On Connectivity of Consecutive-d Digraphs

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Abstract

The concept of a consecutive-d digraph was proposed by Du, Hsu and Hwang as a generalization of de Bruijn digraphs, Kautz digraphs, and their generalizations given by Imase and Itoh and Reddy, Pradhan and Kuhl. In this paper we determine the connectivity of consecutive-d digraphs and study how to modify consecutive-d digraphs to reach maximum connectivity. Our results will generalize and improve several existing results on the connectivity of de Bruijn digraphs, Kautz digraphs and Imas-Itoh digraphs.

1 Introduction

De Bruijn graphs [1], Kautz graphs [2] and their generalizations have been extensively studied [3–20]. It was stated in [5] that these graphs are competitive topological structures for interconnection networks of computers and multiprocessor systems. For a nice survey, the reader is referred to [3] and [4].

For integers d, n, q, r satisfying $0 < d \leq n$, $-n/2 < q \leq n/2$, and $q \neq 0$, a consecutive-d digraph $G(d,n,q,r)$ (as defined in [12]) has n nodes, labeled by integers mod n, with edges from each node i to d consecutive nodes, which are those with labels $qi+r+k \pmod{n}$. The concept of the consecutive-d digraph generalizes many interconnection networks of computers and multiprocessor systems. The generalized de Bruijn digraphs [19,21] and the generalized Kautz digraphs [20] are its two useful subclasses consisting of $G_B(d,n) = G(d,n,d,0)$ and $G_I(d,n) = G(d,n,n-d,n-d)$, respectively. The following results on connectivity of $G_B(d,n)$, $G_I(d,n)$, and $G(d,n,q,r)$ are known.

(1) If $n \geq d^3$, $G_B(d,n)$ and $G_I(d,n)$ are $(d-1)$-connected (Imase, Soneoka and Okada [22].)
(2) If $n \geq d^4$, then $G_I(d,n)$ is d connected iff $(d+1) \mid n$ and $gcd(d,n) > 1$ (Homobono and Peyrat [17].)
(3) If $gcd(n,q) = d$ and $n > d^2$, then $G(d,n,q,r)$ is at least $(d-1)$-connected and it is d-connected iff it has no loop. (Du, Hsu, and Peck [14])

In this paper, we determine the connectivity of $G(d,n,q,r)$ in almost all cases, and as corollaries, remove condition $n \geq d^4$ on n from the result of Homobono and Peyrat [17], significantly relax condition $n \geq d^3$ from the result of Imase, Soneoka and Okada [22]. In addition, we also study how to modify $G_B(d,n)$ to get a d-connected digraph by replacing all loops with a cycle or a set of

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disjoint cycles.

2 Preliminaries

Let $\psi = gcd(q-1, n)$. (note: $\psi = n$ if $q = 1$.) Denote by $(x)_n$ the residue of x modulo n, represented by a number in $\{0, 1, \ldots, n-1\}$. An edge is said to be with k-value i, where $0 \leq i < d$, if it is contained in the subgraph $G(1, n, q, r+i)$. The following lemma can be found in [23].

Lemma 1 *$G(d, n, q, r)$ has the following properties:*

(a) Each node has at most one loop.
(b) $G(d, n, q, r)$ has no loop iff $0 < (r)_\psi \leq \psi - d$.
(c) If $d < \psi$, then all loops of $G(d, n, q, r)$ are with the same k-value.
(d) If $\psi = 1$, then for each k-value there exists exactly one loop with the k-value. If $\psi > 1$, then for each k-value, either there is no loop or there are exactly ψ loops with the k-value. Moreover, if i is a loop-node, then the ψ loop-nodes are i, $i + n/\psi$, $\cdots$, $i + (\psi - 1)n/\psi$. In particular, if $d \geq 2$ then $G(d, n, q, r)$ has either no loop or at least two loops.
(e) If $|q-1| \leq d$ and x is a loop-node, then either $x + \lfloor n/(q-1) \rfloor$ or $x + \lceil n/(q-1) \rceil$ is a loop-node.

In particular, notice that if $d \geq \psi$ then there always exists a loop. Next, we show a new lemma concerning loops in $G_B(d, n)$.

Lemma 2 *Consider $G_B(d, n)$, where $d > 1$. Let x and y be two distinct loop-nodes. Then either $|x - y| = 1$ or $\frac{n}{(d-1)} - 1 \leq |x - y| \leq n - (\frac{n}{(d-1)} - 1)$. Moreover, if $|x - y| = 1$, then the loop at x and the loop at y are with k-values 0 and $d - 1$.*

PROOF. Since x and y are loop-nodes, we have

$$(d-1)x + k \equiv 0 \pmod{n} \tag{1}$$
$$(d-1)y + k' \equiv 0 \pmod{n} \tag{2}$$

for $0 \leq k, k' \leq d - 1$. Without loss of generality, assume $k \leq k'$. Then $0 \leq k' - k \leq d - 1$. Subtracting (2) from (1), we obtain

$$(d-1)(x-y) \equiv k' - k \pmod{n}$$

If $|x - y| = 1$, then $k' - k = d - 1$ and we must have $k' = d - 1$ and $k = 0$. If $|x - y| > 1$, then $(d-1)(x-y) = k' - k + \ell n$ for some nonzero integer ℓ

and hence $|x-y| \geq \frac{n}{(d-1)} - 1$. It is also easy to see that $|l| \leq (d-2)$, so that $|x-y| \leq \frac{|l|}{d-1} + 1 \leq n - (\frac{n}{(d-1)} - 1)$. $\square$

From the above two lemmas, it is easy to see that $G_B(d,n)$ has exactly $d-1+\psi$ loop-nodes. 2ψ of them form ψ pairs of adjacent nodes. The rest of them are isolated. These $d-1$ "groups" of size 1 or 2 are almost evenly distributed in Z_n with "distance" at least $\frac{n}{(d-1)} - 1$ apart.

