ADVANCES IN SWITCHING NETWORKS
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Notes on the Complexity of Switching Networks

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1 Introduction

1.1 Overview

There are various complexity measures for switching networks and communication networks in general. These measures include, but not limited to, the number of switching components, the delay time of signal propagating through the network, the complexity of path selection algorithms, and the complexity of physically designing the network. This chapter surveys the study of the first measure, and partially the second measure. It is conceivable that the number of switching components, or the "size" of a network, affects directly the third and fourth measures.

The most common and most intuitively obvious way to study a particular type of switching networks is to model the network by a graph. It is customary to use a directed acyclic graph with a degsinated set of vertices called *inputs* and another disjoint set of vertices called *outputs*. Depending on the functionality of the network under consideration, this graph has to satisfy certain conditions. Our job is then to determine the minimum number of edges of graphs satisfying these conditions, and construct an optimal one if possible. For example, a typical functionality of a network is *rearrangeability*. The corresponding graph model is the *connectors*, also referred to as *rearrangeable networks*. For different types of networks and their desired functionalities, the reader is referred to [16, 72, 40]. This chapter assumes that the reader is familiar with concepts such as rearrangeability, strictly nonblockingness or wide sense nonblockingness of multistage interconnection networks.

There have been several articles and books discussing aspects of switching network complexity or their associated graphical models. For example, the articles by Pippenger (1990, [72]), and Bien (1989, [17]); the classic book by Beneš (1965, [16]), a book by Hui (1990, [39]), and a recent book by Hwang (1998, [40]). We shall see that this chapter's material, although has some small overlaps with the articles and books above, have been collected in one place for the first time. Many open questions shall also be presented along the way.

We shall attempt to make this chapter as self-contained as possible, requiring mostly elementary background on probability and linear algebra. This objective will certainly affect our choice of which results to be covered. Moreover, we shall also try to select results that are more intuitive, which might not be the best results known on the problems under discussion. Pointers to more advanced results shall be given as needed.

1.2 Basic models

We now give definitions of several important graphs and their components arising in studying the complexity of switching networks, settling down the main terminologies for the rest of the chapters.

Definition 1.1. A bipartite graph $G = (I, O; E)$, where I and O are the bipartitions, is called an (n, d, c)-*expander* if $|I| = |O| = n$, $\Delta(G) = d$, and for every $X \subset I$ such that $|X| \leq n/2$, we have

$$|\Gamma(X)| \geq \left(1 + c\left(1 - \frac{|X|}{n}\right)\right)|X|$$

where $\Gamma(X)$ is the set of neighbors of all vertices X, $\Delta(G)$ denotes the maximum degree of vertices in G, d is called the *density*, and c is called the *expansion rate* of the expander G. It is a *strong* (n, d, c)-*expander* if the above inequality holds for all $X \subseteq I$. A *family of linear expanders of density d and expansion c* is a sequence $\{G_i\}_{i=1}^{\infty}$, where G_i is an (n_i, d, c)-expander and $n_i \to \infty$, $n_{i+1}/n_i \to 1$ as $i \to \infty$.

Definition 1.2. An (n, m)-*network* is an directed acyclic graph (DAG) with n distinguished vertices called *inputs* and a disjoint set of m distinguished vertices called *outputs*. When $n = m$, we call it an n-*network* for short. The *size* of a network is its number of edges. The *depth* of a network is the maximum length of a path from an input to an output.

An (n, m)-network is our main model for a switching network with n inputs and m outputs. With respect to physical switching networks, the vertices of our DAG represent *links* between switching components of the underlying physical networks, and there is an edge between two links if these two links are an input and an output link of some switch. Moreover, the depth of the DAG implicitly represent the delay of signal propagating from inputs to outputs.

A classical objective of studying switching networks of various types was to design these networks with as few switching components as possible. The main hope was that reducing this number also reduces several other complexity measures. Hence, one of our objectives is to find the minimum size (n,m)-network (or n-network for that matter) satisfying certain properties, and construct one with as small a size as possible. We shall see that this objective is usually a trade-off with another objective which is to have a network with small depth, which presumably has small signal propagation delay. Determining this trade-off in some precise manner is thus another research topic.

Definition 1.3. An (n,m)-*concentrator* is an (n,m)-network where $n \geq m$, such that for any subset S of m inputs there exists a set of m vertex disjoint paths connecting S to the outputs. An n-*concentrator* is an (n,n)-concentrator. An (n,m,d)-*concentrator* is an (n,m)-concentrator with at most dn edges. An (n,d)-*concentrator* is an (n,n,d)-concentrator. A *family of linear concentrators of density* d is a sequence $\{G_i\}_{i=1}^{\infty}$ such that each G_i is an (n_i,d)-concentrator, where $n_i \to \infty$ as $i \to \infty$.

Definition 1.4. An n-*superconcentrator* is an n-network with inputs I and outputs O such that for any $S_1 \subseteq I$ and $S_2 \subseteq O$ with $|S_1| = |S_2| = c$, there exist a set of c vertex disjoint paths connecting S_1 to S_2. An (n,k)-*superconcentrator* is an n-superconcentrator with at most kn edges. A *family of linear superconcentrators of density* k is a sequence $\{G_i\}_{i=1}^{\infty}$ such that each G_i is an (n_i,k)-superconcentrator, where $n_i \to \infty$ as $i \to \infty$.

Concentrators and superconcentrators are models for various communication networks and parallel architectures [72, 39]. For example, we can think of concentrators as a model for switching networks in which a set of processors communicate with a set of identical memories. While if a set of processors can request a particular set of memories, we need superconcentrators.

The graphs we just introduced are only part of the set of models we shall discuss. They were defined here to facilitate the history discussion in the next section. Other models shall be defined in associated sections.

1.3 A historical perspective

In this section, we briefly summarize the study of superconcentrators, concentrators and expanders. An objective is to give the reader a feel of why certain results were mentioned or covered in this chapter. Another reason for discussing these graphs' history is that they have rich and deep connections to many other areas of Computer Science and Mathematics, motivating many beautiful and difficult problems. Other complexity models, although shall be discussed later on in the chapter, will not be mentioned here. We shall briefly summarize their development in the corresponding section.

Concentrators were first introduced by Pinsker (1973 [65]) in the context of telephone switching networks. The notion of a superconcentrator, according to Pippenger (1996, [73]), was first introduced by Aho, Hopcroft and Ullman (1975, [1]), who attributed it to conversations with W. Floyd. They wanted to use superconcentrator as a tool to establish non-linear lower bounds for the complexity of circuits computing Boolean functions. Valiant (1975, [85, 84]) constructed linear size superconcentrators using Pinsker's concentrators, thus negating the intention of Aho, Hopcroft and Ullman. Paul, Tarjan and Celoni, continuing the theme, applied superconcentrators to demonstrate the optimality of certain algorithms (1977, [63, 64]). In 1973, Pippenger [66] also independently obtained some initial results on concentrators. From 1977 to 1979, Pippenger [69, 68, 67] and Fan Chung [24] studied the size bounds of minimum sized superconcentrators and concentrators. The concepts of connectors, generalizers, generalized connectors were also introduced by Pippenger in these papers as models for different switching networks.

Pinsker, Valiant, Pippenger and Chung works were roughly based on the same line of reasoning: the probabilistic method. Their proofs of the existence of good concentrators and superconcentrators were based on the existence of good bounded concentrators. The proofs were not constructive and thus of little practical value, although certainly very interesting. In 1973, Margulis [53] gave the first explicit construction of a family of expanders. As we shall see, expanders can be used to construct bounded concentrators; hence, Margulis construction yields an explicit construction of superconcentrators. Margulis construction was fairly technical using deep results from the Representation Theory of finite groups. He showed that there exists a constant $k > 0$ such that for $n = m^2, m \in \mathbb{N}$, each graph $\hat{G}_n$ of his constructed family of graphs is an $(n, 5, k)$-expander. Extending this work, Gabber and Galil (1981, [35]) showed that $k = k_0 = \frac{2-\sqrt{3}}{4}$ works. Their proof was a bit less technical, involving relatively elementary analysis. Moreover, they specified a way to construct a family of $(n, 7, 2k_0)$-expanders (larger density). They also specify how to use linear expanders to construct linear bounded concentrators and then to use the later to construct linear superconcentrators.

As we can see, the explicit constructions were too specialized to the given parameters, and thus not entirely satisfactory. Ron Rivest and S. Bhatt suggested another method which reuse earlier results on probabilistic construction: randomly choose a graph in a clever way, then check to see if the graph is a concentrator or superconcentrator. The main question is obviously that how hard the checking procedure is. Blum, Karp, Vornberger, Papadimitriou and Yannakakis (1981, [19]) gave a negative answer: the check(s) is coNP-complete.

In the meantime, researchers keep working on bounding the best possible size of the graphs. They also limit attention to graphs with a fixed depth. We will discuss more on this later. In 1983, so much excitement were ignited when expanders were used to explicitly construct a "parallel sorting network" which sorts n numbers in $O(\log n)$ time using n parallel processors. The work was done by M. Ajtai, J. Komlós, and E. Szemerédi [3], settling a long standing unsolved problem. This problem looked so impossible that, in fact, Knuth [48] in his previous edition of "The Art of Computer Programming" Vol. 3, stated it as a 50 exercise to show that constructing such a sorting network is impossible.

In 1984, an important idea comes into place as Tanner [83] used association schemes (see [21]), in particular certain class of distance regular graphs arising from finite geometry, to construct better expanders. The basic idea is to characterize the expansion rate using the graph's eigenvalue. Immediately after that, Noga Alon and partially Milman took the idea to its peak by a series of papers ([9, 5, 6, 7, 8]). Alon characterized the expansion rate using the second smallest eigenvalue $\mu(G)$ of the Laplacian of a graph G. Basically, as $\mu(G)$ gets larger the expansion rate is larger. As eigenvalues of real symmetric matrices (as the case with a graph's Laplacian) can be computed easily in polynomial time, we now have another way to "check" if a graph has certain expansion rate. The suggestion by Rivest and Bhatt can now be used. Alon did just that. He showed how to randomly generate expanders. Notice that since the problem is coNP-complete, quite a bit of information is lost in the characterization. (See [46], for example, for a sample research on this loss.) However, this method is fairly practical in certain applications.

Obviously a very natural question to ask is: "how large can $\mu(G)$ be?". Alon and Boppana (mentioned in [7]) proved that for any fixed d and for any infinite family of graphs G with maximum degree d,

$$\limsup \mu(G) \leq d - 2\sqrt{d-1}$$

Nilli (from Israel, 1991, [61]) got an upper bound that does not have any hidden constant, but the term $d-2\sqrt{d-1}$ is still dominating. It is obviously interesting to construct families of expanders which achieve this eigenvalue bound. The special case when $d-1$ is a prime congruent to 1 modulo 4 was done by Lubotzky, Phillips and Sarnak (1988, [51]), and independently by Margulis (1988, [53]). They constructed a family of d-regular graphs with $\lambda \leq 2\sqrt{d-1}$, where $\lambda(G)$ is the second largest eigenvalue of G. Formally, we define

Definition 1.5. A d-regular graph G with n vertices is called a *Ramanujan graph* if $\lambda(G) \leq 2\sqrt{d-1}$. (Or equivalently $\mu(G) \geq d - 2\sqrt{d-1}$.)

Ramanujan graphs are optimal in the sense of the eigenvalue bound. Why were the graphs named after *Ramanujan* ? Basically, the construction was the Cayley graph of certain group, where the generators were chosen to be the solutions to certain representation of integers in quaternary quadratic form. The general formula is unknown, but Ramanujan had a conjecture in 1916 about the formula (well, obviously these kinds of conjectures can only come from Ramanujan), which was proved in several special cases by Eichler (1954, [30]) and Igusa (1959, [43]).

From this point on, a lot more research papers were published on various aspects of these graphs, several of which will be discussed in later section. One big open problem (mentioned in [34]) is

Open Problem 1.6. Find an elementary (e.g. purely combinatorial) proof that certain family of graphs is expanders. For example the ones constructed in [34] or in [51].

Solving this problem would lead to an entirely new wave of research on expanders. The spectral characterization of expansion rate loses a lot of information. The proofs of known explicit constructions were either too complicated, or used the spectral characterization which already reduces the power of the graphs. Calculating eigenvalues, although gives the quantitative measure, sheds no light as to why certain set of graphs are good expanders. A combinatorial proof would be much more satisfactory and would certainly lead to new development in the area.

It is worth mentioning that these graphs also play important roles in areas other than switching networks. To mention a few, for example, Noga Alon (1986, [7]) used geometric expanders (expanders constructed from finite geometry) to deduce a certain strengthening of a theorem of de Bruijn and Erdős on the number of lines determined by a set of points in a finite projective plane. He also obtained some results on Hadamard matrices and constructed some graphs relevant to Ramsey theory. Sipser and Spielman (1996, [81] – see also [82]) constructed asymptotically good linear codes from expanders.

2 The complexity of checking whether a graph is a superconcentrator or concentrator

As we have mentioned in the introduction, if it is possible to check in polynomial time if a graph is a (super)concentrator, then it might be possible to devise a randomized algorithm to generate these graphs. The results in this section are from [19], which says that the checks are both coNP-complete.

