On a Hyperplane Arrangement Problem and Tighter Analysis of an Error-Tolerant Pooling Design

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Abstract

In this paper, we formulate and investigate the following problem: given integers d, k and r where $k > r \geq 1, d \geq 2$, and a prime power q, arrange d hyperplanes on $\mathbb{F}_q^k$ to maximize the number of r-dimensional subspaces of $\mathbb{F}_q^k$ each of which belongs to at least one of the hyperplanes. The problem is motivated by the need to give tighter bounds for an error-tolerant pooling design based on finite vector spaces.

1   Introduction

Designing good error-tolerant pooling design is a central problem in the area of non-adaptive group testing [9], which has many practical applications including DNA library screening [8, 10, 21], multiple access control [5–7, 17, 26], and error correcting/detecting superimposed codes [11–15], to name a few.

To date, there are relatively few papers addressing the problem of designing and analyzing error-tolerant pooling designs [1,3,4,16,19,20,22,24]. In [22], Ngo and Du introduced a non-adaptive pooling design based on finite vector spaces, which was later found to be highly error-tolerant by D'yachkov et a. [10]. The analysis of the design in [10] was not very tight. In this paper, we give a tighter analysis of the design. This is done via formulating a new and very interesting hyperplane arrangement problem on finite fields.

To formally describe our problem, we first need a few definitions. A 01-matrix M is said to be d-disjunct if and only if no column is contained in the union of d others. (Here, columns are viewed as characteristic vectors of sets of rows.) A d-disjunct matrix corresponds precisely to a pooling design which can identify at most d negative items. For the design to tolerate a few errors in outcomes, it is not sufficient for a column to just not be covered by d others. A d^z-disjunct matrix is a matrix where, given any $d + 1$ columns $C_0, C_1, \ldots, C_d$, the set $C_0 \setminus C_1 \cup \cdots \cup C_d$ has at least z elements. It is easy to show that a d^z-disjunct matrix can detect $z - 1$ errors and correct $\lfloor (z - 1)/2 \rfloor$ errors.

The construction in [22] is as follows. Let q be a prime power and m, k, r be integers such that $m > k > r \geq 1$. Let $M_q(m, k, r)$ be the 01-matrix whose rows are indexed by r-dimensional subspaces of $\mathbb{F}_q^m$ and whose columns are indexed by k-dimensional subspaces of $\mathbb{F}_q^m$. $M_q(m, k, r)$ has a 1 in row R and column C if and only if R is a subspace of C. It is easy to see that $M_q(m, k, d)$ is d-disjunct (the containment method by Macula [18]). Later, D'yachkov et al. [10] realized that we do not have to take $r = d$ for $M_q(m, k, r)$ to be d-disjunct (r could be a lot smaller than d, even $r = 1$ works sometimes). Moreover, the construction can, in general, tolerate a lot of errors. Specifically, their main result was that,

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for any $d + 1$ k-dimensional subspaces $C_0, C_1, \ldots, C_d$ of $\mathbb{F}_q^m$, the number of r-dimensional subspaces each of which belongs to C_0 but not other C_i is at least

$$\begin{bmatrix} k \\ r \end{bmatrix}_q - d \begin{bmatrix} k-1 \\ r \end{bmatrix}_q + (d-1) \begin{bmatrix} k-2 \\ r \end{bmatrix}_q,$$

and that the bound is tight for $d \leq q + 1$. Here, for any non-negative integers m, n, $\begin{bmatrix} n \\ m \end{bmatrix}_q$ denotes the Gaussian coefficient, to be defined in the next section.

The number of columns of $M_q(m, k, r)$ is $\begin{bmatrix} m \\ k \end{bmatrix}_q$, exponentially larger than $q + 1$. Hence, it is desirable to devise tight bounds for the case when $q + 1 < d \leq \begin{bmatrix} m \\ k \end{bmatrix}_q - 1$. In this paper, we partially address this problem. In the process, we formulate a new – to the best of the author's knowledge – hyperplane arrangement problem on finite fields.

The rest of this paper is organized as follows. Section 2 motivates the hyperplane arrangement problem and presents preliminary results on the problem. Section 3 gives tighter bounds for the original group testing problem using results from Section 2. Section 4 concludes the paper with additional remarks and a conjecture.

2 An extremal hyperplane arrangement problem on finite fields

2.1 Motivation and notations

Henceforth, we shall use $(a; q)_n$ (or $(a)_n$ for short) to denote the *q-shifted factorial*:

$$(a)_n = (a; q)_n := (1 - a)(1 - aq) \ldots (1 - aq^{n-1}).$$

The q-analogue of a natural number n is denoted by $[n]_q$, and the Gaussian coefficient is denoted by $\begin{bmatrix} n \\ m \end{bmatrix}_q$. They are defined as follows.

$$
\begin{aligned}
[0]_q &:= 0 \\
[n]_q &:= 1 + q + \cdots + q^{n-1}, \ n \geq 1 \\
\begin{bmatrix} n \\ m \end{bmatrix}_q &:= \begin{cases} 0 & \text{when } n < m \\ \frac{(q)_n}{(q)_{n-m}(q)_m} = \frac{(1-q^n)\ldots(1-q^{n-m+1})}{(1-q^m)\ldots(1-q)} & \text{otherwise.} \end{cases}
\end{aligned}
$$

We shall drop the subscript q and write $[n]$ and $\begin{bmatrix} n \\ m \end{bmatrix}$ when there is no potential confusion as to what q is. Our notations are standard in the q-series literature [2].

For any vector space X, let $\overline{X}$ denote the set of all r-dimensional subspaces of X, and $\dim(X)$ the dimension of X. Then, it is well known (see, e.g. [25]) that

$$|\overline{X}| = \begin{bmatrix} \dim(X) \\ r \end{bmatrix}.$$

For any vector spaces X and Y,

$$\overline{X} \cap \overline{Y} = \overline{X \cap Y} \tag{1}$$

because any vector space which is a subspace of X and a subspace of Y is also a subspace of the vector space $X \cap Y$. Note that, in general $X \cup Y$ is not a vector space, and

$$\overline{X} \cup \overline{Y} \neq \overline{\text{span}(X \cup Y)}.$$

The matrix $M_q(m, k, r)$ is d^z-disjunct for

$$z = \min\{|\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d}| \ : \ C_0, C_1, \ldots C_d \text{ are } d + 1 \text{ different } k\text{-dimensional subspaces of } \mathbb{F}_q^m\}.$$

Thus, we want to find k-dimensional subspaces $C_0, C_1, \ldots, C_d$ of $\mathbb{F}_q^m$ that minimizes the quantity

$$|\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d}|.$$

For any $i \in \{1, \ldots, d\}$, let $H_i = C_i \cap C_0$, then

$$
\begin{aligned}
|\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d}| &= |\overline{C_0} \setminus (\overline{C_1} \cap \overline{C_0}) \cup \cdots \cup (\overline{C_d} \cap \overline{C_0})| \\
&= |\overline{C_0} \setminus (\overline{C_1 \cap C_0}) \cup \cdots \cup (\overline{C_d \cap C_0})| \\
&= |\overline{C_0} \setminus \overline{H_1} \cup \cdots \cup \overline{H_d}|.
\end{aligned}
$$