3 Consecutive Runs

In this section we show two lemmas which are important in studying the connectivity of consecutive-d digraphs.

A subset of $\mathbb{Z}_n$ is called a *consecutive run* if its elements can be consecutively numbered mod n. For convenience, we call the d out-edges from the node a *claw* and the set of end points of the claw a *claw-end*. In a consecutive-d digraph a node's claw-end forms a consecutive run of size d. Let $g = gcd(n, q)$. Denote $\bar{i} = \{i, i+n/g, \cdots, i+(g-1)n/g\}$. Then all nodes in $\bar{i}$ have the same set of successors. Each $\bar{i}$ will be called an *orbit*. Denote $\hat{i} = \{ig+r, ig+r+1, \cdots, ig+r+g-1\}$. Then all nodes in $\hat{i}$ have the same set of predecessors. Each $\hat{i}$ will be called a *block*.

A good way to visualize orbit, claws, blocks and consecutive runs is to think of $\mathbb{Z}_n$ as n points $\{0, 1, \ldots n-1\}$ put clockwise on a circle, spaced equally. An orbit is then a set of g equally-spaced points on the circle with interval n/g. Note that $\mathbb{Z}_n$ is partitioned uniformly into orbits. The circle is also partitioned into $\frac{n}{g}$ arcs (or consecutive runs) of size g, which are our blocks. The start of each block ($ig+r$, for some $i \in \mathbb{Z}_n$) is also the start of some claw-end (because there is always an $j \in \mathbb{Z}_n$ such that $jq+r \equiv ig+r \pmod{n}$). Claw-ends from different orbits start at different positions on the $\mathbb{Z}_n$-circle.

Since each orbits is of size g, the in-degree of a node of $G(d,n,q,r)$ must be divisible by g. Thus, if the in-degree of a node is d, then we must have $g \mid d$. It was proved in [14] that $g \mid d$ iff the in-degree of every node is d. Throughout this paper, we assume $g \mid d$, i.e., the in-degree of every node is d. To emphasize this, we may still mention this condition in the statements of lemmas and theorems. Also note that when $g \mid d$, each claw-end contains exactly d/g blocks.

Lemma 3 (Consecutivity Lemma) *Suppose $g \mid d$ and $g < d$. Let C, D, E be a partition of the node set of $G(d,n,q,r)$ such that removal of all nodes in E leaves no path from any node in C to any node in D. Let S be the union of* *all claw ends from nodes in C. If $|E| < d$, then S ($\subseteq C \cup E$) is a consecutive run of size at least $|C| + d - g$.*

PROOF. Suppose there are y orbits which intersect with C. Each such orbit contributes a consecutive run of size d in S, which we shall call a *C-run*. A consecutive run R in S is *maximal* if no other consecutive run in S properly contains R. Let x be the number of maximal consecutive runs in S. Let R be any maximal consecutive run in S, and k be the number of different C-runs which are *entirely* contained in R. Then since each C-run starts at the beginning of some block, we must have $1 + (k-1)g + d - 1 \le |R|$. In other words, each maximal consecutive run R in S contains at most $|R|/g - (d/g - 1)$ different C-runs. Summing over all maximal consecutive runs in S, we get $|S|/g - x(d/g - 1) \ge y$. Hence, S has at least $gy + x(d-g)$ elements. Since $S \subseteq C \cup E$ and $|E| \le d - 1$, we have

$$gy + x(d-g) \le |C| + d - 1.$$

Note that $gy \ge |C|$. If $g = 1$, then it is clear that $x = 1$. If $d > g > 1$, then $d - g \ge d/2$ since $g \mid d$. Thus, $x = 1$. Finally, $x = 1$ implies that S is a consecutive run and $|S| \ge gy + d - g \ge |C| + d - g$. □

From the consecutivity lemma, it is easy to see that $|E| \ge d - g$. This means that if $g \mid d$ and $g < d$, then $G(d, n, q, r)$ is at least $(d-g)$-connected.

Lemma 4 *Let R be a consecutive run of $\mathbb{Z}_n$. Suppose $S = \{a, a+h, \cdots, a + (c-1)h\} \subseteq R$ for some natural numbers a, c and h. Let f be any function so that for each $x \in S$, $f(x)$ is a consecutive run of size d in R. Further assume that $a + ih \in \bigcup_{j=0}^{c-1} f(a + jh)$, $\forall i = 0, \ldots c-1$. Then we have*

(i) If $|f(a+ih) \cap f(a+(i+1)h)| \ge h$ for $i = 0, \cdots, c-2$, then there exists an $i \in \mathbb{Z}_{c-1}$, such that $a + ih \in f(a+ih)$.

