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Optimal Consecutive-k-out-of-$(2k+1)$: G Cycle ⋆

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Abstract. We present a complete proof for the invariant optimal assignment for consecutive-k-out-of-$(2k+1)$: G cycle, which was proposed by Zuo and Kao in 1990 with an incomplete proof, pointed out recently by Jalali, Hawkes, Cui and Hwang.

Key words: Invariant optimal assignment, Consecutive-k-out-of-n: G cycle

1. Introduction

A cyclic consecutive-k-out-of-n: G system $con_C(k, n : G)$ is a cycle of $n(\geqslant k)$ components such that the system works if and only if some k consecutive components all work. Suppose n components with reliabilities $p_{[1]} \leqslant p_{[2]} \leqslant \cdots \leqslant p_{[n]}$ are all exchangeable. How can they be assigned to the n positions on the cycle to maximize the reliability of the system? Kuo, Zhang and Zuo [2] showed that if $k = 2$, then the optimal assignment is *invariant*, i.e., it depends only on the ordering of reliabilities of the components, but not their value. Invariant optimal assignment is very important in practice. In fact, in the real world, one usually knows the ranking of reliabilities of components, but not their exact values. For example, the ages of the components are known and one cannot compute the exact value of reliability from the age of each component. However, one may rank reliabilities of components according to their age by the rule that the older the less reliable. Kuo, Zhang and Zuo [2] also showed that for $k \geqslant 3$ and $n > 2k+1$, $Con_C(k, n : G)$ has no invariant optimal assignment. For $n \leqslant 2k+1$, Zuo and Kuo [3] claimed that there exists an invariant optimal assignment

$$(p_{[1]}, p_{[3]}, p_{[5]}, \ldots, p_{[6]}, p_{[4]}, p_{[2]},).$$

However, Jalali, Hawkes, Cui and Hwang [1] found that their proof is incomplete. In this paper, we give a complete proof for this invariant optimal assignment in case $n = 2k + 1$.

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A similar situation occurred in a line system. A linear consecutive-k-out-of-n: Gsystem $Con_L(k, n : G)$ can be defined in a similar way to the cyclic system $Con_C(k, n : G)$. Kuo, Zhang and Zuo [2] presented an invariant optimal assignment for $Con_L(k, n : G)$ with $n \leqslant 2k$. However, the proof is incomplete too. This was also pointed out by Jalali et al. [1]. In addition, they gave a complete proof for the line system. It is worth pointing out that by setting $p_{[1]} = \cdots = p_{[2k+1-n]} = 0$, the cycle system $Con_C(k, 2k + 1 : G)$ becomes the line system $Con_L(k, n : G)$. Thus, our result yields the result of Jalali et al. [1]. The cycle case is in general much more difficult than the line case, and our proof adopts a new approach different from previous attempts.

2. Main Result

In this section, we show the following.

THEOREM 1. *$Con_C(k, 2k + 1 : G)$ has invariant optimal assignment*

$$(p_{[1]}, p_{[3]}, p_{[5]}, \ldots, p_{[2k+1]}, p_{[2k]}, \ldots, p_{[6]}, p_{[4]}, p_{[2]}).$$

Let $p_1, p_2, \ldots, p_{2k+1}$ be reliabilities of the $2k + 1$ components on the cycle in counter-clockwise direction. For simplicity of the proof, we first assume that

$$0 < p_{[1]} < p_{[2]} < \cdots < p_{[2k+1]} < 1.$$

Our proof is based on the following representation of the reliability of consecutive-k-out-of-n: G cycle for $n \leqslant 2k + 1$.

LEMMA 1. *The reliability of consecutive-k-out-of-n: Gcycle for $n \leqslant 2k+1$ under assignment C can be represented as*

$$R(C) = p_1 \ldots p_n + \sum_{i=1}^{n} q_i p_{i+1} \ldots p_{i+k}$$
$$= p_1 \ldots p_n + \sum_{i=1}^{n} p_i \ldots p_{i+k-1} - \sum_{i=1}^{n} p_i \ldots p_{i+k}$$

where $q_i = 1 - p_i$ and $p_{n+i} = p_i$.