For $|\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d}|$ to be minimized, we can assume that all H_i are hyperplanes of C_0. The number of hyperplanes of C_0 is $\begin{bmatrix} k \\ k-1 \end{bmatrix} = [k]$. Thus, when $d \leq [k]$ we can also assume that the H_i are *different* hyperplanes of C_0; because, given d hyperplanes $H_1, \ldots, H_d$, we can take the span of each of them with a vector $v \notin C_0$ to reconstruct the C_i. For the group testing problem, we will address the case when $d > [k]$ in a later section. In this section, we only consider the case when $d \leq [k]$. Because $|\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d}|$ is minimized when $|\overline{H_1} \cup \cdots \cup \overline{H_d}|$ is maximized, the above discussion motivates the following problem.

Problem 1 (Our Hyperplane Arrangement Problem). Given a k-dimensional vector space C over $\mathbb{F}_q$, and an integer d such that $1 \leq d \leq [k]$, find d hyperplanes $H_1, \ldots, H_d$ of C that *maximizes* the following quantity

$$|\overline{H_1} \cup \cdots \cup \overline{H_d}|.$$

At least, find good upper bounds for the quantity.

The result in [10] can be restated as follows

Theorem 2.1 (D'yachkov et al.). *Given integers $r \leq k - 2$, and $d \leq [k]$. Let $H_1, \ldots, H_d$ be d different hyperplanes of a k-dimensional vector space over $\mathbb{F}_q$. Then,*

$$|\overline{H_1} \cup \cdots \cup \overline{H_d}| \leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (d-1) \begin{bmatrix} k-2 \\ r \end{bmatrix} \tag{2}$$

The bound is tight when $d \leq q + 1$.

2.2 Initial observations

By inclusion-exclusion, we have

$$
\begin{aligned}
|\overline{H_1} \cup \cdots \cup \overline{H_d}| &= \sum_{i=1}^{d} (-1)^{i-1} \sum_{\substack{T \subseteq \{1, \ldots, d\} \\ |T| = i}} \bigcap_{t \in T} \overline{H_t} \\
&= \sum_{i=1}^{d} (-1)^{i-1} \sum_{\substack{T \subseteq \{1, \ldots, d\} \\ |T| = i}} \overline{\bigcap_{t \in T} H_t} \\
&= \sum_{i=1}^{d} (-1)^{i-1} \sum_{\substack{T \subseteq \{1, \ldots, d\} \\ |T| = i}} \begin{bmatrix} \dim \left(\bigcap_{t \in T} H_t \right) \\ r \end{bmatrix} \tag{3}
\end{aligned}
$$

As we will see later, it is not easy to determine the dimension of the intersection of a given number of arbitrary hyperplanes. That is why inclusion-exclusion does not help us directly solve the problem. Next, for any two vector spaces X and Y,

$$\dim(X) + \dim(Y) = \dim(\text{span}(X \cup Y)) + \dim(X \cap Y).$$

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In particular, if X is a hyperplane and Y is a proper subspace of a k-dimensional vector space, then either $Y \subseteq X$ or $\dim(Y) = \dim(X \cap Y) + 1$. To see this, suppose Y has dimension $l \leq k - 1$. If $Y \not\subseteq X$, then $\dim(\text{span}(X \cup Y)) = k$, which implies

$$\dim(Y) = k + \dim(X \cap Y) - \dim(X) = \dim(X \cap Y) + 1.$$

In words, a hyperplane either contains Y or "cut into" Y at one dimension lower than that of Y. This observation leads to the following simple yet important lemma.

Lemma 2.2. *Let $H_1, \ldots, H_x$ be some x hyperplanes of an l-dimensional vector space over $\mathbb{F}_q$ whose intersection is $I = H_1 \cap \cdots \cap H_x$. Let H be any hyperplane not containing I, and set $Y_i = H \cap H_i$, $i \in \{1, \ldots, x\}$. Then, for any subset $S \subseteq \{1, \ldots, x\}$, we have*

$$\dim \left(\bigcap_{i \in S} H_i \right) = \dim \left(\bigcap_{i \in S} Y_i \right) + 1 \tag{4}$$

Proof. Because H does not contain I, H does not contain $\bigcap_{i \in S} H_i$ for any S. Thus,

$$\dim \left(\bigcap_{i \in S} H_i \right) = \dim \left(H \cap \bigcap_{i \in S} H_i \right) + 1 = \dim \left(\bigcap_{i \in S} (H_i \cap H) \right) + 1 = \dim \left(\bigcap_{i \in S} Y_i \right) + 1.$$

$\square$

The Y_i actually are hyperplanes of H. What this lemma tells us is that, the inter-relationship (in terms of dimensions of intersections) between the hyperplanes $H_1, \ldots, H_x$ is the same as the inter-relationship between the hyperplanes $Y_1, \ldots, Y_x$ of H. The hyperplanes $Y_1, \ldots, Y_x$ form a down-scaled picture of $H_1, \ldots, H_x$ inside H.

Consider an i-dimensional subspace X of a k-dimensional vector space S over $\mathbb{F}_q$. Let l be an integer where $i \leq l \leq k$. Then, the number of l-dimensional subspaces of S containing X is $\begin{bmatrix} k-i \\ l-i \end{bmatrix}$. In particular, when $i = k - 2$ the number of hyperplanes that contains X is

$$\begin{bmatrix} k - (k-2) \\ k - 1 - (k-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = q + 1. \tag{5}$$

Lastly, the following identity is the q-analog of the Pascal's triangle identity for binomial coefficients [25]:

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q. \tag{6}$$

2.3 The cases of 4 and 5 hyperplanes

Using the basic observations in the previous section, when there are a constant number of hyperplanes it is possible to enumerate all possible classes of arrangements (with respect to our objective function). In this section, we will compute the objective function for all arrangements of 4 and 5 hyperplanes. These arrangements will serve as the base case to prove generic bounds in the next section.

We will be working on an l-dimensional vector space S over $\mathbb{F}_q$, namely S is isomorphic to $\mathbb{F}_q^l$. For any set of (at least two) hyperplanes $\mathcal{H}$, let $x(\mathcal{H})$ be the maximum number of hyperplanes in $\mathcal{H}$ whose intersection has dimension $l - 2$. Note that $2 \leq x(\mathcal{H}) \leq q + 1$. Also define

$$g(\mathcal{H}) = \left| \bigcup_{H \in \mathcal{H}} \overline{H} \right|.$$

We first consider the 4-hyperplane case.