(ii) If $|f(a+ih) \cap f(a+(i+1)h)| \ge h - 1$ for $i = 0, \cdots, c-2$, then there exists an $i \in \mathbb{Z}_{c-1}$ such that either *$a + ih \in f(a+ih)$* or *$f(a+ih) = \{a + ih + 1, \cdots, a + ih + d\}$ and $f(a+(i+1)h) = \{a + (i+1)h - d, \cdots, a + (i+1)h - 1\}$.*

PROOF.

(i) Let

$$A = \{a + ih \mid x < a + ih,\ \forall x \in f(a+ih)\}$$
$$B = \{a + ih \mid x > a + ih,\ \forall x \in f(a+ih)\}$$

Suppose to the contrary that such an i does not exist; then since $a \in f(a+ih)$ for some i and $a + (c-1)h \in f(a+jh)$ for some j, both A

and B are not empty. It follows that there exists an i such that $a + ih$ and $a + (i + 1)h$ are not both in A nor both in B. If $a + ih \in A$ and $a + (i + 1)h \in B$, then $f(a + ih) \cap f(a + (i + 1)h) = \emptyset$, contradicting $|f(a + ih) \cap f(a + (i + 1)h)| \geq |h| > 0$. If $a + ih \in B$ and $a + (i + 1)h \in A$, then $f(a+ih) \cap f(a+(i+1)h)$ lies between $a+ih+1$ and $a+(i+1)h-1$, so that $|f(a+ih) \cap f(a+(i+1)h)| < h$, contradicting $|f(a+ih) \cap f(a+(i+1)h)| \geq h$, too.

(ii) Similarly, we can prove the second half of the lemma.

□

4 Connectivity

In this section we determine the connectivity of consecutive-d digraphs. The results are described by two theorems, as consequences of which the results of Imase, Soneoka and Okada [22] and Homobono and Peyrat [17] are extended to smaller n. The approaches we use here differ very much from theirs.

Theorem 5 *If $g \mid d$ and $1 < g < d$, then $G(d, n, q, r)$ is at least $(d - g)$-connected and it is d-connected iff it has no loop.*

PROOF. The first half has been proven in the last section. We prove the second half here.

If our graph has a loop at i, then removing $d - 1$ nodes other than i from the claw-end of i disconnects i to the rest of the graph. In other words, if $G(d, n, q, r)$ is d-connected then it has no loop.

For the other direction, let E be a node-cut of the smallest size, which disconnects D from C, i.e. removal of all the nodes in E leaves no path from nodes in C to those in D. Assume $|E| \leq d-1$. We will prove the existence of a loop.

Let S be the union of claw-ends from nodes in C. By Lemma 3, S is a consecutive run of size at least $|C|+d-g$. Without loss of generality, we may assume that all nodes not in S are in D, since otherwise they can be moved into D without increasing the size of the node-cut E. Thus, $S = C \cup E$. Since S is a consecutive run, so is its complement D. The following facts are important in the remainder of the proof.

(i) For any consecutive run R of $\mathbb{Z}_n$ and any orbit O, there are at least $\lfloor \frac{|R|g}{n} \rfloor$ elements of O in R, and at least $g - \lceil \frac{|R|g}{n} \rceil$ elements of O not in R.
(ii) Every claw from E catches some node in D. (Otherwise, E can be decreased, contradicting the minimality of E.)

(iii) If an orbit contains an element of C, then it contains no element of E. (Otherwise, E can be decreased by putting such elements into C.) An orbit having an element in C (E) is called a *C-orbit* (E-orbit). Notice that no claw-end from any C-orbit intersects D.

(iv) D has at most $g-1$ elements in C-orbits. (Otherwise, putting all such elements into C does not change the set E, but makes $|E|+|C|-|S| > g-1$, contradicting $|S| \geq |C|+d-g$.)

Now, for any $k \in \mathbb{Z}_n$, let k^* denote the integer between 0 and $n/2$ such that $k \equiv k^*$ or $-k^* \pmod n$. k^* is called the *magnitude* of k. To prove the existence of a loop, we may assume $q^* \geq d$ since if $q^* < d$, then certainly $\psi < d$, so that a loop must exist by Lemma 1. We consider three cases based on where $|D|$ lies between 0 and n.

Case 1: $(g-a)n/g \leq |D| < (g-a+1)n/g$ for some $a = 0, \cdots, \lfloor g/2 \rfloor$. By (i), each C-orbit contains at least $g-a$ elements in D. Let y be the number of C-orbits, then D has at least $y(g-a)$ elements in C-orbits. By (iv), $y(g-a) \leq g-1$. So, $y=1$. It follows that $|S| = d$. Since $S = C \cup E$, $|E| \leq d-1$ and the claw-end of our C-orbit has size d, every node in C has a loop.