Definition 2.1. A *matcher* is a bipartite graph $B = (V_1, V_2, E)$ such that $|V_1| = |V_2| = 2m$ and that for any m-subset S of V_1, there exists a matching from S into

V_2.

We shall show that deciding whether a bipartite graph is a matcher is coNP-complete, and then deduce as corollaries that checking whether a graph is a concentrator or superconcentrator is also coNP-complete. In this section and throughout the chapter, we will use $[n]$ to denote the set of integers from 1 to n, and $deg(v)$ to denote the degree of a vertex v in some graph.

Theorem 2.2 (Blum et al., 1981 [19]). *Deciding whether a bipartite graph $B = (V_1, V_2, E)$ is a matcher is coNP-complete.*

Proof. We will reduce the complement of the following NP-complete problem to an instance of the matcher problem.
Big-clique. Given a graph (N, A) (N for nodes and A for arcs), with $|N| = 2k$, does is have a clique of size k ?

We can assume $|A| \geq \binom{k}{2}$, since otherwise the answer is clearly NO. Given (N, A), we construct a bipartite graph $B = (V_1, V_2, E)$ as follows.

- $V_1 = N \times [2k]$.
- $V_2 = A \cup \{u_1, \ldots, u_{|A|}\} \cup \bigcup_{n \in N} \{v_1^n, \ldots, v_{c(n)}^n\}$, where $c(n) = 2k - deg(n)$.
- To get E, each vertex $(n, i) \in V_1$, where $n \in N$ and $i \in [2k]$, is connected to the following vertices in V_2:
  - All arcs $a_j \in A$ which are incident to n.
  - All the u-nodes $u_1, u_2, \ldots, u_{\binom{k}{2}-1}$.
  - All nodes v_j^n for $j = 1, \ldots, c(n)$.

Our proof is complete if we can show that (N, A) does not have a clique of size k iff B is a matcher. Notice that

$$|V_2| = 2|A| + (2k)^2 - \sum_{n \in N} deg(n) = 4k^2 = |V_1|.$$

Moreover, by Hall's theorem B is a matcher iff $|\Gamma(S)| \geq |S|$ for all $S \subset V_1, |S| \leq \frac{1}{2}|V_1| = 2k^2$. Thus, it suffices to prove the following claim.
Claim. There exists $S \subset V_1$ such that $|S| \leq 2k^2$ and $|\Gamma(S)| < |S|$ iff (N, A) has a clique of size k.
$(\Rightarrow)$. We first make some observation as follows. Let

$$C = \{n \in N \mid (n, i) \in S \text{ for some } i\}.$$

Let $x = |C|$ and $y = |\{a \in A \mid a = (n_1, n_2) \text{ for some } n_1, n_2 \in C\}|$. Notice that $1 \leq x \leq 2k$, and that

$$\begin{aligned} |\Gamma(S)| &= |\{a \in A \mid a \text{ is incident to some } n \in C\}| + \\ &\quad \binom{k}{2} - 1 + \\ &\quad 2k|C| - \sum_{n \in C} deg(n) \\ &= \binom{k}{2} - 1 + 2kx - y \end{aligned}$$

Now, $|\Gamma(S)| < |S| \leq 2kx$ implies $y > \binom{k}{2} - 1$. However, y is exactly the number of edges of a graph with x vertices, making $y \leq \binom{x}{2}$. Hence, $k \leq x \leq 2k$. Moreover, if $x > k$ then as $\binom{k}{2} - 1 + 2kx - \binom{x}{2}$ is a strictly increasing function in $[1, 2k]$, we get a contradiction:

$$2k^2 \geq |S| > \binom{k}{2} - 1 + 2kx - \binom{x}{2} \geq 1 + \binom{k}{2} - 1 + 2k^2 - \binom{k}{2} = 2k^2$$

Consequently, $x = k$. This implies $y = \binom{k}{2}$, so that C is a k-clique.
$(\Leftarrow)$. Let C be a clique of size k. Let $S = C \times [2k]$, then

$$|\Gamma(S)| = \binom{k}{2} - 1 + 2k^2 - \binom{k}{2} < |S|$$

□

It is easy to see that the problems of determining if a graph is a "concentrator" or "superconcentrator" are in coNP. To prove in polynomial time that a graph is not a concentrator, we only need to present a vertex cut C which separates a set of more than $|C|$ inputs to the outputs. Menger's Theorem ensures that this cut must exists. The proof that a graph is not a super concentrator is similar. Consequently, to show that they are coNP-complete, we can just reduce them to the "matcher" problem.

Corollary 2.3. *Deciding whether a graph G is a concentrator is coNP-complete*

Proof. We reduce "matcher" to "concentrator". Given a bipartite graph $B = (V_1, V_2, E)$ with $|V_i| = 2m$, we construct G by orienting all edges from V_1 to V_2, add a set V_3 of m vertices, and then connect all vertices of V_2 to all vertices of V_3. Clearly B is a matcher iff G is a concentrator. □

Corollary 2.4. *Deciding whether a graph G is a superconcentrator is coNP-complete.*

Proof. In the previous proof, change the size of V_3 to be $2m$. □

Corollary 2.5. *Deciding whether a graph G is an $(n, d, 0)$-expander is coNP-complete.*

3 Expanders

The main objective of this section is to present several results on the eigenvalue characterization of the expansion rate of expanders. Moreover, we shall give a partial answer to the question: "how large can the second smallest eigenvalue of the Laplacian of a graph be?", and give some pointers to results on direct constructions of expanders.

3.1 Algebraic graph theory

We first fix notations and terminologies from algebraic graph theory needed for the rest of the section. The reader is referred to the standard books by Biggs (1993, [18]), Godsil (1993, [36]), Cvetković, Doob, and Sachs (1995, [27]), and Chung (1997, [25]) for more information.

Let $G = (V, E)$ be a simple graph with n vertices, we will always assume the eigenvalues of G are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Those are the eigenvalues of the adjacency matrix A of G. Moreover, $\lambda(G)$ will be used to denote λ_2. Let D be the $n \times n$ diagonal matrix indexed by vertices of G with $(D)_{vv} = deg(v)$; then, the matrix $L := D - A$ is called the *Laplacian* matrix of G. We shall use $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ to denote the eigenvalues of L. In contrast to the λ's, we use $\mu(G)$ to denote μ_{n-1}. The reason for this is that when G is d-regular then $\lambda_i = d - \mu_{n-i+1}$.

Let N be the incident matrix of any orientation H of $G(V, E)$. Let $L^2(V)$ ($L^2(E)$) be the space of real valued functions on V (E), with the usual inner product $\langle f, g\rangle$ and the usual norm $\|f\| = \sqrt{\langle f, f\rangle}$. Note that $L^2(V)$ is isomorphic to $\mathbb{R}^n$ and the Rayleigh quotient for f is $R_A(f) = \frac{\langle Lf, f\rangle}{\|f\|^2}$. Also note that

$$\begin{aligned}
\langle Lf, f\rangle &= \langle N^T N f, f\rangle = \langle Nf, Nf\rangle \\
&= \sum_{(u,v)\in E(H)} (f(u) - f(v))^2 \\
&= \sum_{u\sim v} (f(u) - f(v))^2
\end{aligned}$$

Here $u \sim v$ is used in lieu of $(u, v) \in E$ for convenience. The previous equation implies that L is non-negative definite, thus $\mu_i \geq 0,\ \forall i$. Moreover, it is not hard to see that μ_n is always 0 and that $\mu_{n-1} = 0$ iff G is not connected.

The following bounds on μ_{n-1} will be used quite often.

Proposition 3.1. *Let $f \in L^2(V)$ such that $\sum_v f(v) = 0$, then*

$$\mu(G) \leq \frac{\langle Lf, f\rangle}{\|f\|} = \frac{\displaystyle\sum_{u\sim v}(f(u) - f(v))^2}{\displaystyle\sum_v f^2(v)}$$

In fact, a stronger statement holds

$$\mu(G) = \min_{f \neq 0} \frac{\langle Lf, f\rangle}{\|f\|}$$

with the min *runs over all f satisfying $\sum_v f(v) = 0$.*

Note. $\displaystyle\sum_{u\sim v}(f(u) - f(v))^2$ is sometime called the *Dirichlet sum* of G.

Proof. Let $u_n = \mathbf{1}/\sqrt{n}$ be a unit μ_n-eigenvector of L, then the variational characterization of the eigenvalues (see [37], e.g.) gives

$$\mu_{n-1} = \min_{\substack{0 \neq f \in \mathbb{C}^n \\ f \perp u_n}} R_L(f) = \min_{\substack{0 \neq f \in \mathbb{C}^n \\ f \perp \mathbf{1}}} \frac{\langle Lf, f\rangle}{\|f\|}$$

The condition $f \perp \mathbf{1}$ is the same as $\sum_u f(u) = 0$. ☐

Proposition 3.2. *Suppose G is connected, and $f \in L^2(V)$ is a $\mu(G)$-eigenvector. Let $V^+ := \{v \in V \mid f(v) > 0\}$ and $V^- := V - V^+$. Let $g \in L^2(V)$ be defined by*

$$g(v) = \begin{cases} f(v) & \text{if } v \in V^+ \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have

$$\mu \geq \frac{\displaystyle\sum_{u\sim v}(g(u) - g(v))^2}{\displaystyle\sum_v g^2(v)}$$

Proof. Note that since G is connected, $\mu \neq 0$, making $f \neq 0$. Hence, $V^+ \neq \emptyset$. By definition, we have $(Lf)(v) = \mu f(v), \forall v \in V$. Thus,

$$\mu = \frac{\sum_{v \in V^+} (Lf)(v) f(v)}{\sum_{v \in V^+} f^2(v)}$$

But,

$$\sum_{v \in V^+} f^2(v) = \sum_{v \in V} g^2(v)$$

and,

$$\begin{aligned}
\sum_{v \in V^+} (Lf)(v) f(v) &= \sum_{v \in V^+} \left(d(v) f^2(v) - \sum_{u \in \Gamma(v)} f(v) f(u) \right) \\
&= \sum_{uv \in E(V^+)} (f(u) - f(v))^2 + \sum_{uv \in E(V^+, V^-)} f(u)(f(u) - f(v)) \\
&\geq \sum_{u \sim v} (g(u) - g(v))^2
\end{aligned}$$

which completes our proof. □

3.2 The eigenvalue characterization of expansion rate for regular strong expanders

The results in this section are from Tanner (1984, [83]), Alon and Milman (1985, [9]), and Alon (1986, [6]). As the title of the section indicated, our goal is to show that the larger the expansion rate of a regular strong expander G is, the larger $\mu(G)$ has to be, and vice versa.

We again need to define several types of graphs closely related to expanders: enlargers, magnifiers, and bounded concentrators.

Definition 3.3. An (n, d, ϵ)-*enlarger* is a graph G on n vertices with maximum degree d and $\mu_{n-1}(G) \geq \epsilon$.

Definition 3.4. An (n, d, c)-*magnifier* is a graph $G = (V, E)$ on n vertices, $\Delta(G) = d$ and for every $X \subset V$ with $|X| \leq n/2$, $|\Gamma(X) - X| \geq c|X|$ holds. The *extended double cover* of a graph $G = (V, E)$ with $V = [n]$ is the bipartite graph H on the sets of inputs $X = \{x_1, \ldots, x_n\}$ and outputs $Y = \{y_1, \ldots, y_n\}$ so that $(x_i, y_j) \in E(H)$ iff $i = j$ or $(i, j) \in E(G)$.

Definition 3.5. An $(n, \theta, d, \alpha, c)$-*bounded strong concentrator* is a bipartite graph $G = (I, O; E)$ with n inputs, θn outputs, $\theta < 1$, and at most dn edges, such that if $X \subseteq I$ with $|X| \leq \alpha n$, then $|\Gamma(X)| \geq c|X|$. An (n, θ, d, α)-*bounded concentrator* is an $(n, \theta, d, \alpha, 1)$-*bounded strong concentrator*.

For the purpose of this section, enlargers were introduced just to shorten the sentence: "let G be a graph with maximum degree d and $\mu(G)$ large enough." On the other hand, we need magnifiers because their extended double covers are expanders. Magnifiers are, in a sense, the non-bipartite version of expanders. It is intuitively clear, and will be made precise later, that bounded concentrators are closely related to expanders. The proof of the following lemma is straightforward, hence omitted.

Lemma 3.6. *The double cover of an (n, d, c)-magnifier is an $(n, d{+}1, c)$-expander.*

The following theorem, which is an improved version of Theorem 3.4 in [6], makes our goal precise.

Theorem 3.7. *Let $G = (I, O; E)$ be a d-regular bipartite graph, where $|I| = |O| = n$ and $\mu = \mu(G)$ (= $d - \lambda(G)$). The following hold:*

(i) If G is an (n, d, c) strong expander then $\mu \geq \frac{c^2}{576-48c+2c^2}$.

(ii) If $\mu \geq \epsilon$, then G is an (n, d, c)-expander, where $c = \frac{2d\epsilon-\epsilon^2}{d^2}$.