Proof. The system works if and only if all components work or for some i, the ith component fails and the $(i + 1)$st component, ..., the $(i + k)$th component all work. Since $n \leqslant 2k + 1$, there exists at most one such i. Therefore,

$$R(C) = p_1 \ldots p_n + \sum_{i=1}^{n} q_i p_{i+1} \ldots p_{i+k}.$$

Note that

$$q_i p_{i+1} \ldots p_{i+k} = p_{i+1} \ldots p_{i+k} - p_i \ldots p_{i+k}.$$

This implies the second representation. □

To prove Theorem 1, it suffices to show that in any optimal assignment,

$$(p_i - p_j)(p_{i-1} - p_{j+1}) > 0 \text{ for } 1 < i < j < 2k+1. \tag{1}$$

That is, selecting any component to be labeled p_1, we always have

$$(p_i - p_{2k+2-i})(p_{i+1} - p_{2k+1-i}) > 0 \text{ for } i = 1, \ldots, k. \tag{2}$$

For simplicity of representation, we denote $i' = 2k + 2 - i$. Note that $(k+1)' = k+1$ and $(i')' = i$. Furthermore, without loss of generality, we assume $p_1 > p_{1'}$ throughout this proof. Then the condition (2) can be rewritten as

$$p_i > p_{i'} \text{ for } i = 1, \ldots, k. \tag{3}$$

We will employ an inductive argument to prove these k inequalities in the following ordering:

$$\begin{aligned} p_1 &> p_{1'} \\ p_k &> p_{k'} \\ p_2 &> p_{2'} \\ p_{k-1} &> p_{(k-1)'} \\ &\ldots \end{aligned}$$

Let C_i be the assignment obtained from C by exchanging components i and i' and $C_{i,j,\ldots,y,z} = (C_{i,j}, \ldots, y)_z$. Let C^* be an optimal assignment. We divide this inductive argument into the following lemmas.

LEMMA 2. *Under assumption that $p_1 > p_{1'}$, we must have $p_k > p_{k'}$ and $p_2 \ldots p_{k-1} \geqslant p_{2'} \ldots p_{(k-1)'}$.*

Proof. Consider

$$0 \leqslant R(C^*) - R(C_1^*) = (p_1 - p_{1'})(p_2 \ldots p_k - p_{2'} \ldots p_{k'})q_{k+1}.$$

Since $p_1 > p_{1'}$, we have

$$p_2 \ldots p_k \geqslant p_{2'} \ldots p_{k'}. \tag{4}$$

Note that

$$\begin{aligned} 0 &\leqslant R(C^*) - R(C_k^*) \\ &= (p_k - p_{k'})[(q_{1'}p_1 + q_1 p_{k+1})p_2 \ldots p_{k-1} - (p_{1'}q_1 + q_{1'}p_{k+1})p_{2'} \ldots p_{k-1)'}] \end{aligned} \tag{5}$$

and

$$(q_{1'}p_1 + q_1 p_{k+1}) - (p_{1'}q_1 + q_{1'}p_{k+1}) = (p_1 - p_{1'})(1 - p_{k+1}) > 0.$$

We first claim that $p_k > p_{k'}$. In fact, if $p_2 \dots p_{k-1} \geqslant p_{2'} \dots p_{(k-1)'}$, then it follows from (5) that $p_k > p_{k'}$; If $p_2 \dots p_{k-1} < p_{2'} \dots p_{(k-1)'}$, then it follows from (4) that $p_k > p_{k'}$.

Now, we show that $p_2 \dots p_{k-1} \geqslant p_{2'} \dots p_{(k-1)'}$. Note that

$$
\begin{aligned}
0 \leqslant & R(C^*) - R(C^*_{1,k}) \\
=& [(p_1 p_k - p_{1'} p_{k'}) + (p_k - p_{k'}) p_{k+1} - p_{1'} p_1 (p_k - p_{k'})](p_2 \cdots p_{k-1} - p_{2'} \cdots p_{(k-1)'}) q_{k+1} \\
=& [(p_1 p_k q_{1'} - p_{1'} p_{k'} q_1) q_{k+1} + (1 - p_1 p_{1'})(p_k - p_{k'}) p_2 \cdots p_{k-1} - p_{2'} \cdots p_{(k-1)'}).
\end{aligned}
$$

Moreover, $p_1 q_{1'} - p_{1'} q_1 = p_1 - p_{1'} > 0$ and $p_k > p_{k'}$. Therefore, $p_2 \dots p_{k-1} \geqslant p_{2'} \dots p_{(k-1)'}$. □

To show inductive step, we first prove an equality.