Case 2: $an/g < |D| \leq (a+1)n/g$ for some $a = 0, \cdots, \lfloor g/2 \rfloor - 1$. Again, by (i) we have that of g elements in an orbit, at least a must be in D and at least $g-a-1$ must not be in D. It is not hard to see that there are at least $\lceil \frac{(|D|+d-1)}{g} \rceil$ orbits whose claw-ends intersect D, because each starting point of a block in D is also the starting point of a claw-end and the size of a claw-end, d, is greater than the size of a block, g. These orbits have to be disjoint from C, hence E has at least $(g-a-1)\lceil (|D|+d-1)/g \rceil$ elements. So, $d-1 \geq (g-a-1)\lceil (|D|+d-1)/g \rceil$, i.e.

$$|D| \leq (d-1)(a+1)/(g-a-1) \leq d-1.$$

We will prove that D must contain a loop node. Notice that since S is a union of claw-ends and $g \mid d$, S is also a union of (consecutive) blocks. Hence, D is a union of consecutive blocks, too. Let B be the rightmost block of D (clockwise). If B contains a loop node, then we are done. Thus, we may assume that B has no loop node. For any node i, let $ce(i)$ denote the claw-end from i. The inequality $d \leq q^* \leq n/2$ implies that if for some $i \in B$, $ce(i)$ has its right end point in $D \setminus B$ then the right end point of $ce(i+1)$ is not in B, neither is the left end point of $ce(i+1)$ since B has no loop. In other words, no two consecutive claw-ends from B both intersect D. Hence, B has at most $\lceil g/2 \rceil$ elements whose claw-ends intersect D. Let O be the orbit whose claw-end intersects D only in B. Then $O \subset (D-B) \cup E$. As we have noticed, there are at least $g-a-1$ elements of O not in D. These elements have to be in E, so $|O \cap E| \geq g-a-1 \geq g - (\lfloor \frac{g}{2} \rfloor - 1) - 1 = \lceil g/2 \rceil$.

When B is removed from D, only the nodes whose claw-ends intersect $D \setminus B$ (there are at most $\lceil g/2 \rceil$ of these nodes) have to be moved into E and others can be moved into C. However, all elements in $O \cap E$ can be moved from E to C. Thus, this move does not increase $|E|$. In this way, we can reduce D to have only one block. However, as $g \mid d$ each node in this block has d in-edges. One of them must be from a node in D, which forms a loop.

Case 3: $\lfloor g/2 \rfloor n/g \leq |D| < \lceil g/2 \rceil n/g$. This case exists only for g odd and at most two C-orbits exist by the same argument as that in Case 1. If there exists only one orbit, we can prove, as in Case 1, that each of the nodes in C has a loop. If there are two C-orbits, then each C-orbit must have at most $(g+1)/2$ elements in C since it has at least $(g-1)/2$ elements in D. So, $|E \cup C| \leq d - 1 + 2\frac{g+1}{2} = d + g$. It follows that the claw from each node in a C-orbit can miss only one block in $E \cup C$. If C has no loop, then we must have $|E \cup C| = |S| = d + g$ and hence each C-orbit has exactly $(g+1)/2$ elements in C. Furthermore, $(g+1)/2$ elements of C in a C-orbit must fit in a block which is not contained in any claw-end of this C-orbit. Thus, $1 + (\frac{g+1}{2} - 1)n/g \leq g$. So, $n \leq 2g \leq d$. Thus, $q^* < d$, a loop must exist.

□

Theorem 6 *If $g = 1$, then $G(d, n, q, r)$ is at least $(d-1)$-connected. Moreover, if $n > 3d$ then it is d-connected if and only if none of the following occurs :*

(1) It has a loop.
(2) $r \equiv 1 \pmod{(d+1)}$ and $q \equiv -d \pmod{n}$.
(3) $r \equiv 1 \pmod{(d+1)}$ and $qd \equiv -1 \pmod{n}$.

PROOF. Let E be any node-cut such that removal of all nodes in E leaves no path from C to D. When $d = 1$, the theorem trivially holds; thus, we assume $d > 1 = g$.

We first show that $G(d, n, q, r)$ is $(d-1)$-connected. Suppose $|E| \leq d-1$, then by the consecutivity lemma we have $|C| + d - 1 \leq |S| \leq |C| + |E|$; hence, $|E| = d - 1$ and S is exactly the union of C and E. So, $G(d, n, q, r)$ is at least $(d-1)$-connected. Moreover, both S and D are consecutive runs. It also follows that each claw from D or E contains at least one node in D and each claw from C contains at least a node in C. For the sake of description, let us first introduce some notation.

For any node i, let $l(i)$ (respectively $r(i)$) be the left (right) end point of $ce(i)$ looking clockwise on the $\mathbb{Z}_n$-circle. Let $i, j \in \mathbb{Z}_n$; then we use the phrase *nodes between i and j* to mean all nodes from i to j or from j to i clockwise around the circle, depending on which one has fewer nodes. Let m be the multiplicative inverse of $q \pmod{n}$ and m^* be the magnitude of m. We will prove the theorem by showing the following two claims.

Claim 7 *If $n \geq 3d$, $|E| \leq d-1$ and $|D| \leq |C|$, then $(q^*-1)(|D|-1) < d$ where q^* is magnitude of q. Furthermore, D has a loop-node unless $q \equiv -d \pmod{n}$.*

Claim 8 *If $n > 3d$, $|E| \leq d-1$ and $|C| \leq |D|$, then $(m^*-1)(|C|-1) < d$. Furthermore, C has a loop-node unless $qd \equiv -1 \pmod{n}$.*

Before proving these facts, let us show how the claims enable us to prove the second part of our theorem.

