What do Shannon-type Inequalities, Submodular Width, and Disjunctive Datalog have to do with one another?

Mahmoud Abo Khamis[1] Hung Q. Ngo[1] Dan Suciu[1,2]

[1]LogicBlox Inc. [2]University of Washington

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- A Join Algorithm

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  - subsumes most known bounds.

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- Results on Shannon-type Inequalities

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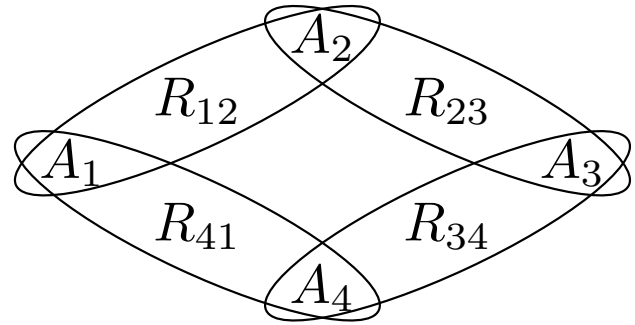

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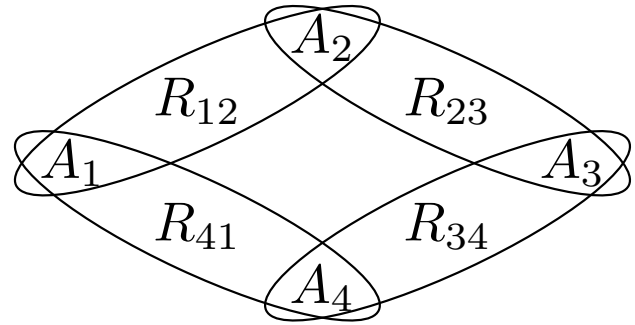

A_1	A_2
a	1
b	1
b	2

A_2	A_3
1	c
1	d
2	c

A_3	A_4
c	3
d	4
d	5

A_4	A_1
3	b
4	a
4	b

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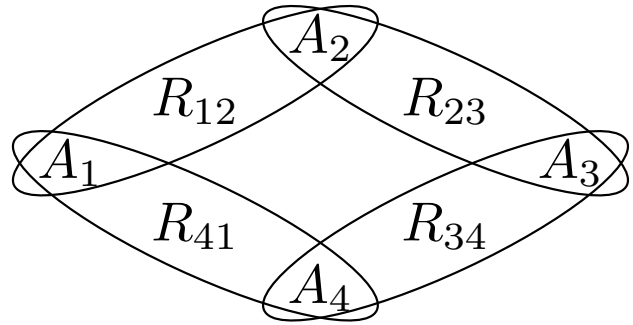

A_1	A_2	A_3	A_4
a	1	d	4
b	1	c	3
b	1	d	4
b	2	c	3

A_1	A_2
a	1
b	1
b	2

A_2	A_3
1	c
1	d
2	c

A_3	A_4
c	3
d	4
d	5

A_4	A_1
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4	a
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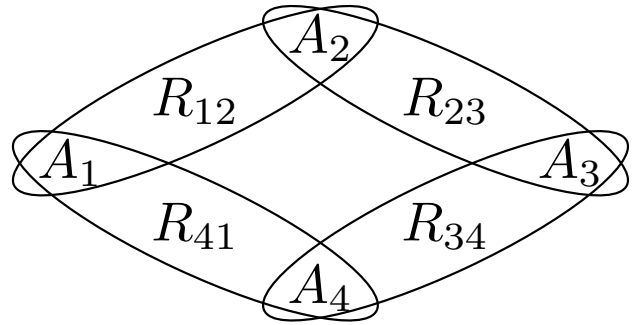

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A_1	A_2
a	1
b	1
b	2

A_2	A_3
1	c
1	d
2	c

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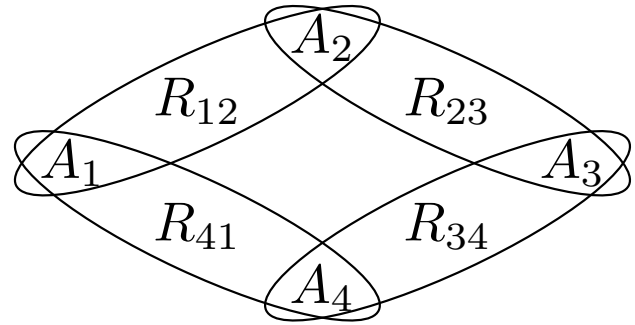

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b	2	c	3	1/4

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b	1	2/4
b	2	1/4

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1	c	1/4
1	d	2/4
2	c	1/4

A_3	A_4	
c	3	2/4
d	4	2/4
d	5	0

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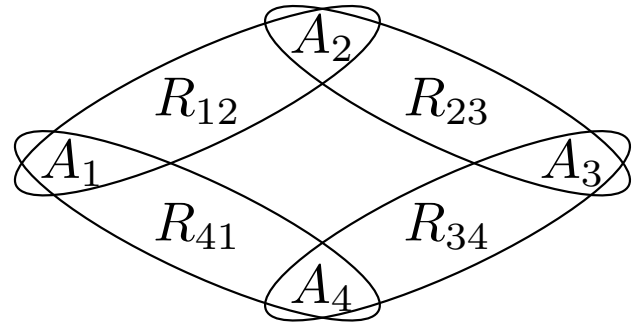

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b	1	d	4	1/4
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$h(A_1A_2A_3A_4) = \log |Q|$