LEMMA 3. *Suppose $m < k/2$. Then*

$$
\begin{aligned}
&\sum_{i=1}^{2m+1} \left(\prod_{j=0}^{k-1} p_{-m+i+j} - \prod_{j=0}^{k} p_{-m+i+j} \right) \\
&\quad = \sum^{m-1} i = 1 \left(\prod_{j=1}^{i} p_k p_{j'} \right) (p_{i+1} q_{(i+1)'} \left(\prod_{j=i+2}^{k-i} p_j \right) \left[1 - \left(\prod_{j=k-i+1}^{k} p_j p_{j'} \right) p_{k+1} \right] \\
&\qquad + \sum_{i=1}^{m} \left(1 - \prod_{j=1}^{i} p_j p_{j'} \right) \left(\prod_{j=i+1}^{k-i} p_j \right) (p_{k-i+1} q_{(k-i+1)'}) \left(\prod_{j=k-i+2}^{k} p_j p_{j'} \right) p_{k+1} \\
&\qquad + q_{1'} p_1 \dots p_k q_{k+1} + \left(\prod_{j=1}^{m} p_j p_{j'} \right) \left(\prod_{j=m+1}^{k-m} p_j \right) \left[1 - \left(\prod_{j=k-m+1}^{k} p_j p_{j'} \right) p_{k+1} \right].
\end{aligned}
$$

Proof. Note that

$$
\begin{aligned}
& p_{1'} p_1 \dots p_{k+1} \\
&\quad = p_{1'} p_1 \dots p_{k+1} q_{k'} + p_{1'} p_1 \dots p_{k+1} p_{k'} \\
&\quad = p_{1'} p_1 \dots p_{k+1} q_{k'} + q_{2'} p_{1'} p_1 \dots p_{k+1} p_{k'} + p_{2'} p_{1'} p_1 \dots p_{k+1} p_{k'} \\
&\quad = \cdots \\
&\quad = \sum_{i=1}^{m-1} \left(\prod_{j=1}^{i} p_j p_{j'} \right) \left(\prod_{j=i+1}^{k-i} p_j \right) (p_{k-i+1} q_{(k-i+1)'}) \left(\prod_{j=k-i+2}^{k} p_j p_{j'} \right) p_{k+1} \\
&\qquad + \sum_{i=1}^{m} \left(\prod_{j=1}^{i} p_j p_{j'} \right) (p_{i+1} q_{(i+1)'}) \left(\prod_{j=i+2}^{k-1} p_j \right) \left(\prod_{j=k-1+1}^{k} p_j p_{j'} \right) p_{k+1} \\
&\qquad + \left(\prod_{j=1}^{m} p_j p_{j'} \right) \left(\prod_{j=m+1}^{k-m} p_j \right) \left(\prod_{j=k-m+1}^{k} p_j p_{j'} \right) p_{k+1}
\end{aligned}
$$