We shall prove this theorem using a sequence of lemmas, with the following plan in mind. To show (i), we will show

G is a strong expander $\rightarrow$ G is a magnifier $\rightarrow$ G is an enlarger (i.e. $\mu(G)$ is large)

Similarly, to prove (ii) we shall show the reverse sequence

G is an enlarger $\rightarrow$ G is a strong expander

Let us now proceed to show that a strong expander is a magnifier.

Lemma 3.8. *Let $G = (I, O; E)$ be an (n, d, c)-strong-expander. Then G is a $(2n, d, \frac{c}{12-c})$-magnifier.*

Proof. Be definition, every $X_1 \subset I$ satisfies

$$|\Gamma(X_1)| \geq \left(1 + c(1 - \frac{|X_1|}{n})\right)|X_1|. \tag{1}$$

In particular,

$$|\Gamma(X_1)| \geq |X_1| \tag{2}$$

When $|X_1| = n/2$, (1) implies $c \leq 2$. Moreover, for every $X_2 \subseteq O$, setting $X_1 = I - \Gamma(X_2)$ in (2) yields $\Gamma(X_2) \geq |X_2|$.

We are to show that for every $X \subset I \cup O$ with size at most n, it must be the case that

$$|\Gamma(X) - X| \geq \frac{c}{12-c}|X|$$

Let $X_1 = X \cap I$ and $X_2 = X \cap O$. The intuition is that when $|X_1|$ is very small comparing to $|X_2|$, then there will be a lot of neighbors of X_2 lying outside of X_1 in I, making $|\Gamma(X) - X|$ large. On the other hand, when $|X_1|$ is relatively large comparing to $|X_2|$, then there will be "many" neighbors of X_1 not in X_2. We now turn this intuition into mathematical rigor.

When $|X_1| \leq |X_2|(1 - \frac{c}{6})$, we have

$$|X| \leq \frac{12-c}{6}|X_2|.$$

Hence,

$$|\Gamma(X) - X| \geq |\Gamma(X_2)| - |X_1| \geq |X_2| - |X_1| \geq \frac{c}{6}|X_2| \geq \frac{c}{12-c}|X|.$$

When $|X_2|(1 - \frac{c}{6}) < |X_1| \leq \frac{n}{2}$, we get

$$|X| < \left(1 + \frac{1}{1-\frac{c}{6}}\right)|X_1| = \frac{12-c}{6-c}|X_1|.$$

This relation, the fact that $c \leq 2$, and (1) yield

$$\begin{aligned}
|\Gamma(X) - X| &\geq |\Gamma(X_1)| - |X_2| \\
&\geq (1 + \frac{c}{2} - \frac{1}{1-\frac{c}{6}})|X_1| \\
&= \frac{4-c}{2}\frac{c}{6-c}|X_1| \\
&> \frac{4-c}{2}\frac{c}{12-c}|X| \\
&\geq \frac{c}{12-c}|X|
\end{aligned}$$

Lastly, when $|X_1| \geq \frac{n}{2}$ it follows that

$$|\Gamma(X) - X| \geq |\Gamma(X_1)| - |X_2| \geq (1 + \frac{c}{2})\frac{n}{2} - \frac{n}{2} = \frac{c}{4}n \geq \frac{c}{4}|X| \geq \frac{c}{12-c}|X|.$$

Here, we use the fact that $c \leq 2$ and that the function $f(x) = (1 + c(1 - \frac{x}{n}))x$ is increasing when $n \geq 2$. □

Next, to complete part (i) of Theorem 3.7 we show that every magnifier has "large" μ.

Lemma 3.9. *Let $G = (V, E)$ be an (n, d, c)-magnifier, then G is an (n, d, ϵ)-enlarger where $\epsilon = \frac{c^2}{2+2(c+1)^2}$.*

Proof. We apply and use notations of Proposition 3.2. Since we have to show that $\mu \geq \epsilon = \frac{c^2}{2+2(c+1)^2}$, it suffices to show that

$$\frac{\sum\limits_{u \sim v}(g(u) - g(v))^2}{\sum\limits_{v} g^2(v)} \geq \frac{c^2}{2 + 2(c+1)^2}$$

This is done by using the maxflow-mincut theorem. Consider a network N with vertex set $\{s\} \uplus V^+ \uplus V \uplus \{t\}$, where s is the source, t is the sink, and $\uplus$ denotes the disjoint union. The edges and their capacities are defined as follows.

(a) For each $u \in V^+$, (s, u) has capacity $cap(s, u) = 1 + c$.

(b) For each pair $(u, v) \in V^+ \times V$, $cap(u, v) = \begin{cases} 1 & \text{if } uv \in E \text{ or } u = v \\ 0 & \text{otherwise.} \end{cases}$

(c) For each $v \in V$, $cap(v, t) = 1$.

We claim that the min-cut of N has capacity $(1 + c)|V^+|$. Consider any cut C. Let $X := \{u \in V^+ \mid (s, u) \notin C\}$. As G has magnifying rate c and $\Gamma_N(X) = X \cup \Gamma_G(X)$, it is easy to see that $|\Gamma_N(X)| \geq (1 + c)|X|$. Moreover, for every $v \in \Gamma_N(X)$ there must be at least one edge with capacity one in C incident to it. All these edges are disjoint, thus the capacity of C is at least $(1+c)(|V^+|-|X|) + |\Gamma_N(X)| \geq (1 + c)|V^+|$. Lastly, the cut $\{(s, u) \mid u \in V^+\}$ has capacity exactly $(1 + c)|V^+|$, proving the claim. By the maxflow-mincut theorem, there exists an orientation $\bar{E}$ of edges in G and a function $h : \bar{E} \to \mathbb{R}$ such that

$$0 \leq h(u, v) \leq 1 \text{ for all } (u, v) \in \bar{E}$$

$$\sum_{v:(u,v)\in\bar{E}} h(u,v) = \begin{cases} 1+c & \text{if } u \in V^+ \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{u:(u,v)\in\bar{E}} h(u,v) \le 1 \text{ for all } v \in V$$

Now, the following is straightforward:

$$\begin{aligned} \sum_{(u,v)\in\bar{E}} h^2(u,v)\big(g(u)+g(v)\big)^2 &\le 2\sum_{(u,v)\in\bar{E}} h^2(u,v)\big(g^2(u)+g^2(v)\big) \\ &= 2\sum_{v\in V} g^2(v)\Bigg(\sum_{u:(u,v)\in\bar{E}} h^2(u,v) + \sum_{u:(v,u)\in\bar{E}} h^2(v,u)\Bigg) \\ &\le 2(1+(1+c)^2)\sum_{v\in V} g^2(v) \end{aligned}$$

and

$$\begin{aligned} \sum_{(u,v)\in\bar{E}} h(u,v)\big(g^2(u)-g^2(v)\big) &= \sum_{u\in V} g^2(u)\Bigg(\sum_{v:(u,v)\in\bar{E}} h(u,v) - \sum_{v:(v,u)\in\bar{E}} h(v,u)\Bigg) \\ &\ge c\sum_{v\in V} g^2(v) \end{aligned}$$

Now, combining all inequalities above along with Cauchy-Schwarz inequality we get

$$
\begin{aligned}
\mu &\geq \frac{\sum_{u\sim v}\big(g(u)-g(v)\big)^2}{\sum_v g^2(v)} \\
&= \frac{\sum_{u\sim v}\big(g(u)-g(v)\big)^2 \sum_{(u,v)\in\bar{E}} h^2(u,v)\big(g(u)+g(v)\big)^2}{\sum_v g^2(v) \sum_{(u,v)\in\bar{E}} h^2(u,v)\big(g(u)+g(v)\big)^2} \\
&\geq \frac{\left(\sum_{(u,v)\in\bar{E}} h(u,v)|g^2(u)-g^2(v)|\right)^2}{2(1+(c+1)^2)\big(\sum_{v\in V} g^2(v)\big)^2} \\
&\geq \frac{1}{2+2(c+1)^2}\left(\frac{\sum_{(u,v)\in\bar{E}} h(u,v)(g^2(u)-g^2(v))}{\sum_{v\in V} g^2(v)}\right)^2 \\
&\geq \frac{c^2}{2+2(c+1)^2}
\end{aligned}
$$

□

The previous two lemmas trivially imply part (i) of Theorem 3.7. Now we are ready to complete part (ii) of Theorem 3.7. We first need a lemma from [83]. Let $G = (I, O; E)$ be a bipartite graph such that $|I| = n$, $|O| = m$, and that $deg(i) = a, \forall i \in I$, $deg(o) = b$, $\forall o \in O$. Let M be an $n \times m$ 01-matrix whose rows are indexed by I and whose columns are indexed by O such that $m_{io} = 1$ if $io \in E$ and $m_{io} = 0$ otherwise. Let $B = MM^T$, then clearly B is real, symmetric, and non-negative definite. As usual, we let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be the eigenvalues of B and $u_1, u_2, \ldots, u_n$ be a set of corresponding orthonormal eigenvectors. Notice that $(B)_{ij}$ is the number of common neighbors of i and j, hence $\sum_j (B)_{ij} = ab, \forall i$. (Each neighbor of i is counted b times in the sum.) This implies $\mathbf{1}/\sqrt{n}$ is an eigenvector of B with corresponding eigenvalue ab. Suppose θ is any eigenvalue of B and e is a θ-eigenvector. Let e_i be a coordinate of e with highest absolute value, then $(Be)_i = \theta e_i$ implies

$$\theta|e_i| = |\sum_j (B)_{ij} e_j| \leq |e_i| \sum_j (B)_{ij} = ab|e_i|$$

Consequently, ab is also the largest eigenvalue of B, i.e. $\theta_1 = ab$. We may thus assume that $u_1 = \mathbf{1}/\sqrt{n}$.

Lemma 3.10. *If $\theta_1 > \theta_2$, then G is an $(n, m/n, a, \alpha, c(\alpha))$-bounded strong concentrator, where*

$$c(\alpha) \geq \frac{a^2}{\alpha(ab - \theta_2) + \theta_2}$$

Proof. Let β be a positive real number not exceeding α. For any $X \subseteq I$ with $|X| = \beta n$, let $x \in \{0,1\}^n$ be X's characteristic vector. Clearly $xx^T = \|x\|^2 = \beta n$. Similarly, let $Y = \Gamma(X)$ and y be its characteristic vector. As for any $o \in O$, $(xM)_o$ is the number of vertices in X adjacent to o, the sum of entries in xM is exactly βna. Hence,

$$\|xM\|^2 = \sum_{o \in O} (xM)_o^2 \geq \left(\frac{\sum_{o \in O} (xM)_o}{|Y|} \right)^2 |Y| = \frac{\beta^2 n^2 a^2}{|Y|} \tag{3}$$

Now, write $x = \gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_n u_n$ we get

$$xB = \gamma_1 \theta_1 u_1 + \gamma_2 \theta_2 u_2 + \cdots + \gamma_n \theta_n u_n.$$

Thus, by orthonormality we have

$$\|xM\|^2 = (xB)x^T = \theta_1 \gamma_1^2 + \theta_2 \gamma_2^2 + \cdots + \theta_n \gamma_n^2 \tag{4}$$

Now,

$$\gamma_1 = x u_1^T = \frac{\sum_i x_i}{\sqrt{n}} = \beta \sqrt{n}$$

Hence, (4) gives

$$\begin{aligned} \|xM\|^2 &= \theta_1 \gamma_1^2 + \sum_{j=2}^{n} \theta_j \gamma_j^2 \\ &\leq ab\beta^2 n + \theta_2 (\|x\|^2 - \gamma_1^2) \\ &= \beta^2 n (ab - \theta_2) + \theta_2 \beta n \end{aligned} \tag{5}$$

Lastly, combining (3) and (5) yield

$$\frac{|\Gamma(X)|}{|X|} = \frac{|Y|}{\beta n} \geq \frac{a^2}{\beta(ab - \theta_2) + \theta_2}$$

As this inequality is true for all $\beta \leq \alpha$, the proof is completed. □

Part (ii) of Theorem 3.7 could now be derived as a corollary of this lemma.