For the forward direction, if $G(d,n,q,r)$ is d-connected, then clearly it has no loop. Furthermore, if $r \equiv 1 \pmod{(d+1)}$ and $q \equiv -d \pmod{n}$, then we can assume $r = x(d+1)+1$ for some $x \in \mathbb{Z}_n$. By definition, x is connected to $\{x+1,\ldots,x+d\}$ and $x+1$ is connected to $\{x-d+1,\ldots,x\}$. Of the d claws containing x, there is exactly one claw not containing $x+1$ which is the claw from $x+1$. Similarly, of the d claws containing $x+1$, there is exactly one claw not containing x which is the claw from x. The remaining $d-1$ claws contain both x and $x+1$. No other claw intersects x or $x+1$. Hence, removing all $d-1$ nodes whose claws intersect both x and $x+1$ disconnects x and $x+1$ from the rest of the nodes, contradicting $G(d,n,q,r)$ being d-connected. We are left to show that condition (3) does not hold. Again, let $r = x(d+1)+1$ for some $x \in \mathbb{Z}_n$. Notice that if $qd \equiv -1 \pmod{n}$, then xd connects to $\{xd+1,\ldots,xd+d\}$ and $xd+d$ connects to $\{xd,\ldots,xd+d-1\}$. Thus, removing $d-1$ nodes $\{xd+1,\ldots,xd+d-1\}$ disconnect the rest of the nodes from xd and $xd+d$. So if $G(d,n,q,r)$ is d-connected then (3) cannot happen either.

For the backward direction, if none of the three conditions holds and $G(d,n,q,r)$ is still not d-connected, then there exists a node cut E with size less than d. When $|C| \leq |D|$, by Claim 7 it must be the case that $q \equiv -d \pmod{n}$. However, by part (b) of Lemma 1 our graph has no loop only if $0 < (r)_\psi \leq \psi - d$. But $\psi = gcd(q-1,n) = gcd(d+1,n)$ which is less than d unless $d+1 = \psi$. It follows that $(r)_\psi = 1$ or $r \equiv 1 \pmod{(d+1)}$, a contradiction. When $|D| \leq |C|$, by Claim 8 it must be the case that $qd \equiv -1 \pmod{n}$, thus $(q-1)d \equiv -d-1 \pmod{n}$. So $\psi \mid d+1$. Similar to the previous case, we conclude that $d+1 = \psi$, which implies $r \equiv 1 \pmod{(d+1)}$, another contradiction. □

Proof of Claim 7 Without loss of generality, we assume $q > 0$. The case when $q < 0$ is symmetric. Notice that $q^* < n/2$ since $g = 1$. Moreover, as we have discussed, $|E| \leq d-1$ implies that both $S = C \cup E$ and D are consecutive runs and $|E| = d-1$.

We first show that for any two nodes i and $i+1$ of D, S cannot fit between $l(i)$ and $r(i+1)$. (When $q < 0$, we will show that S cannot fit between $l(i+1)$ and $r(i)$.) The number of nodes between $l(i)$ and $r(i+1)$ is $q^* + d$. If S fits between them then $|S| = |C \cup E| \leq q^* + d - 2$ because any claw-end from D must also intersect D. If follows that $q^* > |C|$. Moreover, $|D| \leq |C|$ and $n = |D| + |C| + d - 1$ implies that $|C| \geq \frac{n-d+1}{2} \geq |D|$, and $n \geq 3d$ gives us

$$q^* > |C| \geq \frac{n-d+1}{2} \geq (n - \frac{n}{3} + 1)/2 > n/3.$$

Let l and r be the left and right end point of S, respectively. Consider the nodes between $l(i+2)$ and $r(i+3)$ if $i+2$ and $i+3$ are both in D. It follows from $l(i) \notin S$, $r(i+1) \notin S$, and $n/3 < q^* < n/2$ that $l(i+2)$ lies between $r(i+1)$ and $l(i)$, thus $l(i+2) \notin S$. Now, if $l(i+3) \notin S$ then it must be the case that $|D|$ is at least as large as the number of points from $r(i+1)$ to $l(i+3)$, which is $2q^* - (d-2)$. Thus, $2q^* - (d-2) \leq |D| \leq \frac{n-d+1}{2}$, which leads to $q^* \leq \frac{n}{3} - \frac{3}{4}$, contradicting $q^* > \frac{n}{3}$. Consequently, $l(i+3) \in S$. Again, as each claw-end from D intersects D, $r(i+3) \notin S$; hence, the elements of S also lie between $l(i+2)$ and $r(i+3)$. The same conclusion holds if we consider $i-1$ and $i-2$.

Continuing this way, it is obvious that there are at least $k = \lfloor \frac{|D|-1}{2} \rfloor$ different adjacent pairs $(i, i+1)$ of nodes in D such that the elements of S lie between $l(i)$ and $r(i+1)$. Let these pairs be $(i_1, i_1+1), \ldots, (i_k, i_k+1)$. Without loss of generality, assume $l(i_k)$ is closest to l, which means that $r(i_k+1)$ is furthest from r. Since all $r(i_j)$'s are different and not in S, considering the the points between $l(i_k)$ and $r(i_k+1)$ we obtain $q^* + d - 1 - k \geq |S| = n - |D|$. So,

$$q^* \geq n - |D| - d + 1 + k \geq n - |D| - d + 1 + \frac{|D|}{2} - 1 \geq \frac{n}{2} - \frac{1}{4}$$

contradicting $q^* < n/2$.