and

$$p_1\dots p_k - p_{1'}p_1\dots p_k - p_1\dots p_{k+1}$$
$$= q_{1'}p_1\dots p_k q_{k+1} - p_{1'}p_1\dots p_{k+1}.$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{2m+1}\left(\prod_{j=0}^{k-1} p_{-m+i+j} - \prod_{j=0}^{k} p_{-m+i+j}\right)\\
&\quad = \sum_{i=2}^{m} q_{-m+i-1}\prod_{j=0}^{k-1} p_{-m+i+j} + \sum_{i=m+2}^{2m+1} q_{-m+i+k}\prod_{j=0}^{k-1} p_{-m+i+j}\\
&\qquad + p_{-m+1}\dots p_{-m+k} + p_1\dots p_k - p_0\dots p_k - p_1\dots p_{k+1}\\
&\quad = \sum_{i=1}^{m-1}\left(\sum_{j=1}^{i} p_j p_{j'}\right)(p_{i+1}q_{(i+1)'})\left(\prod_{j=i+2}^{k-i} p_j\right)\\
&\qquad + \sum_{i=1}^{m}\left(\prod_{j=i+1}^{k-i} p_j\right)(p_{k-i+1}q_{(k-i+1)'})\left(\prod_{j=ik-i+2}^{k} p_j p_{j'}\right)p_{k+1}\\
&\qquad + \left(\prod_{j=1}^{m} p_j p_{j'}\right)\left(\prod_{j=m+1}^{k-m} p_j\right) + p_1\dots p_k - p_{1'}p_1\dots p_k - p_1\dots p_{k+1}\\
&\quad = \sum_{i=1}^{m-1}\left(\prod_{j=1}^{i} p_j p_{j'}\right)(p_{i+1}q_{(i+1)'})\left(\prod_{j=i+2}^{k-i} p_j\right)\left[1-\left(\prod_{j=k-i+1}^{k} p_j p_{j'}\right)p_{k+1}\right]\\
&\qquad + \sum_{i=1}^{m}\left(1-\prod_{j=1}^{i} p_j p_{j'}\right)\left(\prod_{j=i+1}^{k-i} p_j\right)(p_{k-i+1}q_{(k-i+1)'})\left(\prod_{j=k-i+2}^{k} p_j p'_j\right)p_{k+1}\\
&\qquad + q_{1'}p_1\dots p_k q_{k+1} + \left(\prod_{j=1}^{m} p_j p_{j'}\right)\left(\prod_{j=m+1}^{k-m} p_j\right)\left[1-\left(\prod_{j=k-m+1}^{k} p_j p_{j'}\right)p_{k+1}\right].
\end{aligned}$$

□

LEMMA 4. *Suppose $m < k/2$. If*

$$p_i > p_{i'} \text{ for } i = 1,\dots,m,k-m+1,\dots,k \tag{6}$$

and

$$\prod_{j=m+1}^{k-m} p_j \geqslant \prod_{j=m+1}^{k-m} p_{j'} \tag{7}$$

then

$$p_{m+1} > p_{(m+1)'}$$

and

$$\prod_{j=m+2}^{k-m} p_j \geqslant \prod_{j=m+2}^{k-m} p_{j'}.$$

Proof. We first show $p_{m+1} > p_{(m+1)'}$. For contradiction, suppose $p_{m+1} < p_{(m+1)'}$. It follows from (7) that

$$\prod_{j=m+2}^{k-m} p_j > \prod_{j=m+2}^{k-m} p_{j'}. \tag{8}$$

By Lemma 3,

$$\begin{aligned}
0 \leqslant{}& R(C^*) - R(C^*_{m+1}) \\
={}& \sum_{i=1}^{2m+1}\left(\prod_{j=0}^{k-1} p_{-m+i+j} - \prod_{j=0}^{k} p_{-m+i+j} + \prod_{j=0}^{k-1} p_{(-m+i+j)'} - \prod_{j=0}^{k} p_{(-m+i+j)'}\right) \\
&- \sum_{i=1}^{2m+1}\left[\frac{p_{(m+1)'}}{p_{m+1}}\left(\sum_{j=0}^{k-1} p_{-m+i+j} - \prod_{j=0}^{k} p_{-m+i+j}\right)\right. \\
&\left. + \frac{p_{m+1}}{p_{(m+1)'}}\left(\prod_{j=0}^{k-1} p_{-m+i+j} - \prod_{j=0}^{k} p_{(-m+i+j)'}\right)\right] \\
={}&(p_{m+1} - p_{(m+1)'})\left\{\sum_{i=1}^{m-1}\left(\prod_{j=1}^{i} p_j p_{j'}\right)\left[(p_{i+1}q_{(i+1)'}\left(\prod_{j=i+2}^{m} p_j\right)\left(\prod_{j=m+2}^{k-i} p_j\right)\right.\right. \\
&\left. -(p_{(i+1)'}q_{i+1})\left(\prod_{j=i+2}^{m} p_{j'}\right)\left(\prod_{j=m+2}^{k-i} p_{j'}\right)\right]\left[1 - \left(\prod_{j=k-i+1}^{k} p_j p_{j'}\right) p_{k+1}\right] \\
&+ \sum_{i=1}^{m}\left(1 - \prod_{j=1}^{i} p_i p_{j'}\right)\left[\left(\prod_{j=i+1}^{m} p_j\right)\left(\prod_{j=m+2}^{k-i} p_j\right)(p_{k-i+1}q_{(k-i+1)'})\right. \\
&\left. - \left(\prod_{j=i+1}^{m} p_{j'}\right)\left(\prod_{j=m+2}^{k-i} p_{j'}\right)(p_{(k-i+1)'}q_{k-i+1})\right]\left(\prod_{j=k-i+2}^{k} p_j p_{j'}\right) p_{k+1} \\
&+ (q_{1'}p_1 \cdots p_m p_{m+2} \cdots p_k - q_1 p_{1'} \cdots p_{m'} p_{(m+2)'} \cdots p_{k'})q_{k+1} \\
&\left. + \left(\prod_{j=1}^{m} p_j p_{j'}\right)\left(\prod_{j=m+2}^{k-m} p_j - \prod_{j=m+2}^{k-m} p_{j'}\right)\left[1 - \left(\prod_{j=k-m+1}^{k} p_j p_{j'}\right) p_{k+1}\right]\right\} \\
<{}&0,
\end{aligned}$$