Corollary 3.11. *If $G = (I, O; E)$ is a d-regular bipartite graph with $|I| = |O| = n$ and $\mu = \mu(G)$, then*

(i) G is an (n, d, c)-strong expander, where

$$c = \frac{2d\mu - \mu^2}{d^2}$$

(ii) G is an (n, d, c)-expander, where

$$c = \frac{2d\mu - \mu^2}{d^2 - d\mu + \mu^2/2}$$

Proof. Let M and B be the matrices for G as in Lemma 3.10, and let A be G's adjacency matrix. Also let $\theta_1 \geq \cdots \geq \theta_n$ be the eigenvalues of B and $\lambda_1 \geq \cdots \geq \lambda_{2n}$ be the eigenvalues of A as usual. Since $(B)_{ij} = (MM^T)_{ij}$ counts the number of common neighbors of i and j, while $(A^2)_{ij}$ counts the number of length-2 paths from i to j, it is obvious that

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} = A^2$$

It is standard that as G is bipartite, whenever λ is an eigenvalue of A, then so is $-\lambda$. Moreover, $\lambda_1 = d$ and $\lambda_2 = d - \mu$ since G is d-regular. Thus, $\theta_1 = d^2$ and $\theta_2 = (d - \mu)^2$. Now, for any $X \subseteq I$ with $\alpha = |X|/n$, applying Lemma 3.10 gives

$$\begin{aligned} |\Gamma(X)| &\geq \frac{d^2}{\alpha\big(d^2 - (d-\mu)^2\big) + (d-\mu)^2}|X| \\ &= \left(1 + \frac{(2d\mu - \mu^2)(1-\alpha)}{d^2 - (2d\mu - \mu^2)(1-\alpha)}\right)|X| \\ &\geq \left(1 + \frac{(2d\mu - \mu^2)}{d^2}\left(1 - \frac{|X|}{n}\right)\right)|X| \end{aligned}$$

When $|X| \leq n/2$, we get

$$|\Gamma(X)| \geq \left(1 + \frac{(2d\mu - \mu^2)}{d^2 - d\mu + \mu^2/2}\left(1 - \frac{|X|}{n}\right)\right)|X|$$

□

3.3 On the second eigenvalue of a graph

As we have seen, the expanding rate of a graph increases as its second smallest Laplacian eigenvalue increases. Thus, it makes sense to study how large μ_{n-1} can be. Through out this section, we shall consider only d-regular graphs and let $\mu = \mu(G) = \mu_{n-1}(G) = d - \lambda_2(G)$. Alon and Boppana (mentioned in [7]) proved that for any fixed d and for any infinite family of graphs G with maximum degree d,

$$\limsup \mu(G) \leq d - 2\sqrt{d-1}$$

The bound is sharp when $d-1$ is a prime congruent to 1 modulo 4, as shown by the explicit constructions of Lubotzky, Phillips and Sarnak [51], and independently by Margulis [53]. Alon [7] conjectured the following

Conjecture 3.12 (Alon [7]). As $n \to \infty$, the probability that $\mu(G) \leq d - 2\sqrt{d-1} - o(1)$ goes to 1.

In other words, as n gets large, with very high probability we have $\mu(G) \leq d - 2\sqrt{d-1}$. The conjecture is still open as far as we know. Friedman, Kahn and Szemeédi (1989, [33]) showed that

$$\mu(G) \geq d - 2\sqrt{d-1} - \log d - o(1)$$

as $n \to \infty$. Nilli (1991, [61]) got an upper bound that does not have any hidden constant:

Theorem 3.13 (Nilli, 1991 [61]). *Let G be a graph in which there are two edges with distance at least $2k+2$, and let d be the maximum degree of G. Then,*

$$\mu(G) \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1}$$

Proof. Let (u_1, u_2) and (w_1, w_2) be two edges with distance at least $2k+2$. Let L be the Laplacian matrix of G as usual. Let $U_0 = \{u_1, u_2\}$ and $W_0 = \{w_1, w_2\}$. For $1 \leq i \leq k$, let U_i (resp. W_i) be the set of vertices of distance i from U_0 (resp. W_0). Clearly $U := \cup_{0 \leq i \leq k} U_i$ has distance at least 2 from $W := \cup_{0 \leq i \leq k} W_i$, so that there is no edge joining the two unions. Moreover, it is easy to see that $|U_i| \leq (d-1)|U_{i-1}|$ and $|W_i| \leq (d-1)|W_{i-1}|$.

Let $f : V(G) \to \mathbb{R}$ be defined as

$$f(v) = \begin{cases} a(d-1)^{-i/2} & \text{for } v \in U_i, 0 \leq i \leq k \\ b(d-1)^{-i/2} & \text{for } v \in W_i, 0 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

where $a > 0$, $b < 0$ are real numbers such that $\sum_v f(v) = 0$. The variational characterization yields

$$\mu(G) = \mu_{n-1} \leq \frac{\langle Lf, f\rangle}{\langle f, f\rangle}$$

We also have

$$\begin{aligned}
\langle f, f\rangle &= \sum_{u\in U} f^2(u) + \sum_{w\in W} f^2(w)\\
&= \sum_{i=0}^{k} |U_i| \frac{a^2}{(d-1)^i} + \sum_{i=0}^{k} |W_i| \frac{b^2}{(d-1)^i}\\
&= A_1 + B_1
\end{aligned}$$

where A_1 and B_1 are the first and second sum, respectively. Moreover, as we have mentioned: (a) there are no edges joining U and W; (b) there are no edges connecting U_i or W_i to $V(G) - U \cup W$ if $i \leq k-1$; and (c) there are at most $d-1$ edges joining a vertex of U_i (W_i) to a vertex of U_{i+1} (W_{i+1}), we have

$$\begin{aligned}
\langle Lf, f\rangle &= \sum_{u\sim v} (f(u) - f(v))^2\\
&= \sum_{\substack{u\sim v\\ \{u,v\}\cap U\neq\emptyset}} (f(u) - f(v))^2 + \sum_{\substack{u\sim v\\ \{u,v\}\cap W\neq\emptyset}} (f(u) - f(v))^2\\
&= \sum_{i=0}^{k-1} \sum_{\substack{u\sim v\\ u\in U_i, v\in U_{i+1}}} (f(u) - f(v))^2 + \sum_{i=0}^{k-1} \sum_{\substack{u\sim v\\ u\in W_i, v\in W_{i+1}}} (f(u) - f(v))^2\\
&\quad + \sum_{\substack{u\sim v\\ u\in U_k, v\in V-U\cup W}} (f(u))^2 + \sum_{\substack{u\sim v\\ u\in W_k, v\in V-U\cup W}} (f(u))^2\\
&\leq a^2 \left(\sum_{i=0}^{k-1} |U_i|(d-1)\left(\frac{1}{(d-1)^{i/2}} - \frac{1}{(d-1)^{(i+1)/2}}\right)^2 + |U_k| \frac{d-1}{(d-1)^k}\right) +\\
&\quad b^2 \left(\sum_{i=0}^{k-1} |W_i|(d-1)\left(\frac{1}{(d-1)^{i/2}} - \frac{1}{(d-1)^{(i+1)/2}}\right)^2 + |W_k| \frac{d-1}{(d-1)^k}\right)\\
&= a^2 \left(\sum_{i=0}^{k} \frac{|U_i|}{(d-1)^i}(d - 2\sqrt{d-1}) + (2\sqrt{d-1} - 1)\frac{|U_k|}{(d-1)^k}\right) +\\
&\quad b^2 \left(\sum_{i=0}^{k} \frac{|W_i|}{(d-1)^i}(d - 2\sqrt{d-1}) + (2\sqrt{d-1} - 1)\frac{|W_k|}{(d-1)^k}\right)\\
&= A_2 + B_2
\end{aligned}$$

with A_2 and B_2 defined in the obvious way. To this end, we are left to show that

$$\frac{A_2 + B_2}{A_1 + B_1} \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k+1} =: C$$

We shall show that $A_2/A_1 \leq C$ and $B_2/B_1 \leq C$ instead. Notice that $\frac{|U_i|}{(d-1)^i} \geq \frac{|U_{i+1}|}{(d-1)^{i+1}}$, clearly

$$A_1 = \sum_{i=0}^{k} |U_i| \frac{a^2}{(d-1)^i} \geq a^2(k+1)\frac{|U_k|}{(d-1)^k}$$

hence,

$$\frac{A_2}{A_1} = d - 2\sqrt{d-1} + a^2 \frac{2\sqrt{d-1} - 1}{A_1} \frac{|W_k|}{(d-1)^k} \leq C$$

$B_2/B_1 \leq C$ is proved similarly.

□

Corollary 3.14. *Let G, d and k be defined as in the previous theorem, If G is d regular, then*

$$\lambda_2(G) \geq 2\sqrt{d-1}\left(1 - \frac{1}{k+1}\right) + \frac{1}{k+1}$$

3.4 Explicit Constructions of Expanders

Many practical applications require explicit constructions of expander graphs. Explicit constructions turn out to be a lot more difficult than showing existence. In 1973, Margulis [52] gave the first explicit construction of (strong) expanders. He explicitly constructed a family of bipartite graphs $\{\hat{G}_n\}$ for $n = m^2, m = 1, 2, \ldots,$ and show that there is a constant $k > 0$ such that for each $n = m^2$, $m \in \mathbb{N}$, $\hat{G}_n$ is an $(n, 5, k)$-strong expander. This construction was certainly undesirable as the constant k was not known. Moreover, his proof used deep results from Representation Theory. Angluin (1979, [12]) pointed out that Margulis' method could be used to construct $(n, 3, k')$-strong expanders but the constant k' is also not known. Gabber and Galil (1981, [34]) slightly modified Margulis' construction and used relatively elementary Taylor analysis to show how to construct a family of $(m^2, 5, (2-\sqrt{3})/4)$-strong expanders for each $m \in \mathbb{N}$. They also constructed a family of $(m^2, 7, (2-\sqrt{3})/2)$-strong expanders. Let us mention here their first construction. For each $m \in \mathbb{N}$, construct an $m^2 \times m^2$ bipartite graph

$\bar{G}_{m^2} = (I_m, O_m; E)$ where $I_m = O_m = \mathbb{Z}_m \times \mathbb{Z}_m$ and each vertex $(i,j) \in I_m$ is connected to 5 vertices in O_m defined by the following permutations:

$$\begin{aligned}
\sigma_1(i,j) &= (i,j)\\
\sigma_2(i,j) &= (i,i+j)\\
\sigma_3(i,j) &= (i,i+j+1)\\
\sigma_4(i,j) &= (i+j,j)\\
\sigma_5(i,j) &= (i+j+1,j)
\end{aligned}$$

Here, the additions are done modulo m. As we have mentioned, there is no known elementary proof that this family are expanders with the prescribed expansion rate. This construction was later modified slightly by Jimbo and Maruoka (1987, [44]) and Alon, Galil and Milman (1987, [8]) to obtain better superconcentrators.

As we have discussed in the introduction, after the eigenvalue characterization of expansion rate, the main problem was to construct Ramanujan graphs, which are optimal expanders in the eigenvalue sense. The special case where $d-1$ is a prime congruent to 1 modulo 4 was "solved" by Lubotzky, Phillips, and Sarnak (1988, [51]), and independently by Margulis (1988, [53]). Later, in 1994 Morgenstern [58, 57] constructed for every prime power q many families of $(q+1)$-regular Ramanujan graphs. All these constructions were Cayley graphs of certain groups.

Other works and information on expanders could be found in [4, 50, 11, 2, 46, 79, 17, 25].

4 Concentrators and Superconcentrators

4.1 Linear Concentrators and Superconcentrators

Valiant (1975, [84]) showed that there exists n-superconcentrators of size at most $238n$ and depth $O(\log^2 n)$. Valiant's proof was based on a recursive construction using Pinsker's concentrator (1973, [65]). Pinsker showed that there exist n-concentrators with at most $29n$ edges. A somewhat weaker version of this result was also obtained independently by Pippenger (1973, [66]). In 1977, Pippenger [67] gave a simpler construction of n-superconcentrators with size at most $40n$, maximum degree 16, and depth $O(\log n)$, while Valiant and Pinsker's graphs did not have $O(1)$ degree bound. This was certainly a big improvement. On the same line of reasoning, with finer analysis Fan Chung (1979, [24]) showed that there exists n-concentrators of size at most $27n$ and n-superconcentrators of size at most $38.5n$. Bassalygo (1981, [13]) improved the n-concentrator bound to $20n$ and n-superconcentrator bound to $36n$.

We present here the results of Pippenger [67] and a small generalization given by Gabber and Galil [34], since although it is not the best result, it is fairly intuitive.

Lemma 4.1 (Pippenger, 1977). *For every m, there is a bipartite graph with $6m$ inputs and $4m$ outputs, in which every input has outdegree at most 6, every output has indegree at most 9, and, for every $k \le 3m$ and every set S of k inputs, there exists a matching from S into some k-subset of the outputs.*

Proof. Let $\mathcal{M} := \{0, 1, \ldots, 36m - 1\}$, and π be any permutation on $\mathcal{M}$. Let $G(\pi)$ be the bipartite graph obtained from π by taking $\{0, 1, \ldots, 6m - 1\}$ as inputs, $\{0, 1, \ldots, 4m - 1\}$ as outputs, and $E(G) := \{((x \bmod 6m), (\pi(x) \bmod 4m)) \mid x \in \mathcal{M}\}$.

We way that $G(\pi)$ is *good* if there are no $k \le 3m$, a k-subset A of the inputs, and a k-subset B of the outputs such that $\Gamma(A) \subseteq B$. We would like to look for a good graph, which will satisfy our criteria. The existence of good graphs is shown by proving that the probability of $G(\pi)$ being *bad* (i.e. not good) is strictly less than 1.

