a particular case of (20). On the other hand, the RHS of (20) is a polynomial in t, which is nonnegative on $[a,b]$ so it must admit the representations like (45). Therefore, (20) is equivalent to (45).

APPENDIX II: THE PROOF OF THEOREM 1

First, using the variable change (17) at both sides of (20), (21) leads to the next lemma.

Lemma 4: If $P(t) = \sum_{i=0}^{n} p_i t^i$ admits the representation (20) for some matrices $X \geq 0, Z \geq 0$ then for every $y = [\,y_0 \quad y_1 \quad \cdots \quad y_n\,]^T \in R^{n+1}$,

$$\begin{array}{rcl} \mathbf{p}^T y & = & \langle X, \mathcal{M}_{1[n/2]}(y) - a\mathcal{M}_{[n/2]}(y)\rangle \\ & & +\langle Z, b\mathcal{M}_{[n/2]}(y) - \mathcal{M}_{1[n/2]}(y)\rangle. \end{array} \tag{47}$$

If $P(t) = \sum_{i=0}^{n} p_i t^i$ admits the representation (20) for some matrices $X \geq 0, Z \geq 0$ then for every $y = [\,y_0 \quad y_1 \quad \cdots \quad y_n\,]^T \in R^{n+1}$,

$$\mathbf{p}^T y = \langle X, \mathcal{M}_{1[n/2]}(y) - a\mathcal{M}_{[n/2]}(y)\rangle + \langle Z, \mathcal{M}_{[n/2]}(y)\rangle. \tag{48}$$

Proof of Theorem 1. For the first part, the conic hull of $C_{a,b}$ is obviously characterised by LMIs (23) if we can show that the convex hull of $C_{a,b}$ is characterized by LMIs (23) and $y_0 = 1$.
In what follows, set $k = [n/2]$.
Suppose conv$C_{a,b}$ is the set of y with $y_0 = 1$ satisfying LMIs (23), which is obviously convex. Then $C_{a,b} \subset$ conv$C_{a,b}$ because each $y = [\,1 \quad t \quad \cdots \quad t^n\,]^T \in C_{a,b}$ satisfies LMIs (23):

$$\begin{array}{c} \mathcal{M}_k(y) = \mathcal{M}_k(t) \geq 0, \\ b\mathcal{M}_k(y) - \mathcal{M}_{1k}(y) = (b-t)\mathcal{M}_k(t) \geq 0, \\ \mathcal{M}_{1k}(y) - a\mathcal{M}_k(y = (t-a)\mathcal{M}_k(t) \geq 0. \end{array}$$

From the definition of the convex hull,

$$\text{conv}(C_{a,b}) \subset \text{conv}C_{a,b}$$

follows.
It remains to show

$$\text{conv}C_{a,b} \subset \text{conv}(C_{a,b}).$$

For that, let the support function [15] for any $A \subset R^{n+1}$ be defined as

$$s(A, \mathbf{p}) = \sup_{y \in A}\langle y, \mathbf{p}\rangle \ \forall \mathbf{p} \in R^{n+1}.$$

So

$$s(C_{a,b}, \mathbf{p}) - \sum_{i=0}^{n} p_i t^i \geq 0 \ \forall \ t \in [a,b]$$

and according to Markov Lukacs theorem for each $\mathbf{p}$ there are $X_{\mathbf{p}} \geq 0$ and $Z_{\mathbf{p}} \geq 0$ such that

$$s(C_{a,b}, \mathbf{p}) - \sum_{i=0}^{n} p_i t^i = (t-a)\langle X_{\mathbf{p}}, M_k(t)\rangle + (b-t)\langle Z_{\mathbf{p}}, M_k(t)\rangle.$$

By Lemma 4, it is then true that whenever $y = (1, y_1, ..., y_n)^T \in$ conv$C_{a,b}$ (i.e. y satisfies LMIs (23))

$$\begin{array}{rcl} s(C_{a,b}, \mathbf{p}) - \sum_{i=0}^{n} y_i p_i & = & \langle X_{\mathbf{p}}, \mathcal{M}_{1k}(y) - a\mathcal{M}_k(y)\rangle \\ & & +\langle Z_{\mathbf{p}}, b\mathcal{M}_k(y) - \mathcal{M}_{1k}(y)\rangle \\ & \geq & 0 \end{array}$$

implying $s(C_{a,b}, \mathbf{p}) \geq s(\text{conv}C_{a,b}, \mathbf{p}) \ \forall \ \mathbf{p} \in R^{n+1}$ or equivalently conv$C_{a,b} \subset$ conv$(C_{a,b})$ [15].
The proof for the second part is similar.