Lemma 2.3. *Let* $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ *be a set of* 4 *hyperplanes of* $\mathbb{F}_q^l$.

(i) *If* $x(\mathcal{H}) = 4$, *then*

$$g(\mathcal{H}) = g_1^{(4)} := 4\begin{bmatrix} l-1 \\ r \end{bmatrix} - 3\begin{bmatrix} l-2 \\ r \end{bmatrix}. \tag{7}$$

This case can only hold when $q \geq 3$.

(ii) *If* $x(\mathcal{H}) = 3$, *then*

$$g(\mathcal{H}) = g_2^{(4)} := 4\begin{bmatrix} l-1 \\ r \end{bmatrix} - 5\begin{bmatrix} l-2 \\ r \end{bmatrix} + 2\begin{bmatrix} l-3 \\ r \end{bmatrix}. \tag{8}$$

(iii) *If* $x(\mathcal{H}) = 2$, *then there are two cases:*

$$g(\mathcal{H}) = g_3^{(4)} \quad := \quad 4\begin{bmatrix} l-1 \\ r \end{bmatrix} - 6\begin{bmatrix} l-2 \\ r \end{bmatrix} + 4\begin{bmatrix} l-3 \\ r \end{bmatrix} - \begin{bmatrix} l-4 \\ r \end{bmatrix} \tag{9}$$

$$g(\mathcal{H}) = g_4^{(4)} \quad := \quad 4\begin{bmatrix} l-1 \\ r \end{bmatrix} - 6\begin{bmatrix} l-2 \\ r \end{bmatrix} + 3\begin{bmatrix} l-3 \\ r \end{bmatrix}. \tag{10}$$

Moreover, $g_1^{(4)} \geq g_2^{(4)} \geq g_3^{(4)} \geq g_4^{(4)}$.

Proof. Cases (i) and (ii) follow straightforwardly from the inclusion-exclusion formula (3) and Lemma 2.2. Suppose $x(\mathcal{H}) = 2$, then $W = H_1 \cap H_2 \cap H_3$ has dimension $l - 3$. If H_4 does not contain W then $H_1 \cap H_2 \cap H_3 \cap H_4$ has dimension $l - 4$, and the formula for $g_3^{(4)}$ follows from (3) again. (Note that, when $l \leq 3$ all formulas follow trivially.) Thus, the last case is when $W \subset H_4$. Let $V_i = H_4 \cap H_i$, for $i = 1, 2, 3$. Because $x(\mathcal{H}) = 2$, the V_i are three different hyperplanes of H_4. Moreover, $V_1 \cap V_2 \cap V_3 = W$, and $\dim(V_i) = l - 2$ for $i = 1, 2, 3$. We can compute $g(\mathcal{H})$ as follows, noting Lemma 2.2,

$$
\begin{aligned}
g(\mathcal{H}) \quad &= \quad |\overline{H_1} \cup \overline{H_2} \cup \overline{H_3}| + |\overline{H_4} \setminus \overline{H_1} \cup \overline{H_2} \cup \overline{H_3}| \\
&= \quad |\overline{H_1} \cup \overline{H_2} \cup \overline{H_3}| + |\overline{H_4} \setminus \overline{V_1} \cup \overline{V_2} \cup \overline{V_3}| \\
&= \quad |\overline{H_1} \cup \overline{H_2} \cup \overline{H_3}| + |\overline{H_4}| - |\overline{V_1} \cup \overline{V_2} \cup \overline{V_3}| \\
&= \quad \left(3\begin{bmatrix} l-1 \\ r \end{bmatrix} - 3\begin{bmatrix} l-2 \\ 4 \end{bmatrix} + \begin{bmatrix} l-3 \\ r \end{bmatrix}\right) + \begin{bmatrix} l-1 \\ r \end{bmatrix} - \left(3\begin{bmatrix} l-2 \\ r \end{bmatrix} - 2\begin{bmatrix} l-3 \\ r \end{bmatrix}\right) \\
&= \quad g_4^{(4)}.
\end{aligned}
$$

$\square$

Lemma 2.4. *Let* $\mathcal{H} = \{H_1, H_2, H_3, H_4, H_5\}$ *be a set of* 5 *hyperplanes of* $\mathbb{F}_q^l$.

(i) *If* $x(\mathcal{H}) = 5$, *then*

$$g(\mathcal{H}) = g_1^{(5)} := 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 4\begin{bmatrix} l-2 \\ r \end{bmatrix}. \tag{11}$$

This case can only hold when $q \geq 4$.

(ii) *If* $x(\mathcal{H}) = 4$, *then*

$$g(\mathcal{H}) = g_2^{(5)} := 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 7\begin{bmatrix} l-2 \\ r \end{bmatrix} + 3\begin{bmatrix} l-3 \\ r \end{bmatrix}. \tag{12}$$

This case can only hold when $q \geq 3$.

(iii) If $x(\mathcal{H}) = 3$, then there are three cases:

$$g(\mathcal{H}) = g_3^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 8\begin{bmatrix} l-2 \\ r \end{bmatrix} + 4\begin{bmatrix} l-3 \\ r \end{bmatrix}, \tag{13}$$

$$g(\mathcal{H}) = g_4^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 9\begin{bmatrix} l-2 \\ r \end{bmatrix} + 7\begin{bmatrix} l-3 \\ r \end{bmatrix} - 2\begin{bmatrix} l-4 \\ r \end{bmatrix}, \tag{14}$$

$$g(\mathcal{H}) = g_5^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 9\begin{bmatrix} l-2 \\ r \end{bmatrix} + 5\begin{bmatrix} l-3 \\ r \end{bmatrix}. \tag{15}$$

The last case (of $g_5^{(5)}$) can only hold when $q \geq 3$).

(iv) If $x(\mathcal{H}) = 2$, then there are four cases:

$$g(\mathcal{H}) = g_6^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 10\begin{bmatrix} l-2 \\ r \end{bmatrix} + 10\begin{bmatrix} l-3 \\ r \end{bmatrix} - 4\begin{bmatrix} l-4 \\ r \end{bmatrix}, \tag{16}$$

$$g(\mathcal{H}) = g_7^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 10\begin{bmatrix} l-2 \\ r \end{bmatrix} + 10\begin{bmatrix} l-3 \\ r \end{bmatrix} - 5\begin{bmatrix} l-4 \\ r \end{bmatrix} + \begin{bmatrix} l-5 \\ r \end{bmatrix}, \tag{17}$$

$$g(\mathcal{H}) = g_8^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 10\begin{bmatrix} l-2 \\ r \end{bmatrix} + 9\begin{bmatrix} l-3 \\ r \end{bmatrix} - 3\begin{bmatrix} l-4 \\ r \end{bmatrix}, \tag{18}$$

$$g(\mathcal{H}) = g_9^{(5)} \quad := \quad 5\begin{bmatrix} l-1 \\ r \end{bmatrix} - 10\begin{bmatrix} l-2 \\ r \end{bmatrix} + 6\begin{bmatrix} l-3 \\ r \end{bmatrix}. \tag{19}$$

The last case (of $g_9^{(5)}$ can only hold when $q \geq 3$).