Since S cannot fit between $l(i)$ and $r(i+1)$ for any $i, i+1 \in D$, all nodes between $r(i)$ and $l(i+1)$ (if $q^* \geq d$) or between $l(i+1)$ and $r(i)$ (when $q^* < d$) must be in D. Counting this way and taking into account the fact that both D and S are consecutive runs, it is easy to see that D must have at least $2 + q^* - d + (|D| - 2)q^*$ nodes. Thus $|D| > 1 + q^* - d + (|D| - 2)q^*$, or $(q^* - 1)(|D| - 1) < d$.

If $|D| = 1$, then the node in D is clearly a loop-node. If $|D| \geq 2$, then $q^* - 1 < d$. When $q^* < d$, every pair of claws from adjacent nodes overlap. By Lemma 4, D has a loop-node. When $q^* = d$, D has either a loop-node or a pair of nodes i and $i+1$ such that claws from i and $i+1$ end with $f(i) = \{i+1, \cdots, i+d\}$ and $f(i+1) = \{i-d+1, \cdots, i\}$, respectively. The latter one implies that $r + qi \equiv i + 1 \pmod{n}$ and $r + q(i+1) \equiv i - d + 1 \pmod{n}$. Thus, $q \equiv -d \pmod{n}$, the exceptional case. □

Proof of Claim 8 Since $g = 1$, the claw-ends coming out from C can be ordered so that the second node of each is the first node of the next claw-end. Then, C must consist of nodes with indices $a, a + m^*, \ldots, a + (|C| - 1)m^*$, and these must all lie among the $d - 1 + |C|$ consecutive nodes in $S = C \cup E$. If $m^* = 1$, then the claim holds trivially. If $m^* > 1$, then either all the nodes lying between these nodes of C are in E, so that $(m^* - 1)(c - 1) < d$, or the size of D, $|D|$, is at most $m^* - 1$ so that D can fit between adjacent nodes of C in this order. The lemma is proven if we show that this latter case cannot happen when $n > 3d$.

We use the word "interval" to mean the nodes lying between adjacent nodes of C exclusively in the order above. Clearly $m^* < \frac{n}{2}$, thus *either* every other interval of C contains D, so that $m^* \geq |D| + \lfloor \frac{|C|}{2} \rfloor$, *or* there are 3 consecutive intervals such that the first one or the third one contains D and the other two do not, so that $n - 2m^* \geq |D|$. If the former occurs, then $n/2 \geq |D| + |C|/2 \geq (3/2)|C|$, so that $n \leq |D| + |C| + d - 1 < n/2 + n/6 + n/3 - 1 < n$, a contradiction. If the latter case occurs, then $n \geq 2m^* + |D| \geq 3|D| \geq 3|C|$, so $n \leq |D| + |C| + d - 1 \leq n/3 + n/3 + n/3 - 1 < n$, again a contradiction.

If $|C| = 1$, then the node in C is obviously a loop-node. If $|C| \geq 2$, then $m^* - 1 < d$. When $m^* < d$, by Lemma 4 C contains a loop-node. When $m^* = d$, also by Lemma 4 C has either a loop or a pair of nodes i and $i + m^*$ such that the claws from i and $i + m^*$ end with $\{i + 1, \cdots, i + d\}$ and $\{i, \cdots, i + d - 1\}$, respectively. The latter one implies $r + qi \equiv i + 1 \pmod{n}$ and $r + q(i + m^*) \equiv i \pmod{n}$. Thus, $qd = qm^* \equiv -1 \pmod{n}$, the exceptional case. □

The following corollary removes the condition $n \geq d^4$ from the result of Homobono and Peyrat [17].

Corollary 9 *$G_I(d, n)$ is d-connected iff $gcd(n, d) > 1$ and $(d + 1) \mid n$.*

PROOF. Note that $G_I(d, n)$ has no loop iff $(d+1) \mid n$ by Lemma 1 part (b). When $g = gcd(d, n) > 1$ and $(d+1) \mid n$ we must have $g < d$, thus by Theorem 5, $G_I(d, n)$ is d-connected. Conversely, if $d+1$ does not divide n, then $G_I(d, n)$ is not d-connected because it has a loop; and if $g = 1$ and $(d + 1) \mid n$, then by Theorem 6, $G_I(d, n)$ is not d-connected. Notice that we proved this direction independent of $n > 3d$. Therefore, $G_I(d, n)$ is d-connected iff $gcd(n, d) > 1$ and $(d + 1) \mid n$. □

The following corollary uses a weaker condition, namely $n > d \cdot gcd(n, d)$, instead of the condition $n \geq d^3$ in the result of Imase, Soneoka and Okada [22].

Corollary 10 *If $n > d \cdot gcd(n,d)$, then $G_B(d,n)$ and $G_I(d,n)$ are at least $(d-1)$-connected.*

PROOF. When $gcd(n,d) = d$, $G(d,n,q,r)$ is the line-graph of $G(d,n/d,q,r)$. To see this, consider a digraph $\bar{G}$ with nodes $\bar{0}, \dots, \overline{n'-1}$ where $n' = n/d$, and with edges labeled by $0, \dots, n$; each node $\bar{i}$ has in-edges $i, i+n', \dots, i+(d-1)n'$ and out-edges $qi+r, qi+r+1, \dots, qi+r+d-1$. Clearly, there is an edge from $\bar{i}$ to $\bar{j}$ iff $j \equiv qi+r+k \pmod{n'}$ for some $k = 0, \dots, d-1$. Thus, $\bar{G}$ is isomorphic to $G(d,n',q,r)$. On the other hand, the line graph of $\bar{G}$ is $G(d,n,q,r)$.