Remark. As mentioned in the Remark after Theorem 1, when n is even, $\mathcal{M}_{1[n/2]}(y)$ depends on $(y_0, y_1, ..., y_{n+1})$ and the representation (20) implies (47) for all $(y_0, y_1, ..., y_{n+1})$.

APPENDIX III: THE PROOF FOR THEOREM 3

For any $+\infty > |C_k| := \max_{\tilde{x}\in C_k} ||\tilde{x}||$, it is obvious that $C_k \subset |C_k|.O_N$.
Take $\mu := \epsilon/4|C_k|$ and let $\{p^{(i)}:\ p^{(i)} \in R^N,\ ||p^{(i)}|| = 1,\ i = 1, 2, ..., \bar{N}\}$ be any μ-net of the unit ball in R^N, i.e. for any $||p|| = 1$ there is p^i such that

$$||p - p^{(i)}|| \leq \mu. \tag{49}$$

It follows from [13] that there are $U^{(i)} \geq 0$ and $V^{(i\ell)} \geq 0$ such that

$$s(C_k, p^{(i)}) - \langle p^{(i)}, \tilde{x}\rangle + \epsilon/2 = \langle U^{(i)}, M_{\bar{N}}(x)\rangle + \sum_{\ell=1}^{L}\langle V^{(i\ell)}, M_{\ell(\bar{N}-1)}(x)\rangle, \tag{50}$$

where $M_{\ell(\bar{N}-1)}(x) = g_\ell(x)M_{\bar{N}-1}$ for some $\bar{N} < +\infty$.
Therefore, for $\overline{\langle p^{(i)}, y\rangle}$ defined from $\langle p^{(i)}, \tilde{x}\rangle$ by the variable change (35) one has

$$s(C_k, p^{(i)}) - \overline{\langle p^{(i)}, y\rangle} + \epsilon/2 = \langle U^{(i)}, M_{\bar{N}}(y)\rangle + \sum_{\ell=1}^{L}\langle V^{(i\ell)}, M_{\ell(\bar{N}-1)}(y)\rangle.$$

Now, for

$$\text{conv}_\epsilon C_k := \text{Proj}_{R^N}\{y:\ M_{\bar{N}}(y) \geq 0,\ M_{\ell(\bar{N}-1)}(y) \geq 0\} \cap |C_k|.O_N \tag{51}$$

one has

$$s(C_k, p^{(i)}) - \langle p^{(i)}, \bar{y}\rangle + \epsilon/2 = s(C_k, p^{(i)}) - \overline{\langle p^{(i)}, y\rangle} + \epsilon/2 \geq 0 \quad \forall\ \bar{y} \in \text{conv}_\epsilon C_k. \tag{52}$$

Then for each $p \in O_N$ there is $p^{(i)} \in O_N$ satisfying (49), so whenever $\bar{y} \in \text{conv}_\epsilon C_k$,

$$\begin{aligned} s(C_k + \epsilon O_N, p) - \langle p, \bar{y}\rangle &= \\ s(C_k, p) - \langle p, \bar{y}\rangle + \epsilon &= \\ (s(C_k, p) - s(C_k, p^{(i)})) + (s(C_k, p^{(i)}) - \langle p^{(i)}, \bar{y}\rangle) & \\ +(\langle p^{(i)}, \bar{y}\rangle - \langle p, \bar{y}\rangle) + \epsilon &\geq \\ -2\mu|C_k| + c(C_k, p^{(i)}) - \langle p^{(i)}, \bar{y}\rangle + \epsilon &= \\ c(C_k, p^{(i)}) - \langle p^{(i)}, \bar{y}\rangle + \epsilon/2 &\geq 0 \end{aligned}$$

implying $\text{conv}_\epsilon C_k \subset \text{conv}(C_k) + \epsilon.O_N$, showing (38).

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