Any k-subset A (B) of the inputs (outputs) corresponds uniquely to a $6k$-subset ($9k$-subset) $\mathcal{A}$ ($\mathcal{B}$) of $\mathcal{M}$ (but not vice versa.) Moreover, $\Gamma(A) \subseteq B$ iff $\pi(x) \in \mathcal{B}, \forall x \in \mathcal{A}$. For fixed $\mathcal{A}$ and $\mathcal{B}$, there are $(9k)_{6k}(36m-6k)!$ permutations π which satisfy this condition. Hence, let P_m be the probability of $G(\pi)$ being *bad*, we have

$$\begin{aligned} P_m &\le \sum_{k=1}^{3m} \binom{6m}{k}\binom{4m}{k}\frac{(9k)_{6k}(36m-6k)!}{(36m)!} \\ &= \sum_{k=1}^{3m} \frac{\binom{6m}{k}\binom{4m}{k}\binom{9k}{6k}}{\binom{36m}{6k}} \\ &=: I_m \end{aligned}$$

What we want to show is $I_m < 1$. As I_m has $3m$ terms, we first want to bound the largest term. Combinatorially, it's easy to see that

$$\binom{36m}{6k} \ge \binom{6m}{k}\binom{4m}{k}\binom{26m}{4k}$$

Hence,

$$I_m \le \sum_{k=1}^{3m} \frac{\binom{9k}{6k}}{\binom{26m}{4k}} = \sum_{k=1}^{3m} L_k, \text{ where } L_k := \frac{\binom{9k}{6k}}{\binom{26m}{4k}}$$

It is not difficult to check that the sequence $1/L_k$ is unimodal, thus the largest L_k is either L_1 or L_{3m}.

As $3mL_1 < 1$ trivially, we can assume $L_{3m} > L_1$, so that I_m is at most

$$3mL_{3m} = 3m\frac{\binom{27m}{18m}}{\binom{26m}{12m}} = \frac{(27m)!(12m)!(14m)!}{(18m)!(9m)!(26m)!}$$

To this end, we use the following two inequalities. The first one is a good version of Stirling's formula (see Robbins [78]).

$$\sqrt{2\pi n}e^{-n}n^n \leq n! \leq e^{\frac{1}{12n}}\sqrt{2\pi n}e^{-n}n^n$$

and

$$e^x \leq \frac{1}{1-x}$$

Applying these inequalities yields $3mL_{3m} < 1$ for all $m \geq 3$. The cases where $m \leq 2$ are trivial. □

Corollary 4.2. *For every m, there is a bipartite graph with $4m$ inputs and $6m$ outputs, in which every input has outdegree at most 9, every output has indegree at most 6, and, for every $k \leq 3m$ and every set S of k outputs, there exists a matching from S into some k-subset of the inputs.*

Let $s(n)$ be the minimum size of an n-superconcentrator, and $\theta(n) := 4\lceil \frac{n}{6} \rceil$. We first obtain a recursive inequality for $s(n)$.

Lemma 4.3. *For any n, $s(n) \leq 13n + s(\theta(n))$.*

Proof. Let $m = \lceil \frac{n}{6} \rceil$, and G and G' be the graphs of Lemma 4.1 and Corollary 4.2 respectively. Let S' be a $\theta(n)$-superconcentrator with $s(\theta(n))$ edges. Construct an n-superconcentrator as shown in Figure 1. Clearly S has size at most $13n + s\big(\theta(n)\big)$.

□

Now we are ready for the main result:

Theorem 4.4 (Pippenger, 1977). *We have $s(n) \leq 39n + O(\log n)$. In fact, $s(n) \leq 40n$.*

Proof. The ternary Beneš network (see, e.g. [16]) gives

$$s(n) \leq 3n(2\lceil \log_3 n \rceil - 1)$$

because the Beneš network is rearrangeable, it is certainly a superconcentrator. This gives $s(n) \leq 39n$ for $n \leq 3^7 = 2187$.

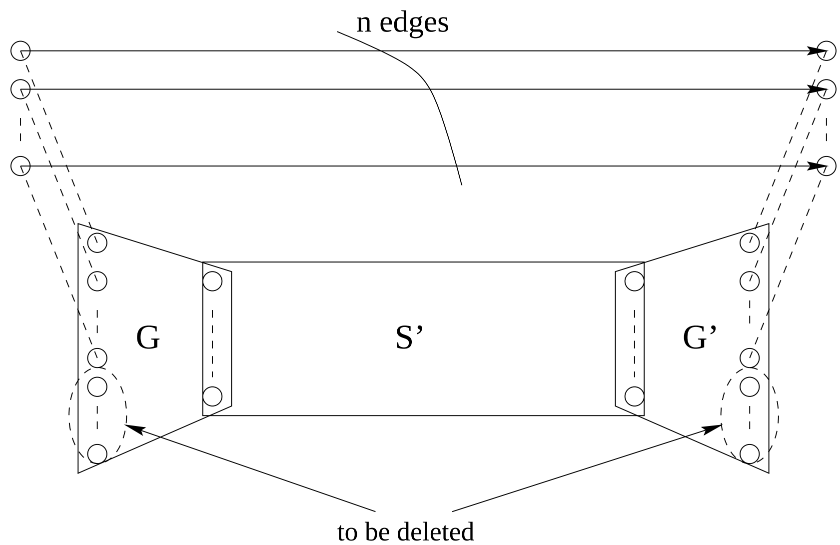

Figure 1: Recursive construction of an n-superconcentrator from a $\theta(n)$-superconcentrator

For larger values of n, we shall use the previous lemma. Define $\theta^0(n) = n$, $\theta^{t+1}(n) = \theta(\theta^t(n))$. Pick t such that $\theta^t(n) > 3^7 \geq \theta^{t+1}(n)$, then apply Lemma 4.3 $t+1$ times, we get

$$s(n) \leq 13(\theta^0(n) + \theta^1(n) + \cdots + \theta^t(n)) + s(\theta^{t+1}(n))$$

It is easy to show by induction on t that $\theta^t(n) \leq \left(\frac{2}{3}\right)^t n + 8$, which implies

$$s(n) \leq 39n + 104(t+3)$$

Moreover, as $\theta(n) \leq \frac{4384}{6561}n$, we have $3^7 < \theta^t(n) \leq \left(\frac{4384}{6561}\right)^t n$. Hence, $t = O(\log n)$ because $\frac{4384}{6561} < 1$. Finer analysis on t shows that $s(n) \leq 40n$. $\square$

Notice that the graph G of Lemma 4.1 is nothing but a $(6m, 2/3, 6, 1/2)$-bounded concentrator. The above construction can be straightforwardly generalized as follows.

Lemma 4.5 (Gabber-Galil, 1981 [34]). *A family of n-superconcentrator of density $\frac{2k+1}{1-\theta}$ can be constructed if for each n an $(n, \theta, k, 1/2)$ bounded concentrator is given.*

As we have already mentioned, Bassalygo (1981, [13]) improved this bound further to $20n$, however we shall not discuss that result here. One might wonder what is known about the lower bound of $s(n)$. Lev and Valiant (1983, [49]) provided the best known so far, whose proof is omitted here.

Theorem 4.6 (Lev and Valiant, 1983). *An n-superconcentrator whose inputs have indegree 0 and outputs have outdegree 0 has size at least $5n - o(n)$.*

The proof of this lower bound is quite involved, and certainly more difficult than the upper bound proof. It is fairly disturbing that the gap between the lower and upper bounds remain quite large. Let us put it as an open problem.

Open Problem 4.7. Close the gap of $5n - o(n) \leq s(n) \leq 20n + o(n)$.

4.2 Superconcentrators with a given depth

The following functions often show up in probabilistic arguments concerning the problem.

Definition 4.8. Let $O^*(f(n))$ denote $f(n)n^{o(1)}$.

Definition 4.9. Let

$$\log^* n := \min\{l \geq 0 \mid \underbrace{\log \ldots \log n}_{l} \leq 1\}$$

where the logarithms are to base 2. By induction on k, define

$$\log^{*(k)} n := \log^{\overbrace{* \cdots *}^{k}} n := \min\{l \geq 0 \mid \underbrace{\log^{\overbrace{* \cdots *}^{k-1}} \ldots \log^{\overbrace{* \cdots *}^{k-1}} n}_{l} \leq 1\}$$

The question of finding the minimum size of an n-superconcentrator with a given depth k was raised by Pippenger (1982, [71]). Let us denote this function by $s(n,k)$. Clearly $s(n,1) = n^2$. In the same paper, Pippenger showed that $\Omega(n \log n) = s(n,2) = O(n \log^2 n)$. Dolev, Dwork, Pippenger and Wigderson (1983, [29]) found that for even depth at least 4,

$$s(n,2k) = \Theta\big(n \log^{*(k-1)} n\big). \tag{6}$$

Rather surprisingly, Pudlák (1994, [76]) showed that for $k \geq 2$, it is also true that

$$s(n,2k+1) = \Theta\big(n \log^{*(k-1)} n\big). \tag{7}$$

Hence, when the depth is at least 4 the extra "odd" level does not help improve the superconcentrator size. Alon and Pudlák (1994, [10]) filled part of the gap by determining the minimum size for depth 3, proving

$$s(n,3) = \Theta(n \log n \log n) \tag{8}$$

They also improved Pippenger's lower bound for depth 2 superconcentrators to $s(n,2) = \Omega\big(n(\log n)^{\frac{3}{2}}\big)$. The only one case left where $s(n,k)$ has not been determined (up to a constant factor) is when $k = 2$, rather weird. One would think that the depth-2 case would be easier than larger depth cases. Finally, Radhakrishnan and Ta-Shma (2000, [77]) have determined the last value:

$$s(n,2) = \Theta\left(\frac{n \log^2 n}{\log \log n}\right) \tag{9}$$

We describe here only the initial results of Pippenger on $s(n,2)$ to get a feel of what is going on. In superconcentrators of depth 2, every path has length at most 2. By adding a vertex into every path of length 1, we increase the size of the superconcentrator by at most a factor of 2, which is irrelevant for our purposes. Hence, we may assume that the set of vertices V of our n-superconcentrator can be partitioned into three disjoint subset $V = I \uplus U \uplus O$, where U is the set of middle vertices called *links*.

Let $\mathcal{N}$ be an n-superconcentrator of size N and depth 2. Let X (Y) be a random set of inputs (outputs) where each $v \in X$ $(v \in Y)$ appears independently with probability p. Let the random variable x (resp. y) denote the cardinality of X (resp. Y). As $\mathcal{N}$ is a superconcentrator, there exists a set of $m = \min\{x, y\}$ vertex disjoint paths joining X and Y. We first get a lower bound for $E[m]$.

Lemma 4.10. *With the notations just introduced, we have*

$$E[m] \geq np + O\big((np)^{\frac{2}{3}}\big)$$

Proof. Applying Markov's inequality, with $\epsilon \in [0,1]$ to be chosen, we get

$$\begin{aligned} E[m] &\geq np(1-\epsilon)P[m \geq np(1-\epsilon)] \\ &= np(1-\epsilon)P[x \geq np(1-\epsilon), y \geq np(1-\epsilon)] \\ &= np(1-\epsilon)P^2[x \geq np(1-\epsilon)] \end{aligned} \tag{10}$$

As x is the sum of n random indicators, $E[x] = np$ and $Var[x] = np(1-p)$. Now, Chebyshev's inequality gives

$$\begin{aligned} P[x < np(1-\epsilon)] &\leq P[|x - np| \geq np\epsilon] \\ &\leq \frac{Var[x]}{(np\epsilon)^2} \\ &\leq \frac{1}{np\epsilon^2} \end{aligned} \tag{11}$$

Combining inequalities (10) and (11), then set $\epsilon = \left(\frac{2}{np}\right)^{\frac{1}{3}}$ gives

$$\begin{aligned} E[m] &\geq np(1-\epsilon)\left(1 - \frac{1}{np\epsilon^2}\right)^2 \\ &\geq np(1-\epsilon)\left(1 - \frac{2}{np\epsilon^2}\right) \\ &= np\left(1 - (\frac{2}{np})^{\frac{1}{3}}\right)^2 \\ &\geq np\left(1 - 2(\frac{2}{np})^{\frac{1}{3}}\right) \\ &= np + O\left((np)^{\frac{2}{3}}\right) \end{aligned}$$

□

For each $u \in U$, let f_u and g_u be the number of edges directed into and out of u respectively. Then clearly

$$N = \sum_u (f_u + g_u)$$

We say that a link u is *useful* if it is on a path from an input of X to an output of Y. Let Z be the set of useful links and $z = |Z|$. The useful links are "useful" in the proof of the following theorem.