Moreover, $g_1^{(5)} \geq g_2^{(5)} \geq g_3^{(5)} \geq g_4^{(5)} \geq g_5^{(5)}$; and $g_4^{(5)} \geq g_6^{(5)} \geq g_7^{(5)} \geq g_8^{(5)} \geq g_9^{(5)}$. Also, $g_5^{(5)} \geq g_6^{(5)}$ when $q \geq 4$.

Proof. • Cases (i) and (ii) follow straightforwardly from the inclusion-exclusion formula (3) and Lemma 2.2.

• Suppose $x(\mathcal{H}) = 3$, and assume $V = H_1 \cap H_2 \cap H_3$ has dimension $l - 2$. Let $V_i = H_4 \cap H_i$, for $i = 1, 2, 3$, and $U = H_1 \cap H_2 \cap H_3 \cap H_4$. Since $x(\mathcal{H}) = 3$, H_4 does not contain V and thus $\dim(U) = l - 3$ by Lemma (2.2). We consider three cases as follows.

Case 1: H_5 contains some V_i for $i = 1, 2, 3$. Note that H_5 cannot contain two different V_i because the span of two different V_i is exactly H_4. Without loss of generality, assume $V_1 \subset H_5$. In this case H_1, H_2, H_3, H_4 intersect H_5 at 3 different hyperplanes (of H_5), because H_1 and H_4 intersect H_5 at the same hyperplane V_1. Lemma 2.2 and the inclusion-exclusion formula (3) gives

$$\begin{aligned}
|\overline{H_1} \cup \cdots \cup \overline{H_5}| &= |\overline{H_1} \cup \overline{H_2} \cup \overline{H_3}| + |\overline{H_4} \setminus \overline{H_4 \cap H_1} \cup \overline{H_4 \cap H_2} \cup \overline{H_4 \cap H_3}| + \\
&\quad |\overline{H_5} \setminus \overline{H_5 \cap H_1} \cup \cdots \cup \overline{H_5 \cap H_4}| \\
&= \left(3\begin{bmatrix} l-1 \\ r \end{bmatrix} - 2\begin{bmatrix} l-2 \\ r \end{bmatrix}\right) + 2\left(\begin{bmatrix} l-1 \\ r \end{bmatrix} - 3\begin{bmatrix} l-2 \\ r \end{bmatrix} + 2\begin{bmatrix} l-3 \\ r \end{bmatrix}\right) = g_3^{(5)}.
\end{aligned}$$

Case 2: H_5 contains U but does not contain any W_i for $i = 1, 2, 3$. In this case, H_1, H_2, H_3, H_4 intersect H_5 at 4 different hyperplanes all of which contains U. It follows that

$$\begin{aligned}
|\overline{H_1} \cup \cdots \cup \overline{H_5}| &= |\overline{H_1} \cup \overline{H_2} \cup \overline{H_3}| + |\overline{H_4} \setminus \overline{H_4 \cap H_1} \cup \overline{H_4 \cap H_2} \cup \overline{H_4 \cap H_3}| + \\
&\quad |\overline{H_5} \setminus \overline{H_5 \cap H_1} \cup \cdots \cup \overline{H_5 \cap H_4}| \\
&= \left(3\begin{bmatrix} l-1 \\ r \end{bmatrix} - 2\begin{bmatrix} l-2 \\ r \end{bmatrix}\right) + \left(\begin{bmatrix} l-1 \\ r \end{bmatrix} - 3\begin{bmatrix} l-2 \\ r \end{bmatrix} + 2\begin{bmatrix} l-3 \\ r \end{bmatrix}\right) + \\
&\quad \left(\begin{bmatrix} l-1 \\ r \end{bmatrix} - 4\begin{bmatrix} l-2 \\ r \end{bmatrix} + 3\begin{bmatrix} l-3 \\ r \end{bmatrix}\right) = g_5^{(5)}.
\end{aligned}$$

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Case 3: H_5 does not contain U. This is the situation of Lemma 2.2. We have

$$
\begin{aligned}
|\overline{H_1} \cup \cdots \cup \overline{H_5}| &= |\overline{H_1} \cup \cdots \cup \overline{H_4}| + |\overline{H_5} \setminus \overline{H_5 \cap H_1} \cup \cdots \cup \overline{H_5 \cap H_4}| \\
&= \left(4 \begin{bmatrix} l-1 \\ r \end{bmatrix} - 5 \begin{bmatrix} l-2 \\ r \end{bmatrix} + 2 \begin{bmatrix} l-3 \\ r \end{bmatrix} \right) + \\
&\quad \left(\begin{bmatrix} l-1 \\ r \end{bmatrix} - 4 \begin{bmatrix} l-2 \\ r \end{bmatrix} + 5 \begin{bmatrix} l-3 \\ r \end{bmatrix} - 2 \begin{bmatrix} l-4 \\ r \end{bmatrix} \right) = g_4^{(5)}.
\end{aligned}
$$

- If $x(\mathcal{H}) = 2$, then $W = H_1 \cap H_2 \cap H_3$ has dimension $l - 3$. The formula for $g_6^{(5)}$ comes from the case when H_4 and H_5 both contain W; $g_7^{(5)}$ is obtained when $W \subset H_4$ but $W \not\subset H_5$ or vice versa; $g_8^{(5)}$ is obtained when W is neither a subspace of H_4 or H_5 and H_5 does not contain the intersection $U = H_1 \cap H_2 \cap H_3 \cap H_4$, and $g_9^{(5)}$ is obtained when W is neither a subspace of H_4 nor H_5, yet H_5 does contain U. The computation is similar to the previous case.

$\square$

2.4 Tighter bounds and the packing arrangement

We first consider the simplest case when $r = 1$. The total number of lines (i.e. 1-dimensional subspaces) of a k-dimensional vector space S over $\mathbb{F}_q$ is

$$
\begin{bmatrix} k \\ 1 \end{bmatrix} = [k] = \frac{q^k - 1}{q - 1}.
$$

Let V be any $k-2$-dimensional subspace of S, and $H_1, \ldots, H_{q+1}$ be the set of all hyperplanes containing V. Then, the inclusion-exclusion formula (3) gives

$$
|\overline{H_1} \cup \cdots \cup \overline{H_{q+1}}| = (q+1) \begin{bmatrix} k-1 \\ 1 \end{bmatrix} - q \begin{bmatrix} k-2 \\ 1 \end{bmatrix} = (q+1) \frac{q^{k-1} - 1}{q - 1} - q \frac{q^{k-2} - 1}{q - 1} = \frac{q^k - 1}{q - 1}.
$$

The following theorem follows immediately.