It was proved in [14] that if $g \mid d$, then $G(d,n,q,r)$ is at least $(d-1)$-line-connected and it is d-line-connected iff it has no loop. This implies that if $g = d$ and $n > d^2$, then $G(d,n,q,r)$ is at least $(d-1)$-connected and it is d-connected iff it has no loop. This fact and Theorem 6 allow us to assume that $1 < gcd(n,d) < d$.

Consider the proof of Theorem 5. We show $|E| \geq d-1$. In Case 1, if $|C| = 1$, $|E| \geq d-1$; if $|C| \geq 2$, then we must have $1 + n/g \leq d$ since only one C-orbit exists. Thus, $n \leq g(d-1)$, a contradiction. In Case 2, by the reduction, we may assume that D has only one block. Since $n > gd \geq (g-1)d$, exactly one claw from D intersects D. However, there are d claws intersecting D. Realize that $d-1$ of them must come from E, i.e. $|E| \geq d-1$. In Case 3, if there is only one C-orbit, then it is similar to that in Case 1. If two C-orbits exist, then each C-orbit contains at least $(g-1)/2$ elements of C. Clearly $E \cup C$ must have at least n/g elements in this case. Any consecutive run of size n/g in $E \cap C$ contains exactly 2 elements of C and the rest are in E. So, if $|E| \leq d-2$, then $n/g \leq 2 + |E| = d$, a contradiction. □

5 Modification of $G_B(d,n)$

A purpose of this study is to find good candidates for the topological structure of communication networks. Here is a basic problem: Given the number of nodes and an upper bound on degree, find a digraph to achieve the smallest diameter and largest connectivity. Suppose that G is a digraph with n nodes and each node of G has in-degree and out-degree at most d. By a simple calculation, it was shown that the diameter of G is at least $\lceil \log_d n(d-1)+1 \rceil - 1$ [9]. In general, for given n and d, determining whether a digraph exists to achieve this lower bound of diameter is not an easy job. However, if we allow a difference of one from the optimal value, then the generalized de Bruijn digraphs $G_B(d,n)$ and the generalized Kautz digraphs $G_I(d,n)$ meet the requirement (see [19–21]). A question is, could these graphs be modified to have largest connectivity? In fact, loops do nothing to contribute to the connectivity. One can "improve" them by deleting whatever loops occur according to the defining formulae, replacing them by a single cycle or several disjoint cycles. This improvement has been studied for $d = 2$ in [11, 24].

From Theorem 6, we see that $G_I(d,n)$ can be at most $(d-1)$-connected without a loop. So, the improvement does not always exist for $G_I(d,n)$. However, it almost always exists for $G_B(d,n)$. We give this result in this section. We first prove a lemma.

Lemma 11 *Let $n > \max(5d, d^2+1)$ and $d \geq 2$. Suppose that E is a node-cut of size at most $d-1$ in $G_B(d,n)$, such that removal of the nodes in E leaves no path from any node in C to any node in D; then* either

(1) C has a loop node, and the number of nodes between any loop node in E and any loop node in C is at most $2d-1$, or
(2) D has a loop node, and the number of nodes between any loop node in E and any loop node in D is at most $2d-1$.

PROOF. First, assume $gcd(n,d) = 1$. Since $1 < d < d^2+1 < n$, we have $q^* \neq 1$, $m^* \neq 1$, $d \not\equiv -d \pmod{n}$ and $d^2 \not\equiv -1 \pmod{n}$. Moreover, by the consecutivity lemma, the set S has $|C|+d-1$ elements and both $S = C \cup E$ and D are consecutive runs. In this case, $|E|$ is indeed minimum, so every claw-end from E must intersect D. Consider the proof of Theorem **??**. If $|D| \leq |C|$, then by Claim 7 D has a loop node and $|D| \leq d$. A claw-end from an E-loop node has to intersect D, so that the number of nodes between an E-loop node and any node in D is at most $2d-1$. If $|C| \leq |D|$, then by Claim 8 C has a loop node and $|C| \leq d$. Hence $|S| \leq 2d-1$. Consequently, the number of nodes between any two nodes in $E \cup C$ is at most $2d-1$.