Theorem 4.11.

$$N \geq \frac{2}{3} n \log n + O(n)$$

Proof. The probability that a link u is connected from X is $1-(1-p)^{f_u} \le f_u p$. Similarly, the probability that u is connected to Y is $\le g_u p$. Hence,

$$P[u \in Z] \le \min\{1, f_u g_u p^2\} \le \min\{1, (f_u+g_u)^2 p^2/4\}$$

This implies

$$E[z] = \sum_{u\in U} P[u\in Z] \le \sum_{u\in U} \min\{1, (f_u+g_u)^2p^2/4\}$$

As it is trivial that $E[z] \ge E[m]$, we have

$$np + O\big((np)^{\frac{2}{3}}\big) \le E[m] \le E[z] \le \sum_{u\in U} \min\{1, (f_u+g_u)^2p^2/4\}$$

To this end, let $k = \lfloor \log n \rfloor$, set $p = 2^{-i}$, multiply both sides of the above equation by 2^i, and then sum it over $1 \le i \le k$, we get

$$n\log n + O(n) \le \sum_{i=1}^{k}\sum_{u} \min\{2^i, (f_u+g_u)^2 2^{-i}/4\} \tag{12}$$

For a fixed $u \in U$, let $j = \lfloor \log(f_u+g_u) \rfloor$ and $t = \log(f_u+g_u) - j$, then $0 \le t < 1$, and $f_u + g_u = 2^{j+t}$. As the function $2^t + 2^{1-t}$ is convex, we get

$$\begin{aligned}\sum_{i=1}^{k} \min\{2^i, (f_u+g_u)^2 2^{-i}/4\} &\le \sum_{i=1}^{j} 2^i + \sum_{i=j+1}^{k} 2^{2j+2t-i} \\ &\le (f_u+g_u)(2^{1-t}+2^t) \\ &\le 3(f_u+g_u)\end{aligned} \tag{13}$$

Consequently, equations (12) and (13) yield

$$n\log n + O(n) \le \sum_u \frac{3}{2}(f_u+g_u) = \frac{3}{2}N$$

□

4.3 Explicit Constructions and other results

We have seen how to use bounded concentrators to construct superconcentrators. Moreover, it is possible to construct bounded concentrators from expanders, as the following lemma shows. Consequently, explicit constructions of expanders induce explicit constructions of superconcentrators.

Lemma 4.12 (Gabber-Galil, 1981 [34]). *Let $p > 1$ be a fixed integer, and $\theta = \frac{p}{p+1}$. Suppose for each $n \in \mathbb{N}$ such that θn is an even integer we can construct an $(\theta n, k, \frac{2}{p-1})$-expander, then we can construct a family of superconcentrators with density $(2k+3)p + 1 + \epsilon$, where $\epsilon \to 0$ as $n \to \infty$.*

Proof. Firstly, for each n divisible by $(p+1)$ such that $np/(p+1)$ is even, we construct an $(n, \theta, (k+1)\theta, 1/2)$-bounded concentrator $G = (I, O; E)$. By definition, G has n inputs and θn outputs. The inputs are partitioned into two parts: a large part L containing θn vertices and a small part S containing $n/(p+1)$ vertices. The large part is connected to the output by a $(\theta n, k, 2/(p-1))$-expander. Each vertex in S is connected to exactly p consecutive outputs, so that the neighbors of vertices in S are completely disjoint. To show that G is an $(n, \theta, (k+1)\theta, 1/2)$-bounded concentrator, let X be any subset of I with $|X| \leq |I|/2$. Let $l = |X \cap L|$ and $s = |X \cap S|$. We need to show that $|\Gamma(X)| \geq |X|$. When $s \geq |X|/p$, S is connected to at least $|X|$ outputs, so that the inequality holds trivially. When $s < |X|/p$, we must have $l \geq r = \lceil \frac{p-1}{p}|X| \rceil$. Since $|X| \leq |I|/2$, we have

$$r = \left\lceil \frac{p-1}{p}|X| \right\rceil \leq \left\lceil \frac{p-1}{2p}|I| \right\rceil = \left\lceil \frac{p^2-1}{2p^2}|L| \right\rceil \leq \frac{1}{2}|L|$$

Thus, let R be a subset of L with r elements we can now use the fact that (L, O) is an $(\theta n, k, 2/(p-1))$-expander to get $|\Gamma(R)| \geq |X|$. The simple details are omitted.

Secondly, given any n, let n' be the smallest integer $> n$ such that $\theta n'$ is even and construct an $(n', \theta, (k+1)\theta, 1/2)$-bounded concentrator as above. Delete from the small part of this concentrator's inputs $n' - n$ vertices, turning it into an $(n, \theta + \epsilon_n, (k+1)\theta, 1/2)$-bounded concentrator, where $\epsilon_n = O(\frac{n'-n}{n}) = o(1)$. Lemma 4.5 now completes the proof. ☐

Gabber and Galil used this lemma and their explicitly constructed expanders to construct families of superconcentrators with density 271.8. This was improved by Chung to 261.5 and later by Buck (1986, [22]) to 190. Jimbo and Maruoka (1987, [44]) and Alon, Galil and Milman (1987, [8]) improved Gabber-Galil expanders slightly to obtained superconcentrators of density 122.74. The Ramanujan graphs allow Lubotzky et al. to reduce this to 78. Pippenger pointed out that density 64 is possible using double covers of 8-regular Ramanujan graphs, whose explicit constructions could be found in Morgenstern (1994, [57]). Morgenstern (1995, [59]) also explicitly constructed a family of bounded concentrators not using expanders. This construction yields density 66. All linear superconcentrators constructed above have logarithmic depths. Wigderson and Zuckerman (1999, [88]) constructed a linear-sized superconcentrator with sub-logarithmic depth: $(\log n)^{2/3+o(1)}$.

Open Problem 4.13. The probabilistic bound of $20n$ remains quite far apart from the best explicit construction bound of $64n$. Thus, an open question is to construct n-superconcentrators of size $< 64n$, and as close to $20n$ as possible.

As for limited depth, Meshulam (1984, [55]) constructed depth 2 superconcentrators of size $3n^{3/2} + O(n^{17/12})$, while Wigderson and Zuckerman [88] constructed one with size $O^*(n)$. Since connectors (defined in the next section) are also superconcentrators, the explicit constructions of n-connectors of depths k yield explicit constructions of n-superconcentrators of the same depth. For depth $k = 2j + 1$, n-connectors were constructed with size $O(n^{1+\frac{1}{(j+1)}})$, and depth $k = 2j$, $j \geq 2$ with size $O(n^{1+\frac{2}{3j-1}})$. More information can be found in the next section.

It is worthwhile to mention that there are several variations of concentrators and superconcentrators, which are also models for different types of switching networks. These include self-routing superconcentrators [73], partial concentrators [41, 42, 38], and natural bounded concentrators [59].

5 Connectors

In this section, We discuss the graphical models for rearrangeable, strictly nonblocking and wide-sense nonblocking networks.

Definition 5.1. An *n-connector* is an n-network with inputs I and outputs O so that for any one-to-one correspondence φ between I and O, there exist a set of n vertex disjoint paths joining i to $\varphi(i)$ for each input $i \in I$. When φ is restricted to those of the form $i_k \to o_{k+j \pmod n)+1}$, for some j, the n-connector is called an *n-shifter*.

Definition 5.2. An *generalized n-connector* is an n-network so that: given any one-to-many correspondence ϕ between inputs and disjoint sets of outputs, there exists a set of vertex disjoint trees joining i to the set $\phi(i)$ for all i with $\phi(i)$ defined.

Definition 5.3. An *generalized n-concentrator* is an n-network satisfying the condition that given any correspondence between inputs a nonnegative integers summing up to at most n, there exists a set of vertex disjoint trees that join each input to the corresponding number of distinct outputs.

Generalized concentrators used to be widely referred to as *generalizer*. Here we adopt this terminology from [31] because it is consistent with our overall use of terminologies. Clearly when each corresponding integer is 0 or 1, generalized concentrators are concentrators.

The n-connectors are the graphical version of a rearrangeable networks, while generalized n-connectors are the version for multicast rearrangeable networks. We shall also use the term *rearrangeable n-connector* for n-connector, to emphasize the difference of this network with strictly nonblocking connector and wide-sense nonblocking connector, which are to be defined shortly. Rearrangeable n-connector used to be called *rearrangeable n-network*. As the names suggested, these connectors are graphical versions of strictly nonblocking networks and wide-sense nonblocking networks, respectively.

Definition 5.4. A *strictly non-blocking n-connector* (SNB n-connector) is an n-connector with input set I and output set O, such that for any $i \in I$, $o \in O$ and a set $\mathcal{P}$ of vertex disjoint paths from $I - \{i\}$ to $O - \{o\}$, there is a path from i to o which is vertex disjoint from $\mathcal{P}$.

To formally define wide-sense nonblockingness (WSNBness), we need to settle several other technical concepts, whose counter parts in switching network are clear from the names.

Let $\mathcal{N}$ be a network with input set I and output set O. A *route* in $\mathcal{N}$ is a directed path from an input to an output. Two routes are *compatible* if they share only an initial segment, which could be empty. A *state* of $\mathcal{N}$ is a set of pairwise compatible routes. The set of all states of $\mathcal{N}$ could be partially ordered by inclusion. Hence, we can speak of a state S_1 being *contained in* a state S_2. A vertex or an edge in a state is *busy* if it is in some route of the state, and *idle* otherwise.

A *connection request* is an element (i, o) in $I \times O$. A connection request (i, o) is said to be *satisfied* by a route R if R originates from i and ends at o. A *generalized connection assignment* (GCA) is a set of connection requests, whose outputs are disjoint. A GCA is *realized* by a state if each request of the GCA is satisfied by some route of the state.

A GCA is (a, f)-limited if it contains at most a requests, of which at most f have a common input. *A state is (a, f)-limited* if the maximal GCA it realizes is (a, f)-limited. Often we speak of the maximal GCA realized by a state S as *the* GCA realized by S. A connection request (i, o) is *(a, f)-limited in a state S* if o is idle, and the GCA obtained by adjoining (i, o) into the GCA realized by S is (a, f)-limited.

Now we are ready to place the formal definition of WSNBness.

Definition 5.5. A *WSNB (a, f)-limited generalized connector* is a network for which there exists a collection $\mathcal{S}$ of states called the *safe states,* such that:

(i) The empty state $\emptyset$ is in in $\mathcal{S}$.

(ii) If $S \in \mathcal{S}$, then any state contained in S is also in $\mathcal{S}$.

(iii) Given $S \in \mathcal{S}$ and any (a, f)-limited connection request (i, o) in S, there exists $S' \in \mathcal{S}$ such that $S' \supset S$ and that S' has a route satisfying (i, o).

A *WSNB a-limited connector* is a WSNB $(a, 1)$-limited generalized connector. A *WSBN connector* is a ∞-limited connector. The prefixes "(n, m)-" and "n-" could be appended with the obvious meaning.

5.1 Rearrangeable connectors

As we have mentioned, connectors are models for rearrangeable networks. We put "rearrangeable" in front of "connectors" to emphasize the difference with SNB connectors and WSNB connectors. Let $c(n)$ denote the minimum size of an n-connector, and $c(n, k)$ be the minimum size of n-connectors with depth k. Pippenger and Valiant (1976, [74]) showed that

$$3n \log_3 n \leq c(n) \leq 6n \log_3 n + O(n)$$

Pippenger (1980, [70]) improved the lower bound of $c(n)$ to be $c(n) \geq 6 \log_6 n + O(n)$, and adopted a comment from the referee James Shearer of his paper to get

$$c(n) \geq \frac{45}{7} n \log_6 n + O(n) \tag{14}$$

which is the best lower bound known so far for $c(n)$.

In the case of connectors with a given depth, it is clear that $c(n, 1) = n^2$. When $k = 2$, de Bruijn, Erdős and Spencer (1974, [28]), while solving a problem of van Lint (1973, [86]), used a probabilistic argument to show $c(n, 2) = O(n^{\frac{3}{2}} \sqrt{\log n})$. Pippenger and Yao (1982, [75]) used an argument from Pippenger and Valiant (1976, [74]) to show

$$c(n, k) = \Omega(n^{1+\frac{1}{k}}) \tag{15}$$

and another probabilistic argument to prove

$$c(n, k) = O(n^{1+\frac{1}{k}} (\log n)^{\frac{1}{k}}) \tag{16}$$

which implies the results by de Bruijn et al.