a contradiction. The last inequality holds because it follows from (6) and (8) that every term in $\{\cdots\}$ is positive.

Now, we show

$$\prod_{j=m+2}^{k-m} p_j \geqslant \prod_{j=m+2}^{k-m} p_{j'}.$$

By Lemma 3, we have

$$\begin{aligned}
0 \leqslant\, & R(C^*) - R(C^*_{m+2,\dots,m-k}) \\
= & \left(\prod_{j=m+2}^{k-m} p_j - \prod_{j=m+2}^{k-m} p_{j'}\right) \\
& \cdot \left\{ \sum_{i=1}^{m-1} \left(\prod_{j=1}^{i} p_j p_{j'}\right) \left[(p_{i+1} q_{(i+1)'} \left(\prod_{j=i+2}^{m+1} p_j\right) \left(\prod_{j=k-m+1}^{k-i} p_j\right) \right.\right. \\
& \left. - (p_{(i+1)'} q_{i+1}) \left(\prod_{j=i+2}^{m+1} p_{j'}\right) \left(\prod_{j=k-m+1}^{k-i} p_{j'}\right) \right] \left[1 - \left(\prod_{j=k-i+1}^{k} p_j p_{j'}\right) p_{k+1} \right] \\
& + \sum_{i=1}^{m} \left(1 - \prod_{j=1}^{i} p_j p_{j'}\right) \left[\left(\prod_{j=i+1}^{m+1} p_j\right) \left(\prod_{j=k-m+1}^{k-i} p_j\right) (p_{k-i+1} q_{(k-i+1)'}) \right. \\
& \left. - \left(\prod_{j=i+1}^{m+1} p_{j'}\right) \left(\prod_{j=k-m+1}^{k-i} p_{j'}\right) (p_{(k-i+1)'} q_{k-i+1}) \right] \left(\prod_{j=k-i+2}^{k} p_j p_{j'}\right) p_{k+1} \\
& + (q_{1'} p_1 \dots p_{m+1} p_{k-m+1} \cdot p_k - q_1 p_{1'} \dots p_{(m+1)'} p_{(k-m+1)'} \dots p_{k'}) q_{k+1} \\
& \left. + \left(\prod_{j=1}^{m} p_j p_{j'}\right) (p_{m+1} - p_{(m+1)'} \left[1 - \left(\prod_{j=k-m+1}^{k} p_j p_{j'}\right) p_{k+1} \right] \right\}.
\end{aligned}$$