Theorem 2.5. *When $r = 1$ and $d \geq q + 1$, the maximum value of $|\overline{H_1} \cup \cdots \cup \overline{H_d}|$ is exactly $[k]$, the total number of lines in S. One way to obtain this maximum is to have $q + 1$ of the hyperplanes contain a $(k - 2)$-dimensional subspace of S.*

For the rest of this section, we can assume $r \geq 2$. We first give a particular arrangement called the *packing arrangement* which proves to be optimal in certain cases.

Definition 2.6 (Packing Arrangement). Suppose $1 + q < d \leq 1 + q + q^2$. Let S be the k-dimensional vector space that the hyperplanes belong to. Let V be any $(k - 2)$-dimensional subspace of S, and W be any $(k - 3)$-dimensional subspace of V. The *packing arrangement* of d hyperplanes is an arrangement in which $q + 1$ hyperplanes, say $H_1, \ldots, H_{q+1}$, all contain V and the rest of the hyperplanes contain W.

We could define the packing arrangement for larger values of d. However, for the purposes of this paper $d \leq 1 + q + q^2$ is sufficient. The following lemma tells us the "cost" of this arrangement.

Lemma 2.7. *Consider $1 + q < d \leq 1 + q + q^2$, and let $H_1, \ldots, H_d$ be in the packing configuration. Then,*

$$
|\overline{H_1} \cup \cdots \cup \overline{H_d}| = d \begin{bmatrix} k-1 \\ r \end{bmatrix} - \left(d(q+1) - (1 + q + q^2) \right) \begin{bmatrix} k-2 \\ r \end{bmatrix} + q(d - q - 1) \begin{bmatrix} k-3 \\ r \end{bmatrix}. \quad (20)
$$

Proof. Without loss of generality, assume $H_1, \ldots, H_{q+1}$ intersect at a $(k-2)$-dimensional subspace V and the rest of the hyperplanes contain a $(k-3)$-dimensional subspace $W \subset V$. Consider any H_i where $q+1 < i \le d$. Let $V_j = H_i \cap H_j$, for $j \in \{1, \ldots, i-1\}$. Note that all V_j contain W; and, due to Lemma 2.2 it is easy to see that $V_1, \ldots, V_{q+1}$ are *different* hyperplanes of H_i. Moreover, the total number of hyperplanes in H_i that contain W is exactly $1 + q$. Hence,

$$\{V_1, \ldots, V_{q+1}\} = \{V_1, \ldots, V_{i-1}\}.$$

Consequently,

$$
\begin{aligned}
|\overline{H_i} \setminus \overline{H_1} \cup \cdots \cup \overline{H_{i-1}}| &= |\overline{H_i} \setminus \overline{H_1 \cap H_i} \cup \cdots \cup \overline{H_{i-1} \cap H_i}| \\
&= |\overline{H_i} \setminus \overline{V_1} \cup \cdots \cup \overline{V_{i-1}}| \\
&= |\overline{H_i} \setminus \overline{V_1} \cup \cdots \cup \overline{V_{q+1}}| \\
&= |\overline{H_i}| - |\overline{V_1} \cup \cdots \cup \overline{V_{q+1}}| \\
&= \begin{bmatrix} k-1 \\ r \end{bmatrix} - (q+1) \begin{bmatrix} k-2 \\ r \end{bmatrix} + q \begin{bmatrix} k-3 \\ r \end{bmatrix}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left| \bigcup_{i=1}^{d} \overline{H_i} \right| &= \left| \bigcup_{i=1}^{q+1} \overline{H_i} \right| + \sum_{i=q+2}^{d} |\overline{H_i} \setminus \overline{H_1} \cup \cdots \cup \overline{H_{i-1}}| \\
&= (q+1) \begin{bmatrix} k-1 \\ r \end{bmatrix} - q \begin{bmatrix} k-2 \\ r \end{bmatrix} + (d-q-1) \left(\begin{bmatrix} k-1 \\ r \end{bmatrix} - (q+1) \begin{bmatrix} k-2 \\ r \end{bmatrix} + q \begin{bmatrix} k-3 \\ r \end{bmatrix} \right) \\
&= d \begin{bmatrix} k-1 \\ r \end{bmatrix} - \left(d(q+1) - (1+q+q^2) \right) \begin{bmatrix} k-2 \\ r \end{bmatrix} + q(d-q-1) \begin{bmatrix} k-3 \\ r \end{bmatrix}.
\end{aligned}
$$

$\square$

Theorem 2.8. *Suppose $d \le [k]$. Consider any d hyperplanes $H_1, \ldots, H_d$ of a k-dimensional vector space S over $\mathbb{F}_q$. Let x be a maximal number of hyperplanes intersecting in a $(k-2)$-dimensional subspace $V \subset C$. Then,*

$$
|\overline{H_1} \cup \cdots \cup \overline{H_d}| \le d \begin{bmatrix} k-1 \\ r \end{bmatrix} + \begin{bmatrix} k-2 \\ r \end{bmatrix} - d \begin{bmatrix} k-3 \\ r \end{bmatrix} + x(x-d-1) \left(\begin{bmatrix} k-2 \\ r \end{bmatrix} - \begin{bmatrix} k-3 \\ r \end{bmatrix} \right) \quad (21)
$$

Proof. Without loss of generality, assume $H_1, \ldots, H_x$ intersect at V of dimension $(k-2)$, and no other H_i contains V. We invoke Lemmas 2.2 and formula (3) again. Since H_i does not contain V, it is easy to see that, for $1 \le j \le x$, the vector spaces $V_j = H_i \cap H_j$ are all distinct with dimension one less than H_j. Also, the intersection of the V_j has dimension one less than V. It follows that

$$
|\overline{H_i} \setminus \overline{H_1} \cup \cdots \cup \overline{H_x}| = |\overline{H_i} \setminus \overline{V_1} \cup \cdots \cup \overline{V_x}| = \begin{bmatrix} k-1 \\ r \end{bmatrix} - x \begin{bmatrix} k-2 \\ r \end{bmatrix} + (x-1) \begin{bmatrix} k-3 \\ r \end{bmatrix}.
$$

Consequently,

$$
\begin{aligned}
\left| \bigcup_{i=1}^{d} \overline{H_i} \right| &= \left| \bigcup_{i=1}^{x} \overline{H_i} \right| + \sum_{i=x+1}^{d} |\overline{H_i} \setminus \overline{H_1} \cup \cdots \cup \overline{H_{i-1}}| \\
&\le \left| \bigcup_{i=1}^{x} \overline{H_i} \right| + \sum_{i=x+1}^{d} |\overline{H_i} \setminus \overline{H_1} \cup \cdots \cup \overline{H_x}| \\
&= x \begin{bmatrix} k-1 \\ r \end{bmatrix} - (x-1) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (d-x) \left(\begin{bmatrix} k-1 \\ r \end{bmatrix} - x \begin{bmatrix} k-2 \\ r \end{bmatrix} + (x-1) \begin{bmatrix} k-3 \\ r \end{bmatrix} \right) \\
&= d \begin{bmatrix} k-1 \\ r \end{bmatrix} + \begin{bmatrix} k-2 \\ r \end{bmatrix} - d \begin{bmatrix} k-3 \\ r \end{bmatrix} + x(x-d-1) \left(\begin{bmatrix} k-2 \\ r \end{bmatrix} - \begin{bmatrix} k-3 \\ r \end{bmatrix} \right)
\end{aligned}
$$

$\square$

We get Theorem 2.1 for free.