Now, assume $1 < gcd(n,d) < d$. Notice that in the proof of Theorem 5, the minimality of $|E|$ is assumed. Here, we do not assume it. However, by Corollary 10, $|E| = d-1$ is indeed minimum. The difference is that D may not be consecutive. To meet the assumption $S = C \cup E$ in the proof of Theorem 5, we have to move at most $g-1$ elements from C into D. Those elements are in $A = (C \cup E) \setminus S$ and cannot have a loop. So, the movement affects only the sizes of C and D. Let $C' = C \setminus A$ and $D' = D \cup A$. Now, consider the proof of Theorem 5 applied to E, C' and D'. In case 1, every node in C' has a loop, so that C has a loop. Moreover, $|S| = d$ so $|C'| = 1$ and $|C| \leq 1+g-1 < d$. Clearly the loop node of C' is within d of every node in E. The claw-end from a loop node i in A has to intersect S, so i is also within $2d-1$ of every node in E. In sum, the number of nodes between a C-loop node and an E-loop node is at most $2d-1$. In case 2, $|D'| \leq d-1$ and D' has a loop, so that $|D| \leq d-1$ and D has a loop. The cardinality of E is minimum so every claw-end from E intersects D'. It follows that every E-loop node is within $2d-1$ from every loop node in D ($\subseteq D'$). In case 3, of g elements in an orbit, $\lceil \frac{g}{2} \rceil - 1$ must not be in D' and clearly are elements of E. Moreover, just as we have noted in the proof of Theorem 5, there are at least $\lceil \frac{|D'|+d-1}{g} \rceil$ orbits whose claw-ends intersect D'; therefore, $|E| \geq \lceil \frac{|D'|+d-1}{g} \rceil$. This gives us

$$\lfloor \frac{g}{2} \rfloor \frac{n}{g} \leq |D'| \leq \frac{g|E|}{\lceil \frac{g}{2} - 1 \rceil} - (d-1).$$

Since g in this case must be odd, $g = gcd(d, n) > 1$, and $|E| \leq d-1$, it is easy to see that this contradicts $n > \max(5d, d^2+1)$.

Finally, we consider the case of $gcd(n, d) = d$. Note that $G_B(d, n)$ is the line-graph of $G_B(d, n/d)$. Thus, E gives a line-cut of size at most $d-1$ for $G_B(d, n/d)$. However, it was proven in [23] that such a line-cut must be incident to a node of $G_B(d, n/d)$, which implies that C or D is a singleton. So, the lemma holds. $\square$

A digraph is called a *modified* $G(d, n, q, r)$ if it is constructed from $G(d, n, q, r)$ by connecting all loop-nodes into disjoint cycles with sizes at least two and deleting all loops. The modification is said to be *cyclic* if all loop-nodes are connected into a single cycle. The modification is said to be *simple* if there is no multiple edge in the resultant simple graph.

Theorem 12 *When* $n > 2d(d-1)$ *and* $d \geq 4$*, there exists a cyclically-modified* $G_B(d, n)$ *of connectivity* d.

PROOF. Consider two loop-nodes x and y where the number of nodes between them is at least $2d$. When (1) in Lemma 11 occurs, $x \in C$ implies $y \in D$. When (2) in Lemma 11 occurs, $x \in D$ implies $y \in C$. This means that as long as all loop-nodes are connected by a cycle (or disjoint cycles) with edges of "distance" at least $2d-1$, the node-cut E of size less than d will no longer exist in the modified graph. Hence, the connectivity becomes d. We next show the existence of such a modification. Consider a graph H with node set consisting of all loop-nodes of $G_B(d, n)$ and an edge between x and y exists iff x and y are at a distance at least $2d-1$ from each other. If H is Hamiltonian, then the theorem is proved. We prove the Hamiltonian property of H by showing that minimum degree $\delta(H)$ of H is at least half the number of its nodes. Consider any loop-node i of $G_B(d, n)$. As we have mentioned in section 2, except for the possible loop node right next to i, all other loop nodes are at least $n/(d-1) - 1 \geq 2d-1$ from it, i.e the number of nodes between them is at least $2d$. Hence, $\delta(H) \geq d-2+\psi$. It is easy to see that when $d \geq 4$, $d-2+\psi \geq \frac{d-1+\psi}{2}$. $\square$

Notice that as $n > \max(5d, d^2+1)$, our cyclic modification is also simple. The next theorem relaxes the conditions on n and d a bit further.

Theorem 13 *Let $\psi = gcd(d-1, n)$. If $1 < \psi < d-1$, then for $n \geq d^2$ and $d \geq 2$, there exists a simply-modified $G_B(d,n)$ of connectivity d.*

PROOF. For each k-value such that $\psi \mid k$, there are exactly ψ loop-nodes which are evenly distributed with distance n/ψ. Note that $n/\lambda \geq 2n/d \geq 2d$. We connect each loop-node x to another loop-node $x + n/\psi$. Then, all loop-nodes are connected by several disjoint cycles of size ψ, and all edges are in the graph H of the proof of Theorem 12. Finally, we notice that the above connections produce no multiple edges. The details are easy to verify. □

6 Discussions

In this paper, we have determined the connectivity of consecutive-d digraphs $G(d,n,q,r)$ in almost all cases, and studied how to modify these graphs to maximize connectivity. Our results generalized and improved existing results on de Bruijn digraphs, Kautz digraphs, and their generalizations.

There are still, however, a few small gaps in our characterization of the connectivity of $G(d,n,q,r)$. In particular, several problems remained to be solved:

(a) When $\gcd(q,n) = d$ and $n \leq d^2$, what are the necessary and sufficient conditions for $G(d,n,q,r)$ to be d-connected.
(b) When $\gcd(q,n) = 1$ and $n \leq 3d$, what are the necessary and sufficient conditions for $G(d,n,q,r)$ to be d-connected.
(c) When $1 < \gcd(q,n) < d$, what are the necessary and sufficient conditions for $G(d,n,q,r)$ to be $(d-i)$-connected, where $0 < i < \gcd(q,n)$.

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