It follows from (6) and $p_{m+1} > p_{(m+1)'}$ that every term in $\{\cdots\}$ is positive. Therefore,

$$\prod_{j=m+2}^{k-m} p_j \geqslant \prod_{j=m+2}^{k-m} p_{j'}. \qquad \square$$

LEMMA 5. *Suppose $m < k/2$. Then*

$$\begin{aligned}
& \sum_{i=1}^{2m+2} \prod_{j=0}^{k-1} p_{-m+i+j} - \sum_{i=1}^{2m+2} \prod_{j=0}^{k} p_{-m-1+i+j} \\
& = \sum_{i=1}^{m} \left(\prod_{j=1}^{i} p_j p_{j'}\right) (p_{i+1} q_{(i+1)'}) \left(\prod_{j=i+2}^{k-i} p_j\right) \left[1 - \left(\prod_{j=k-i+1}^{k} p_j p_{j'}\right) p_{k+1} \right] \\
& + \sum_{i=1}^{m} \left(1 - \prod_{j=1}^{i} p_j p_{j'}\right) \left(\prod_{j=i+1}^{k-i} p_j\right) (p_{k-i+1} q_{(k-i+1)'}) \left(\prod_{j=k-i+2}^{k} p_j p_{j'}\right) p_{k+1} \\
& + q_{1'} p_1 \dots p_k q_{k+1} \\
& + \left(1 - \prod_{j=1}^{m} p_j p_{j'}\right) \left(\prod_{j=m+1}^{k-m} p_j\right) \left(\prod_{k-m+1}^{k} p_j p_{j'}\right) p_{k+1}.
\end{aligned}$$

Proof. It is similar to the proof of Lemma 3. □

LEMMA 6. *Suppose $m < k/2$. If*

$$p_i > p_{i'} \text{ for } i = 1, \ldots, m+1, k-m+1, \ldots, k \tag{9}$$

and

$$\prod_{j=m+2}^{k-m} p_j \geqslant \prod_{j=m+2}^{k-m} p_{j'} \tag{10}$$

then

$$p_{k-m} > p_{(k-m)'} \tag{11}$$

and

$$\prod_{j=m+2}^{k-m-1} p_j \geqslant \prod_{j=m+2}^{k-m-1} p_{j'}. \tag{12}$$

Proof. We first show (11). For contradiction, suppose $p_{k-m} < p_{(k-m)'}$. By (10),

$$\prod_{j=m+2}^{k-m-1} p_J > \prod_{j=m+2}^{k-m-1} p_{j'}. \tag{13}$$

By Lemma 5,

$$\begin{aligned}
0 \leqslant\, & R(C^*) - R(C^*_{k-m}) \\
=\, & (p_{k-m} - p_{(k-m)'}) \Bigg\{ \sum_{i=1}^{m} \left(\prod_{j=1}^{i} p_j p_{j'} \right) \Bigg[(p_{i+1} q_{(i+1)'}) \left(\prod_{j=i+2}^{k-m-1} p_j \right) \left(\prod_{j=k-m+1}^{k-i} p_j \right) \\
& - (p_{(i+1)'} q_{i+1} \left(\prod_{j=i+2}^{k-m-1} p_{j'} \right) \left(\prod_{j=k-m+1}^{k-i} p + j' \right) \Bigg] \Bigg[1 - \left(\prod_{j=k-i+1}^{k} p_j p_{j'} \right) p_{k+1} \Bigg] \\
& + \sum_{i=1}^{m} \left(1 - \prod_{j=1}^{i} p_j p_{j'} \right) \Bigg[\left(\prod_{j=i+1}^{k-m-1} p_j \right) \left(\prod_{j=k-m+1}^{k-i} p_j \right) (p_{k-i+1} q_{(k-i+1)'}) \\
& - \left(\prod_{j=i+1}^{k-m-1} p_{j'} \right) \left(\prod_{j=k-m+1}^{k-i} p_{j'} \right) (p_{(k-i+1)'} q_{k-i+1}) \Bigg] \left(\prod_{j=k-i+2}^{k} p_j p_{j'} \right) p_{k+1} \\
& + (q_{1'} p_1 \cdots p_{k-m-1} p_{k-m+1} \cdots p_k - q_1 p_{1'} \cdots p_{(k-m-1)'} p_{(k-m+1)'} \cdots p_{k'}) q_{k+1} \\
& + \left(1 - \prod_{j=1}^{m} p_j p_{j'} \right) \left(\prod_{j=m+1}^{k-m-1} p_j - \prod_{j=m+1}^{k-m-1} p_{j'} \right) \left(\prod_{k-m+1}^{k} p_j p_{j'} \right) p_{k+1} \Bigg\} \\
& < 0,
\end{aligned}$$

a contradiction.