Corollary 2.9 (Same as Theorem 2.1). *Suppose $2 \leq d \leq q+1$. Then, for any d hyperplanes $H_1, \ldots, H_d$ of a k-dimensional vector space C over $\mathbb{F}_q$ we have*

$$|\overline{H_1} \cup \cdots \cup \overline{H_d}| \leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (d-1) \begin{bmatrix} k-2 \\ r \end{bmatrix}. \tag{22}$$

Moreover, there exists an arrangement of hyperplanes achieving the right hand side.

Proof. Without loss of generality, suppose $H_1, \ldots, H_x$ intersect at some $(k-2)$-dimensional subspace V, and no other H_i contains V. Note that $2 \leq x \leq d$. Thus $x(x-d-1) \leq -d$. The triangle identity (6) gives

$$\begin{bmatrix} k-2 \\ r \end{bmatrix} - \begin{bmatrix} k-3 \\ r \end{bmatrix} = q^{k-2-r} \begin{bmatrix} k-3 \\ r-1 \end{bmatrix} \geq 0.$$

Relation (21) implies

$$\begin{aligned}
|\overline{H_1} \cup \cdots \cup \overline{H_d}| &\leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} + \begin{bmatrix} k-2 \\ r \end{bmatrix} - d \begin{bmatrix} k-3 \\ r \end{bmatrix} - d \left(\begin{bmatrix} k-2 \\ r \end{bmatrix} - \begin{bmatrix} k-3 \\ r \end{bmatrix} \right) \\
&= d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (d-1) \begin{bmatrix} k-2 \\ r \end{bmatrix}.
\end{aligned}$$

The inequality is tight because equality can be obtained by choosing d hyperplanes $H_1, \ldots, H_d$ all of which contain a $(k-2)$-dimensional subspace V. $\square$

Theorem 2.10. *Suppose $d \geq q+2$ and $k > r \geq 2$. Then, for any d hyperplanes $H_1, \ldots, H_d$ of a k-dimensional vector space S over $\mathbb{F}_q$ we have*

$$|\overline{H_1} \cup \cdots \cup \overline{H_d}| \leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (2d-3) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (d-2) \begin{bmatrix} k-3 \\ r \end{bmatrix}. \tag{23}$$

Moreover, when $d = q+2$ the packing arrangement achieves the bound.

Proof. Without loss of generality, suppose $H_1, \ldots, H_x$ intersect at some $(k-2)$-dimensional subspace V, and no other H_i contains V. Note that, in this case $2 \leq x \leq q+1$, and thus $x(x-d-1) \leq 2(2-d-1) = -2(d-1)$. Relation (21) implies

$$\begin{aligned}
|\overline{H_1} \cup \cdots \cup \overline{H_d}| &\leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} + \begin{bmatrix} k-2 \\ r \end{bmatrix} - d \begin{bmatrix} k-3 \\ r \end{bmatrix} - 2(d-1) \left(\begin{bmatrix} k-2 \\ r \end{bmatrix} - \begin{bmatrix} k-3 \\ r \end{bmatrix} \right) \\
&= d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (2d-3) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (d-2) \begin{bmatrix} k-3 \\ r \end{bmatrix}.
\end{aligned}$$

When $d = q+2$, we only have to verify that the right hand side of (23) is the same as that of (20), which is mechanical. $\square$

Theorem 2.11. *Suppose $d \geq q+3$ and $k > r \geq 2$. Then, for any d hyperplanes $H_1, \ldots, H_d$ of a k-dimensional vector space S over $\mathbb{F}_q$ we have*

$$|\overline{H_1} \cup \cdots \cup \overline{H_d}| \leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (3d-7) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (2d-6) \begin{bmatrix} k-3 \\ r \end{bmatrix}. \tag{24}$$

Moreover, when $d = q+3$ the packing arrangement achieves the bound.

9

Proof. Let $x = x(\{H_1, \ldots, H_d\})$. Without loss of generality, suppose $H_1, \ldots, H_x$ intersect at some $(k-2)$-dimensional subspace V, and no other H_i contains V. Consider two cases as follows.

Case 1: $x \geq 3$. Note that, $x \leq q + 1 \leq d - 2$, and thus $x(x - d - 1) \leq 3(3 - d - 1) = -3(d - 2)$. Relation (21) implies

$$
\begin{aligned}
\left| \overline{H_1} \cup \cdots \cup \overline{H_d} \right| & \leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} + \begin{bmatrix} k-2 \\ r \end{bmatrix} - d \begin{bmatrix} k-3 \\ r \end{bmatrix} - 3(d-2) \left(\begin{bmatrix} k-2 \\ r \end{bmatrix} - \begin{bmatrix} k-3 \\ r \end{bmatrix} \right) \\
& = d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (3d-7) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (2d-6) \begin{bmatrix} k-3 \\ r \end{bmatrix}.
\end{aligned}
$$

Case 2: $x = 2$. Applying Lemma 2.3 with $l = k$ we get

$$
\left| \overline{H_1} \cup \overline{H_2} \cup \overline{H_3} \cup \overline{H_4} \right| \leq \max\{g_3^{(4)}, g_4^{(4)}\} = g_3^{(4)} = 4 \begin{bmatrix} k-1 \\ r \end{bmatrix} - 6 \begin{bmatrix} k-2 \\ r \end{bmatrix} + 4 \begin{bmatrix} k-3 \\ r \end{bmatrix} - \begin{bmatrix} k-4 \\ r \end{bmatrix}.
$$

Consider any H_i with $i > 4$. For $j = 1, 2, 3, 4$, let $V_j = H_i \cap H_j$. Then, the V_j are four different hyperplanes of H_i, because $x = 2$. Applying Lemma 2.3 with $l = k - 1$ we get

$$
\left| \overline{V_1} \cup \overline{V_2} \cup \overline{V_3} \cup \overline{V_4} \right| \geq \min\{g_1^{(4)}, g_2^{(4)}, g_3^{(4)}, g_4^{(4)}\} = g_4^{(4)} = 4 \begin{bmatrix} k-2 \\ r \end{bmatrix} - 6 \begin{bmatrix} k-3 \\ r \end{bmatrix} + 3 \begin{bmatrix} k-4 \\ r \end{bmatrix}.
$$