In this section, we shall present the proofs of the two relations (14) and (15). First, we need a famous result which was conjectured by Minc (1963, [56]) and proved by Brègman (1973, [20]). Let A be a 01-square matrix of order n, and S_n the symmetric group of order n as usual. The *permanent* of A: per A is defined to be

$$\text{per } A = \sum_{\pi \in S_n} \prod_{i=1}^{n} (A)_{i\pi(i)}$$

Theorem 5.6 (Brègman-Minc Theorem, 1973). *Let A be an $n \times n$ 01-matrix where row i sums to r_i, $i = 1, \ldots, n$. Then*

$$\textit{per } A \leq \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}$$

Theorem 5.7 (Pippenger, 1980 [70]).

$$c(n) \geq \frac{45}{7} n \log_6 n + O(n)$$

Proof. Let $G = (I, O; V, E)$ be an n-connector with input set i, output set O, vertex set V and edge set E. We can assume each $v \in V$ has indegree at least 2 and outdegree at least 2. Now, if some vertex v has two inarcs $u_1 v$, $u_2 v$ and 2 outarcs vw_1 and vw_2, then we could delete v and add edges $u_1 w_1$, $u_1 w_2$, $u_2 w_1$ and $u_2 w_2$ without changing the rearrangeability of the graph and without increasing the number of edges. Consequently, we may assume each vertex in $V - I \cup O$ has total degree at least 5. Let π be any one to one correspondence between I and O. Let $G' = (V', E')$ be the graph obtained from G by gluing together v and $\pi(v)$ for every $v \in I$, and add a loop to every vertex $v \in V - I \cup O$. Let $d_G(v)$ be the total degree of vertex v in G, then

$$|E| = \frac{1}{2} \left(\sum_{v \in I \cup O} d_G(v) + \sum_{v \notin I \cup O} d_G(v) \right)$$

and,

$$\begin{aligned}
|E'| &= \frac{1}{2} \left(\sum_{v \in I \cup O} d_G(v) + \sum_{v \notin I \cup O} d_G(v) + 2|V - I \cup O| \right) \\
&\leq \frac{1}{2} \left(\sum_{v \in I \cup O} d_G(v) + \sum_{v \notin I \cup O} d_G(v) \right) + \frac{1}{5} \sum_{v \notin I \cup O} d_G(v) \\
&\leq \frac{7}{10} \left(\sum_{v \in I \cup O} d_G(v) + \sum_{v \notin I \cup O} d_G(v) \right) \\
&= \frac{7}{5} |E|
\end{aligned}$$

Let C be the set of cycle decompositions of G', then since G is rearrangeable, each I to O matching results in a different cycle decomposition in C. Hence $|C| \geq$

$n!$. Clearly, $|C| = \text{per } A$ where A is the adjacency matrix of G', i.e. $A_{uv} = 1$ iff $(u, v) \in E'$ and 0 otherwise. Let r_u be the sum of row u of A, then $\sum_v r_v = |E'|$. It is easy to see that the function $\log(x!)/x^2$ over the positive integers is decreasing if $x \geq 3$ and increasing when $x \leq 3$, namely $\log(x!)/x^2$ gets its maximum at $x = 3$. Theorem 5.6 gives

$$\begin{aligned} |E'| &= \sum_v r_v \geq \frac{3^2}{\log 6} \sum_{v \in V'} \frac{\log(r_v!)}{r_v} \\ &= 9 \sum_{v \in V'} \frac{\log_6(r_v!)}{r_v} \\ &\geq 9 \log_6(\text{per } A). \end{aligned}$$

Thus,

$$\begin{aligned} |E| &\geq \frac{5}{7}|E'| \\ &\geq \frac{45}{7} \log_6(\text{per } A) \\ &= \frac{45}{7} \log_6 |C| \\ &\geq \frac{45}{7} \log_6(n!) \\ &\geq \frac{45}{7} n \log_6 n + O(n) \end{aligned}$$

□

It should be noted that using the same trick for the case where G is undirected gives a lower bound for the size of an undirected n-connector:

$$|E| \geq \frac{18}{5} n \log_6(n) + O(n)$$

Theorem 5.8 (Pippenger-Yao, 1982 [75]). *An n-shifter of depth k has at least $kn^{1+\frac{1}{k}}$ edges.*

Proof. Let $T_k(n)$ be a directed rooted tree with n leaves and depth k where all edges directed to the direction of the leaves. Let $P_1, \ldots, P_n$ be the n paths from the root to the leaves of $T_k(n)$. Let

$$\Delta(T_k(n)) := \sum_{j=1}^{n} \sum_{v \in P_j} outdeg(v)$$

We first show that $\Delta(T_k(n)) \geq kn^{1+\frac{1}{k}}$ by induction on k.

As $\Delta(T_1(n)) = n^2$, the case $k = 1$ is trivial. For $k \geq 2$, supposed the root has degree d, which is connected to d subtrees $T^i_{k-1}(n_i)$, where the tree T_i has n_i leaves, for $1 \leq i \leq d$. Then, since the function $x^{1+\frac{1}{k-1}}$ is convex in x, we have

$$
\begin{aligned}
\Delta(T_k(n)) &= dn + \sum_{i=1}^{d} \Delta(T^i_{k-1}(n_i)) \\
&\geq dn + \sum_{i=1}^{d} (k-1) n_i^{1+\frac{1}{k-1}} \\
&\geq dn + (k-1)d \left(\frac{\sum_{i=1}^{d} n_i}{d} \right)^{1+\frac{1}{k-1}} \\
&\geq dn + d(k-1) \left(\frac{n}{d} \right)^{1+\frac{1}{k-1}}.
\end{aligned}
$$

Lastly, straightforward calculus completes the induction:

$$
dn + d(k-1) \left(\frac{n}{d} \right)^{1+\frac{1}{k-1}} \geq kn^{1+\frac{1}{k}}.
$$

Now, let G be an n-shifter. By definition, for each $j = 1, \ldots, n$ there are n vertex disjoint paths P_{ij} joining each input $i \in I$ to output $o \in O$, where $o = i + j \pmod{n} + 1$. Fix i, vary j from 1 to n and assemble the P_{ij} into a tree, keeping only the initial common segments. Call the resulting tree T_i, then T_i has n leaves and depth k. We thus have $\Delta(T_i) \geq kn^{1+\frac{1}{k}}$. Let

$$
\mu(i,j,e) := \begin{cases} 1 & \text{if } e \text{ is an arc from a node of } P_{ij} \\ 0 & \text{otherwise} \end{cases}
$$

then

$$
\sum_{e \in E} \mu(i,j,e) \geq \sum_{v \in P_{ij}} d_{T_i}(v)
$$

with strict inequality when some v along P_{ij} got split in T_i. Consequently,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{e\in E}\mu(i,j,e) &\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v\in P_{ij}} d_{T_i}(v)\\ &= \sum_{i=1}^{n}\Delta(T_i)\\ &\geq kn^{2+\frac{1}{k}}\end{aligned}$$

Lastly, as G is rearrangeable, $P_{1j},\ldots,P_{nj}$ are vertex disjoint. Thus,

$$\sum_{i=1}^{n}\mu(i,j,e)\leq 1$$

This implies

$$kn^{2+\frac{1}{k}} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{e\in E}\mu(i,j,e) \leq \sum_{j=1}^{n}\sum_{e\in E} 1 \leq n|E|$$

which completes the proof. □

Corollary 5.9.

$$c(n,k)=\Omega(n^{1+\frac{1}{k}})$$

Proof. Any n-connector is an n-shifter □

5.2 Non-blocking connectors

In this section, we discuss results on SNB and WSNB connectors. Let $t(n,k)$ be the minimum size of a SNB n-connector of depth k. Let $w(n,k)$ be the minimum size of a WSNB n-connector of depth k. When there is no limitation on the depth, we use $t(n)$ and $w(n)$ respectively.

The multistage Clos network (see [16]) is strictly nonblocking, and thus it gives an upper bound of $O(n^{1+\frac{1}{k}})$ for $t(n,2k)$ and $t(n,2k-1)$, $k\geq 1$. In other words, the extended Clos network gives

$$t(n,k)=O(n^{1+1/\lfloor\frac{k+1}{2}\rfloor}). \quad (17)$$

Moreover, a SNB n-connector is certainly a rearrangeable n-connector, thus the result of Pippenger and Yao mentioned in Corollary 5.9 of the previous section implies that $t(n,k) \geq c(n,k) = \Omega(n^{1+\frac{1}{k}})$. Friedman (1988, [32]) gave the only other known lower bound on SNB n-connector of a given depth k:

$$t(n,k) = \Omega(n^{1+\frac{1}{k-1}}), \tag{18}$$

closing the gap when $k = 2, 3$. Concerning $t(n)$, Pippenger (1978, [69]) showed by a probabilistic argument that

$$t(n) \leq 90n \log_3 n. \tag{19}$$

improving a previous bound of $66 \log n$ by Bassalygo and Pinsker (1973, [14]). Shannon (1950, [80]) was the first to show that $c(n) = \Omega(n \log n)$. The best lower bound for $c(n)$ was shown in Theorem 5.7, which implies

$$t(n) = \frac{45}{7} n \log_6 n + O(n) \tag{20}$$

Open Problem 5.10. Close the gap between the lower bound $\Omega(n^{1+\frac{1}{k-1}})$ and upper bound $O(n^{1+1/\lfloor\frac{k+1}{2}\rfloor})$ of $t(n,k)$ when $k \geq 4$.

A SNB network is also a WSNB network, hence $w(n) \leq t(n)$ and $w(n,k) \leq t(n,k)$. A WSNB network is rearrangeable, hence $w(n) \geq c(n)$ and $w(n,k) \geq c(n,k)$. Thus, we already had

$$w(n,k) = \Omega(n^{1+\frac{1}{k}}). \tag{21}$$

Feldman, Friedman, and Pippenger (1988, [31]) showed that a WSNB generalized connector with a fixed depth k exists, whose size is $O(n^{1+1/k}(\log n)^{1-1/k})$, namely

$$w(n,k) = O(n^{1+\frac{1}{k}}(\log n)^{1-\frac{1}{k}}) \tag{22}$$

Open Problem 5.11. Close the gap between the lower bound $\Omega(n^{1+\frac{1}{k}})$ and upper bound $O(n^{1+\frac{1}{k}}(\log n)^{1-\frac{1}{k}})$ of $w(n,k)$.

In this section, we give an improved version of the result by Friedman, who showed relation (18).

Let $G = (I, O; V, E)$ be a SNB n-connector. Assume that V can be partitioned into stages $V_0 = I, V_1, \ldots, V_k = O$, by adding more edges if necessary, as doing so would not increase the lower bound. For any vertex $v \in V_i$, let $D_L(v)$ ($D_R(v)$) be the set of vertices in V_{i-1} (V_{i+1}) which are connected to v. Also, define $d_L(v) := |D_L(v)|$ and $d_R(v) := |D_R(v)|$.

Let us first gives a lower bound of $|E|$ when $k = 2$.

Lemma 5.12. *For any $i \in I$, $o \in O$, let $V_{io} := D_R(i) \cap D_L(o)$, then*

$$\sum_{v \in V_{io}} \left(\frac{1}{d_L(v)} + \frac{1}{d_R(v)} \right) \geq 1 + \frac{1}{n}$$

Proof. Assume $V_{io} = \{v_1, \ldots, v_m\}$. Let σ and τ be two rearrangements of $\{1, \ldots, m\}$ such that

$$d_L(v_{\sigma_1}) \leq d_L(v_{\sigma_2}) \leq \cdots \leq d_L(v_{\sigma_m})$$

and

$$d_R(v_{\tau_1}) \leq d_R(v_{\tau_2}) \leq \cdots \leq d_R(v_{\tau_m}).$$

We first claim that there is a $j \leq m$ such that either $d_L(v_{\sigma_j}) \leq j$ or $d_R(v_{\tau_j}) \leq j$. Assume for contradiction that $d_L(v_{\sigma_j}) \geq j+1$ or $d_R(v_{\tau_j}) \geq j+1$ for all $j = 1, \ldots, m$. As the request (i, o) must be routed through one of $v_1, \ldots, v_m$, if we could find a state of G not involving i and o which uses all vertices $v_1, \ldots, v_m$, then we reach a contradiction. As $d_L(v_{\sigma_j}) \geq j+1$ or $d_R(v_{\tau_j}) \geq j+1$ for all $j = 1, \ldots, m$, it is easy to find a matching from some subset $\{i_{\sigma_1}, \ldots, i_{\sigma_m}\}$ of $I - \{i\}$ onto V_{io} and another matching from V_{io} onto some subset $\{o_{\sigma_1}, \ldots, o_{\sigma_m}\}$ of $O - \{o\}$. The set of paths $P_j = i_{\sigma_j} \to v_{\sigma_j} \to o_{\sigma_j}$ use up all vertices in V_{io}, contradicting the fact that G is SNB.

Now, let j be a number $\leq m$ such that $d_L(v_{\sigma_j}) \leq j$ or $d_R(v_{\tau_j}) \leq j$. Notice that $1/d_L(v_i) \geq 1/n$ and $1/d_R(v_i) \geq 1/n$, for all $i = 1, \ldots, n$. Thus,

$$\begin{aligned}
\sum_{v \in V_{io}} \left(\frac{1}{d_L(v)} + \frac{1}{d_R(v)} \right) &\geq \left(\frac{1}{d_L(v_{\sigma_1})} + \cdots + \frac{1}{d_L(v_{\sigma_j})} \right) + \frac{m-j}{n} + \\
&\quad \left(\frac{1}{d_R(v_{\tau_1})} + \cdots + \frac{1}{d_R(v_{\tau_j})} \right) + \frac{m-j}{n} \\
&\geq \frac{j}{d_L(v_{\sigma_j})} + \frac{j}{d_R(v_{\tau_j})} + \frac{2m-2j}{n} \\
&\geq 1 + \frac{j}{n} + \frac{2m-2j}{n} \\
&\geq 1 + \frac{1}{n} \qquad (23)
\end{aligned}$$

□

Theorem 5.13. *Let $G = (I, O; V, E)$ be a strictly non-blocking n-connector of depth 2, then G has at size at least $n^2 + n$. Moreover, there exists a strictly non-blocking n-connector of depth 2 and size exactly $n^2 + n$.*

Proof. Given the lemma above, the proof of this theorem is straight forward, as shown below.

$$\begin{aligned} n^2+n=n^2\left(1+\frac{1}{n}\right) &\leq \sum_{i\in I, o\in O}\sum_{v\in V_{io}}\left(\frac{1}{d_L(v)}+\frac{1}{d_R(v)}\right) \\ &= \sum_{v\in V_1}\left(\frac{1}{d_L(v)}+\frac{1}{d_R(v)}\right)|\{(i,o)\mid v\in V_{io}\}| \\ &= \sum_{v\in V_1}\left(\frac{1}{d_L(v)}+\frac{1}{d_R(v)}\right)d_L(v)d_R(v) \\ &= |E| \end{aligned}$$

We could construct a SNB n-connector of depth 2 and size n^2+n by setting $|V_0|=|V_1|=|V_2|=n$, connecting V_0 to V_1 by a perfect matching using n edges, and connecting every vertex in V_1 to every vertex in V_2 using n^2 more edges. The resulting graph is clearly a SNB n-connector. □

This line of reasoning could be extended to find a lower bound for $t(n,k)$ with $k\geq 3$.