Now, we prove (12). Consider

$$\begin{aligned}
0 \leqslant{} & R(C^*) - R(C^*_{m+2,\dots,k-m-1}) \\
={} & \left(\prod_{j=m+2}^{k-m-1} p_j - \prod_{j=m+2}^{k-m-1} p_{j'}\right) \\
& \cdot \left\{ \sum_{i=1}^{m} \left(\prod_{j=1}^{i} p_j p_{j'}\right) \left[(p_{i+1} q_{(i+1)'} \left(\prod_{j=i+2}^{m+1} p_j\right) \left(\prod_{j=k-m}^{k-1} p_j\right) \right.\right. \\
& \left. - (p_{(i+1)'} q_{i+1}) \left(\prod_{j=i+2}^{m+1} p_{j'}\right) \left(\prod_{j=k-m}^{k-i} p_{j'}\right)\right] \left[1 - \left(\prod_{j=m-i+1}^{k} p_j p_{j'}\right) p_{k+1}\right] \\
& + \sum_{i=1}^{m} \left(1 - \prod_{j=1}^{i} p_j p_{j'}\right) \left[\left(\prod_{j=i+1}^{m+1} p_j\right) \left(\prod_{j=k-m}^{k-1} p_j\right) (p_{k-i+1} q_{(k-i+1)'}) \right. \\
& \left. - \left(\prod_{j=i+1}^{m+1} p_{j'}\right) \left(\prod_{j=k-m}^{k-i} p_{j'}\right) (p_{(k-i+1)'} q_{k-i+1}) \right] \left(\prod_{j=k-i+2}^{k} p_j p_{j'}\right) p_{k+1} \\
& + (q_{1'} p_1 \dots p_{m+1} p_{k-m} \dots p_k - q_1 p_{1'} \dots p_{(m+1)'} p_{(k-m)'} \dots p_{k'}) q_{k+1} \\
& \left. + \left(1 - \prod_{j=1}^{m} p_j p_{j'}\right) (p_{m+1} p_{k-m} - p_{(m+1)'} p_{(k-m)'}) \left(\prod_{k-m+1}^{k} p_j p_{j'}\right) p_{k+1} \right\}.
\end{aligned}$$

It follows from (9) and (11) that every term in $\{\dots\}$ is positive. Thus, (12) holds. □

By Lemmas 2, 4 and 6, we know that (3) holds. Therefore, Theorem 1 is proved for $0 < p_{[1]} < \cdots < p_{[2k+1]} < 1$.

Finally, we deal with the case that some equality signs hold in $0 \leqslant p_{[1]} \leqslant p_{[2]} \leqslant \cdots \leqslant p_{[2k+1]} \leqslant 1$. If $p_{[1]} = p_{[2]} = \cdots = p_{[2k+1]}$, then Theorem 1 is trivially true. If there exists i, $1 \leqslant i < 2k+1$, such that $p_{[i]} < p_{[i+1]}$, then for sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
0 &< p_{[1]} + \varepsilon < \cdots < p_{[i]} + i\varepsilon < p_{[i+1]} \\
&\quad - (2k+1-i)\varepsilon < \cdots < p_{[2k+1]} - \varepsilon < 1.
\end{aligned}$$

For them, we already proved the optimality of assignment C^* in Theorem 1, that is, for any assignment C, $R(C^*) \geqslant R(C)$. Now, we can complete our proof of Theorem 1 by setting $\varepsilon \to 0$.

3. Discussion

Zuo and Kuo [3] proved only that $R(C^*) \geqslant R(C^*_i)$ for any i. This cannot imply that $R(C^*) \geqslant R(C)$ for any assignment C. Thus, their proof is incomplete.

Note that $Con_C(k, n : G)$ for $n \leqslant 2j$ cannot be induced from $Con_C(k, 2k+1 : G)$. Moreover, our inductive argument does not work in cycle system $Con_C(k, n : G)$ for $n \leqslant 2k$. Thus, some additional techniques are required to solve the general case.

References

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