Hence,

$$
\left| \overline{H_i} \setminus \overline{H_1} \cup \overline{H_2} \cup \overline{H_3} \cup \overline{H_4} \right| = |\overline{H_i}| - |\overline{V_1} \cup \overline{V_2} \cup \overline{V_3} \cup \overline{V_4}| \leq \begin{bmatrix} k-1 \\ r \end{bmatrix} - 4 \begin{bmatrix} k-2 \\ r \end{bmatrix} + 6 \begin{bmatrix} k-3 \\ r \end{bmatrix} - 3 \begin{bmatrix} k-4 \\ r \end{bmatrix}.
$$

Putting them all together, we get

$$
\begin{aligned}
\left| \bigcup_{i=1}^{d} \overline{H_i} \right| & = \left| \overline{H_1} \cup \overline{H_2} \cup \overline{H_3} \cup \overline{H_4} \right| + \sum_{i=5}^{d} \left| \overline{H_i} \setminus \overline{H_1} \cup \overline{H_2} \cup \overline{H_3} \cup \overline{H_4} \right| \\
& \leq \left(4 \begin{bmatrix} k-1 \\ r \end{bmatrix} - 6 \begin{bmatrix} k-2 \\ r \end{bmatrix} + 4 \begin{bmatrix} k-3 \\ r \end{bmatrix} - \begin{bmatrix} k-4 \\ r \end{bmatrix} \right) + \\
& \quad (d-4) \left(\begin{bmatrix} k-1 \\ r \end{bmatrix} - 4 \begin{bmatrix} k-2 \\ r \end{bmatrix} + 6 \begin{bmatrix} k-3 \\ r \end{bmatrix} - 3 \begin{bmatrix} k-4 \\ r \end{bmatrix} \right) \\
& = d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (4d-10) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (6d-20) \begin{bmatrix} k-3 \\ r \end{bmatrix} - (3d-12) \begin{bmatrix} k-4 \\ r \end{bmatrix}
\end{aligned}
$$

It is easy to see that, when $k > r \geq 2$ the last expression is at most the right hand side of (24).

Lastly, when $d = q + 3$ the fact that the packing arrangement achieves the bound (24) is straightforward. $\qquad \square$

Theorem 2.12. *Suppose $d \geq q + 4$ and $k > r \geq 2$. Then, for any d hyperplanes $H_1, \ldots, H_d$ of a k-dimensional vector space S over $\mathbb{F}_q$ we have*

$$
\left| \overline{H_1} \cup \cdots \cup \overline{H_d} \right| \leq d \begin{bmatrix} k-1 \\ r \end{bmatrix} - (4d-13) \begin{bmatrix} k-2 \\ r \end{bmatrix} + (3d-12) \begin{bmatrix} k-3 \\ r \end{bmatrix}. \tag{25}
$$

Moreover, when $d = q + 4$ the packing arrangement achieves the bound.

Proof. The proof is similar to the previous theorem with three cases to consider: $x \geq 4$, $x = 3$, and $x = 2$. This time we make use of Lemma 2.4 and its various relations. $\qquad \square$

3 Tighter analysis of $M_q(m, k, r)$

The results of the previous section help us analyze the $M_q(m, k, r)$ construction. Firstly, we show that $M_q(m, k, 1)$ is not a good design when $d \geq q + 1$. The result is a direct corollary of Theorem 2.5.

Corollary 3.1. *When $d \geq q + 1$, then $M_q(m, k, 1)$ is not d-disjunct.*

Proof. Let C_0 be a k-dimensional subspace of $\mathbb{F}_q^m$. Let $H_1, \ldots, H_{q+1}$ be hyperplanes of C_0 chosen according to Theorem 2.5. Let v be any vector in $\mathbb{F}_q^m$ not belonging to C_0. For $i = 1, \ldots, d$, let $C_i = \mathrm{span}\{H_i, v\}$. Choose arbitrarily k-dimensional subspaces $C_{q+2}, \ldots, C_d$. Then, it is easy to see that $\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d} = \emptyset$. $\qquad\square$

Secondly, the number of columns of $M_q(m, k, r)$ is $\begin{bmatrix} m \\ k \end{bmatrix}$, which is exponentially larger than $[k]$, the number of hyperplanes in a k-dimensional vector space. The following theorem shows a limit of the pooling design.

Theorem 3.2. *If $d \geq [k]$, then $M_q(m, k, r)$ is not d-disjunct.*

Proof. Consider any k-dimensional subspace C_0 of $\mathbb{F}_q^m$. Let $H_1, \ldots, H_{[k]}$ be the set of all hyperplanes of C_0. Let v be any vector in $\mathbb{F}_q^m \setminus C_0$. For any $i = 1, \ldots, [k]$, define $C_i = \mathrm{span}\{v, H_i\}$. For $i = [k] + 1, \ldots, d$, choose k-dimensional subspaces C_i arbitrarily as long as they have not been chosen before. Then, $|\overline{C_0} \setminus \overline{C_1} \cup \cdots \cup \overline{C_d}| = 0$, namely $M_q(m, k, r)$ is not d-disjunct. This is because any r-dimensional subspace of C_0 is also a subspace of some H_i, $i = 1, \ldots [k]$; thus, it is also an r-dimensional subspace of C_i. $\qquad\square$

Henceforth, we only need to consider the case when $r \geq 2$ and $q + 2 \leq d \leq [k]$. The following corollaries follow from Theorems 2.10, 2.11, and 2.12, respectively.

Corollary 3.3. *When $d \geq q + 2$ and $m > k > r \geq 2$, $M_q(m, k, r)$ is d^z-disjunct, where*

$$z = \begin{bmatrix} k \\ r \end{bmatrix} - d \begin{bmatrix} k - 1 \\ r \end{bmatrix} + (2d - 3) \begin{bmatrix} k - 2 \\ r \end{bmatrix} - (d - 2) \begin{bmatrix} k - 3 \\ r \end{bmatrix}$$

Moreover, the construction is exactly d^z-disjunct when $d = q + 2$.

Corollary 3.4. *When $d \geq q + 3$ and $m > k > r \geq 2$, $M_q(m, k, r)$ is d^z-disjunct, where*

$$z = \begin{bmatrix} k \\ r \end{bmatrix} - d \begin{bmatrix} k - 1 \\ r \end{bmatrix} + (3d - 7) \begin{bmatrix} k - 2 \\ r \end{bmatrix} - (2d - 6) \begin{bmatrix} k - 3 \\ r \end{bmatrix}.$$

Moreover, the construction is exactly d^z-disjunct when $d = q + 3$.

Corollary 3.5. *When $d \geq q + 4$ and $m > k > r \geq 2$, $M_q(m, k, r)$ is d^z-disjunct, where*

$$z = \begin{bmatrix} k \\ r \end{bmatrix} - d \begin{bmatrix} k - 1 \\ r \end{bmatrix} + (4d - 13) \begin{bmatrix} k - 2 \\ r \end{bmatrix} - (3d - 12) \begin{bmatrix} k - 3 \\ r \end{bmatrix}.$$

Moreover, the construction is exactly d^z-disjunct when $d = q + 4$.