Lemma 5.14. *For any pair $i\in I$ and $o\in O$, let $\mathcal{P}$ be the set of all paths from i to o, and let*

$$\begin{aligned} A_{io} &:= \{v\in V_1\cap V(\mathcal{P})\} \\ B_{io} &:= \{v\in V_{k-1}\cap V(\mathcal{P})\} \end{aligned}$$

Then,

$$\sum_{v\in A_{io}}\frac{1}{d_L(v)}+\sum_{v\in B_{io}}\frac{1}{d_R(v)}\geq 1+\frac{1}{n}.$$

Proof. Let $m=|A_{io}|$ and $m'=|B_{io}|$. Suppose $A_{io}=\{v_1,\ldots,v_m\}$ and $B_{io}=\{u_1,\ldots,u_{m'}\}$. Let σ and τ be two permutations on $\{1,\ldots,m\}$ and $\{1,\ldots,m'\}$, respectively, such that the arrays $\{d_L(v_{\sigma_i})\}$ and $\{d_R(u_{\tau_i})\}$ are weakly increasing. Similar to Lemma 5.12, we claim that there is either a $j\leq m$ such that $d_L(v_{\sigma_j})\leq j$, or a $j'\leq m'$ such that $d_R(v_{\tau'_j})\leq j'$. Assume otherwise, then there are distinct vertices $\{i_1,\ldots,i_m\}\subseteq I-\{i\}$ which could be matched one to one onto A_{io}, and distinct vertices $\{o_1,\ldots,o_{m'}\}\subseteq O-\{o\}$ which could be matched one to one onto B_{io}. Now, connect A_{io} to B_{io} by a maximal set of vertex disjoint paths $\mathcal{F}$. Make $\mathcal{F}$ a set of vertex disjoint paths from I to O by adjoining $\{i_1,\ldots,i_m\}$

and $\{o_1, \ldots, o_{m'}\}$ as possible. Clearly, there does not exist a new path from i to o disjoint from $\mathcal{F}$. Without loss of generality, we assume there is a $j \leq m$ such that such that $d_L(v_{\sigma_j}) \leq j$. We have

$$\begin{aligned}
\sum_{v \in A_{io}} \frac{1}{d_L(v)} + \sum_{u \in B_{io}} \frac{1}{d_R(u)} &\geq \left(\frac{1}{d_L(v_{\sigma_1})} + \cdots + \frac{1}{d_L(v_{\sigma_j})}\right) + \frac{m'}{n} \\
&\geq \frac{j}{d_L(v_{\sigma_j})} + \frac{m'}{n} \\
&\geq 1 + \frac{1}{n} \qquad (24)
\end{aligned}$$

□

Theorem 5.15. *Let $G = (I, O; V, E)$ be a strictly non-blocking n-connector of depth $k \geq 3$, then*

$$|E| \geq \frac{1}{8} n(n+2)^{\frac{1}{(k-1)}}$$

Proof. Let

$$\begin{aligned}
A &:= \{v \in V \mid d_L(v) \geq \frac{4|E|}{n}\} \\
B &:= \{v \in V \mid d_R(v) \geq \frac{4|E|}{n}\}, \text{ and} \\
V' &:= A \cup B
\end{aligned}$$

Since $\sum_v d_L(v) = \sum_v d_R(v) = |E|$, $|A| \leq \frac{n}{4}$ and $|B| \leq \frac{n}{4}$, which imply $|V'| \leq \frac{n}{2}$.

Let $\mathcal{P}$ be a maximal set of vertex disjoint paths from I to O such that each path $P \in \mathcal{P}$ hits at least one member of V'.

Let $\bar{V} := V - V' \cup V(\mathcal{P})$, then the induced subgraph of G on $\bar{V}$: $G|_{\bar{V}} = (\bar{I}, \bar{O}; \bar{V}, \bar{E})$ is strictly non-blocking because any path from I to O which is vertex disjoint from $\mathcal{P}$ does not hit V' due to the maximality of $\mathcal{P}$. Also note that $\bar{I}$ and $\bar{O}$ each has at least $\frac{n}{2}$ vertices, and that d_L and d_R of each $v \in \bar{V}$ are at most $\frac{4|E|}{n}$. We assume $I|_{\bar{V}}$ and $O|_{\bar{V}}$ both have size exactly $n/2$, removing some vertices in $\bar{I}$ and/or $\bar{O}$ if necessary. Let $\bar{V}_i = V_i \cap \bar{V}$, and for any pair $i \in \bar{I}$ and $o \in \bar{O}$, let

$$\begin{aligned}
\bar{A}_o &:= \text{set of vertices in } \bar{V}_1 \text{ which can reach } o \\
\bar{B}_i &:= \text{set of vertices in } \bar{V}_{k-1} \text{ reachable from } i
\end{aligned}$$

Then, clearly $|\bar{A}_o| \leq (4|E|/n)^{k-1}$ and $|\bar{B}_i| \leq (4|E|/n)^{k-1}$. We also define $\bar{A}_{io}$ and $\bar{B}_{io}$ in the sense of Lemma 5.14, then by Lemma 5.14

$$\begin{aligned}
\left(\frac{n}{2}\right)^2 (1 + \frac{2}{n}) &\leq \sum_{i \in \bar{I}, o \in \bar{O}} \left(\sum_{v \in \bar{A}_{io}} \frac{1}{d_L(v)} + \sum_{v \in \bar{B}_{io}} \frac{1}{d_R(v)} \right) \\
&= \sum_{o \in \bar{O}} \left(\sum_{v \in \bar{A}_o} (\#\ i\text{'s connected to } v) \frac{1}{d_L(v)} \right) + \\
& \quad \sum_{i \in \bar{I}} \left(\sum_{v \in \bar{B}_i} (\#\ o\text{'s connected to } v) \frac{1}{d_R(v)} \right) \\
&= \sum_{o \in \bar{O}} |\bar{A}_o| + \sum_{i \in \bar{I}} |\bar{B}_i| \\
&\leq n\,(4|E|/n)^{k-1}
\end{aligned}$$

This inequality and $k \geq 3$ imply

$$|E| \geq \frac{1}{4} \left(\frac{1}{4}\right)^{\frac{1}{k-1}} n(n+2)^{\frac{1}{k-1}} \geq \frac{1}{8} n(n+2)^{\frac{1}{k-1}}$$

□

Remark 5.16. The original theorem from Friedman gives

$$|E| \geq \frac{1}{32} n^{1 + \frac{1}{(k-1)}}$$

There are room for improvement. Firstly the constant 4 in the definition of A and B might still be optimized. Secondly, we could do something better than just considering the second and the next-to-last stage of G.

5.3 Generalized connectors and generalized concentrators

Let $gc(n)$, $gt(n)$, $gw(n)$ and $g(n)$ be the minimum sizes of generalized n-connectors, generalized SNB n-connectors, generalized WSNB n-connectors, and generalized n-concentrators, respectively. Naturally, we also use $gc(n,k)$, $gt(n,k)$, $gw(n,k)$, and $g(n,k)$ to denote the minimum size of the graphs have depth-k.

As we have mentioned in the previous section, Pippenger and Valiant (1976, [74]) showed that

$$3n \log_3 n \leq c(n) \leq 6n \log_3 n + O(n)$$

For $n = 2^k$, Ofman (1965, [62]) implicitly showed

$$gc(n) \leq 10n \log_2 n + O(n)$$

While if $n = 3^k$, Thompson (1977, in a tech report at CMU) showed

$$gc(n) \leq 12n \log_3 n + O(n)$$

Pippenger (1978, [68]) connect the two functions by showing

$$gc(n) = c(n) + O(n)$$

In this proof, Pippenger needed a bound for generalized n-concentrators and showed that generalized n-concentrators with at most $120n$ edges exist. Fan Chung [24] improved this bound to $118.5n$, more precisely

$$g(n) \leq 118.5n + O(\log n) \tag{25}$$

Dolev, Dwork, Pippenger and Wigderson (1983, [29]) probabilistically showed

$$gc(n, k) = O((n \log n)^{1+\frac{1}{k}}) \tag{26}$$

and constructively proved

$$gc(n, 3j - 2) = O(n^{1+\frac{1}{j}}) \tag{27}$$

Masson and Jordan (1972, [54]), and Nassimi and Sahni (1982, [60]) gave two different constructions which proves

$$gc(n, 3) = O(n^{\frac{5}{3}}) \tag{28}$$

Kirkpatrick, Klawe and Pippenger (1985, [47]) explicitly constructed generalized n-connectors to show

$$gc(n, 3) = O(n^{\frac{3}{2}}(\log n)^{\frac{1}{2}}) \tag{29}$$

and extended this to

$$gc(n, 2k - 1) = O(n^{1+\frac{1}{k}}(\log n)^{\frac{k-1}{2}}) \tag{30}$$

The upper bound for $gc(2k - 1)$ differs very little from that of the best explicit construction bound for $c(n, 2k - 1)$ (see next section).

It is easy to see that $gc(n, k) \leq gw(n, k) \leq gt(n, k)$. Feldman, Friedman, and Pippenger (1988, [31]) showed that

$$gw(n, k) = O(n^{1+\frac{1}{k}}(\log n)^{1-\frac{1}{k}}) \tag{31}$$

which in effect gives the best upper bound so far for $gc(n, k)$ also. There is no known upper bound for $gt(n, k)$.

Open Problem 5.17. Find an upper bound for $gt(n, k)$.

Note also that the corresponding lower bounds of $c(n, k)$, $w(n, k)$ and $t(n, k)$ yield lower bounds for their generalized versions.

5.4 Explicit Constructions

Rearrangeable n-connectors with size $O(n \log n)$ were constructed by Beizer [15], and rediscovered by Beneš (1965, [16]), Joel (1968, [45]) and Waksman (1968, [87]). Pippenger (1978, [69]) slightly generalized these constructions to get the bound of $6n \log_3 n$. In the same paper, Pippenger also showed how the classical construction by Slepian, Duguid and LeCorre can be used to construct n-connectors with depth $(2j + 1)$ and size $O(n^{1+\frac{1}{j+1}})$. Another construction based on combinatorial designs by Richards and Hwang (1985, []) gives n-connectors of depth 2 and size $O(n^{5/3}$ (see also [31]). This construction can be used with the previous construction to construct n-connectors of depth $2j$ and size $O(n^{1+2/(3j-1)})$.

Clos (1953, [26]), Cantor (1971, [23]) and Pippenger (1978, [69]) showed that $t(n, 2k - 1) = O(n^{1+\frac{1}{k}})$ by explicit constructions. That $t(n) = O(n(\log n)^2)$ was shown by explicit constructions of Cantor [23] and its generalization by Pippenger [69].

Masson and Jordan (1972, [54]) constructed WSNB generalized n-connector of depth 3 and size $O(n^{5/3})$. Pippenger (1973, [66]) and Nassimi and Sahni (1982, [60]) constructed WSNB generalized connectors with depth $j^2 - 3j + 3$ and size $O(n^{1+2/j})$, implying the results by Masson and Jordan. Note that any construction of SNB network yields also a WSNB network.

Dolev, Dwork, Pippenger and Wigderson (1983, [29]) constructed generalized n-connectors of depth $(3k - 2)$ and size $O(n^{1+\frac{1}{k}})$. Kirkpatrick, Klawe and Pippenger (1985, [47]) explicitly constructed generalized n-connectors of depth 3 and size $O(n^{\frac{3}{2}}(\log n)^{\frac{1}{2}})$, and of depth $2k - 1$ and size $O(n^{1+\frac{1}{k}}(\log n)^{\frac{k-1}{2}})$.

Feldman, Friedman, and Pippenger (1988, [31]) constructed WSNB generalized n-connectors with depth 2 and size $O(n^{5/3})$, and with depth 3 and size $O(n^{11/7})$. Wigderson and Zuckerman (1999, []) constructed depth k WSNB generalized connectors of size $n^{1+\frac{1}{k}+o(1)}$, which is within a factor of $n^{o(1)}$ to the optimal bound of equation (22)).

6 Conclusions

In this chapter, we have surveyed studies on the complexity of switching networks, mostly on the tradeoff between the size and the depth of various types of switching networks. The graph models of different switching networks were investigated: expanders, concentrators, superconcentrators, expanders, rearrangeable, strictly non-blocking and wide-sense non-blocking connectors, plus their generalizations.

Researches on these graphs were thoughroughly collected, with more intuitive results presented. Many open questions were also specified, which hopefully will help practitioners new to this field identify research problems.

This research area is certainly very interesting, has deep connection to many different areas of Mathematics and Computer Science. Algebraic Graph Theory and Probabilistic Method are the two popular tools which are used to deal with questions arising from switching networks. Obviously, researches on switching networks have also enriched techniques and problems in the former two fields.

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