Remark 3.6. Note that, to apply all the three corollaries above, we only need to find the range of d which makes $z > 0$. It turns out that this range is quite large, and the task is mechanical. We omit this step here. Also, it is straightforward to check that the bounds in Theorems 2.10, 2.11, and 2.12 are better than that of Theorem 2.1.

4 Discussions

It is very natural to ask the converse of our hyperplane arrangement problem, leading to the following:

Problem 2 (Second Hyperplane Arrangement Problem). Given a k-dimensional vector space C over $\mathbb{F}_q$, and an integer d such that $1 \leq d \leq [k]$, find d hyperplanes $H_1, \ldots, H_d$ of C that *minimizes* the following quantity

$$|\overline{H_1} \cup \cdots \cup \overline{H_d}|.$$

Historically, there have been quite a lot of studies on hyperplane arrangements. The extremal problems such as the problem of dividing a space into as many regions as possible given a fixed number of hyperplanes are mostly on infinite vector spaces. Arrangement problems and results on finite fields mostly are about algebraic and structural information (Möbius functions, Poincaré polynomials, ...) or topological structures. The reader is referred to [23] for a good treatment of such problems. Our two hyperplane arrangement problems are new, to be best of the author's knowledge.

It is possible to show that the packing arrangement is the best for $d = q + 5$ $(q \geq 3)$ and so on, but the current method becomes too tedious to be useful. We conjecture that the packing arrangement is best for Problem 1 when $1 + q < d \leq 1 + q + q^2$. We also leave open Problem 2 at this point.

References

[1] M. AIGNER, *Searching with lies*, J. Combin. Theory Ser. A, 74 (1996), pp. 43–56.

[2] G. E. ANDREWS, *The theory of partitions*, Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.

[3] D. J. BALDING, W. J. BRUNO, E. KNILL, AND D. C. TORNEY, *A comparative survey of non-adaptive pooling designs*, in Genetic mapping and DNA sequencing (Minneapolis, MN, 1994), Springer, New York, 1996, pp. 133–154.

[4] D. J. BALDING AND D. C. TORNEY, *Optimal pooling designs with error detection*, J. Combin. Theory Ser. A, 74 (1996), pp. 131–140.

[5] A. E. F. CLEMENTI, A. MONTI, AND R. SILVESTRI, *Selective families, superimposed codes, and broadcasting on unknown radio networks (extended abstract)*, in Proceedings of the Twelfth Annual ACM-SIAM Symposium on Discrete Algorithms (Washington, DC, 2001), Philadelphia, PA, 2001, SIAM, pp. 709–718.

[6] A. DE BONIS AND U. VACCARO, *Efficient constructions of generalized superimposed codes with applications to group testing and conflict resolution in multiple access channels*, in Algorithms—ESA 2002, vol. 2461 of Lecture Notes in Comput. Sci., Springer, Berlin, 2002, pp. 335–347.

[7] ———, *Constructions of generalized superimposed codes with applications to group testing and conflict resolution in multiple access channels*, Theoret. Comput. Sci., 306 (2003), pp. 223–243.

[8] D.-Z. DU, F. HWANG, W. WU, AND T. ZNATI, *New construction for transversal design*, Journal of Computational Biology, 13 (2006), pp. 990–995.

[9] D.-Z. DU AND F. K. HWANG, *Combinatorial group testing and its applications*, World Scientific Publishing Co. Inc., River Edge, NJ, 1993.

[10] A. D'YACHKOV, F. HWANG, A. MACULA, P. VILENKIN, AND C. W. WENG, *A construction of pooling designs with some happy surprises*, Journal of Computational Biology, 12 (2005), pp. 1129–1136.

[11] A. G. D'YACHKOV, A. J. MACULA, AND V. V. RYKOV, *New applications and results of superimposed code theory arising from the potentialities of molecular biology*, in Numbers, information and complexity (Bielefeld, 1998), Kluwer Acad. Publ., Boston, MA, 2000, pp. 265–282.

[12] A. G. D'YACHKOV, A. J. MACULA, JR., AND V. V. RYKOV, *New constructions of superimposed codes*, IEEE Trans. Inform. Theory, 46 (2000), pp. 284–290.

[13] A. G. D'YACHKOV AND V. V. RYKOV, *A survey of superimposed code theory*, Problems Control Inform. Theory/Problemy Upravlen. Teor. Inform., 12 (1983), pp. 229–242.

[14] W. H. KAUTZ AND R. C. SINGLETON, *Nonrandom binary superimposed codes*, IEEE Trans. Inf. Theory, 10 (1964), pp. 363–377.

[15] H. K. KIM AND V. LEBEDEV, *On optimal superimposed codes*, J. Combin. Des., 12 (2004), pp. 79–91.

[16] E. KNILL, W. J. BRUNO, AND D. C. TORNEY, *Non-adaptive group testing in the presence of errors*, Discrete Appl. Math., 88 (1998), pp. 261–290.

[17] X. MA AND L. PING, *Coded modulation using superimposed binary codes*, IEEE Trans. Inform. Theory, 50 (2004), pp. 3331–3343.

[18] A. J. MACULA, *A simple construction of d-disjunct matrices with certain constant weights*, Discrete Math., 162 (1996), pp. 311–312.

[19] ———, *Error-correcting nonadaptive group testing with d^e-disjunct matrices*, Discrete Appl. Math., 80 (1997), pp. 217–222.

[20] S. MUTHUKRISHNAN, *On optimal strategies for searching in presence of errors*, in Proceedings of the Fifth Annual ACM-SIAM Symposium on Discrete Algorithms (Arlington, VA, 1994), New York, 1994, ACM, pp. 680–689.

[21] H. Q. NGO AND D.-Z. DU, *A survey on combinatorial group testing algorithms with applications to DNA library screening*, in Discrete mathematical problems with medical applications (New Brunswick, NJ, 1999), vol. 55 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Amer. Math. Soc., Providence, RI, 2000, pp. 171–182.

[22] ———, *New constructions of non-adaptive and error-tolerance pooling designs*, Discrete Math., 243 (2002), pp. 161–170.

[23] P. ORLIK AND H. TERAO, *Arrangements of hyperplanes*, vol. 300 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1992.

[24] J. K. PERCUS, O. E. PERCUS, W. J. BRUNO, AND D. C. TORNEY, *Asymptotics of pooling design performance*, J. Appl. Probab., 36 (1999), pp. 951–964.

[25] J. H. VAN LINT AND R. M. WILSON, *A course in combinatorics*, Cambridge University Press, Cambridge, second ed., 2001.

[26] J. K. WOLF, *Born again group testing: multiaccess communications*, IEEE Transaction on Information Theory, IT-31 (1985), pp. 185–191.