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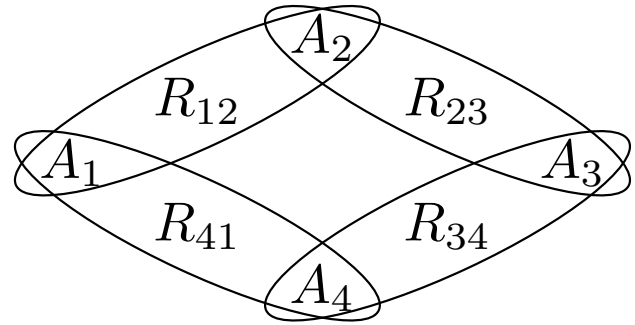

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A_1	A_2	
a	1	1/4
b	1	2/4
b	2	1/4

$h(A_1A_2) \leq \log|R_{12}|,$

A_2	A_3	
1	c	1/4
1	d	2/4
2	c	1/4

$h(A_2A_3) \leq \log|R_{23}|,$

A_3	A_4	
c	3	2/4
d	4	2/4
d	5	0

$h(A_3A_4) \leq \log|R_{34}|,$

A_4	A_1	
3	b	2/4
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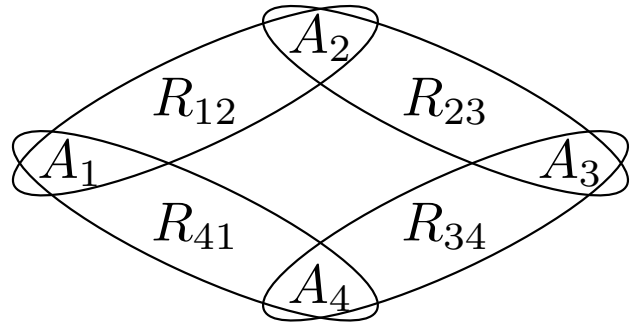

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$h(A_1A_2) \le \log|R_{12}|, \quad h(A_2A_3) \le \log|R_{23}|, \quad h(A_3A_4) \le \log|R_{34}|, \quad \ldots$

$h(A_2|A_1 = \text{‘a’}) \le \log|\sigma_{A_1=\text{‘a’}}R_{12}|, \quad h(A_2|A_1 = \text{‘b’}) \le \log|\sigma_{A_1=\text{‘b’}}R_{12}|, \quad \ldots$

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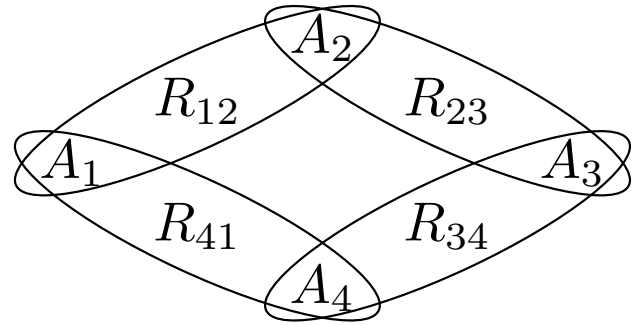

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$h(A_2|A_1) \le \log \max_x |\sigma_{A_1=x}R_{12}|$

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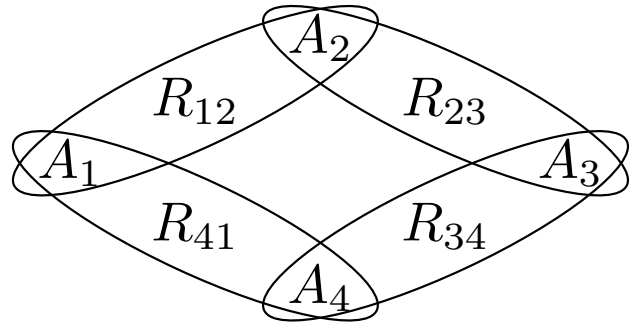

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$$h(A_2|A_1) \le \log \underbrace{\max_x |\sigma_{A_1=x}R_{12}|}_{\deg_{R_{12}}(A_2|A_1)}$$

Size Bounds: Input

- A full conjunctive query

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Bound Idea

$\log |Q| \quad \leq \quad$ maximum $\quad h(A_1, \ldots, A_n)$

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$$\log |Q| \quad \leq \quad \text{maximum} \quad h(A_1, \ldots, A_n)$$
$$\text{over all} \quad \textit{entropies } h$$

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Bound Idea

$$\log |Q| \quad \leq \quad \text{maximum} \quad h(A_1, \ldots, A_n)$$
$$\text{over all} \quad \textit{entropies } h$$
$$\text{such that} \quad h \textit{ satisfies} \text{ degree constraints of } Q$$

Size Bounds: Preliminaries

- HDC is the set of functions $h : 2^{[n]} \to \mathbb{R}_+$ *satisfying* the degree constraints

$$\mathsf{HDC} \stackrel{\text{def}}{=} \Big\{ h \quad | \quad h(Y|X) \le \log N_{Y|X}, \quad \forall (X, Y, N_{Y|X}) \Big\}$$

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- Γ_n^* is the set of *entropic* functions
- $\overline{\Gamma}_n^*$ is the topological closure of Γ_n^*
- Γ_n is the set of *polymatroids*, i.e. functions $h : 2^{[n]} \to \mathbb{R}_+$ satisfying

$$h(X \cup Y) + h(X \cap Y) \le h(X) + h(Y), \qquad X, Y \subseteq [n] \qquad \text{(submodularity)}$$
$$h(X) \le h(Y), \qquad X \subseteq Y \subseteq [n] \qquad \text{(monotonicity)}$$
$$h(\emptyset) = 0 \qquad \text{(strictness)}$$

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- $\overline{\Gamma}_n^*$ is the topological closure of Γ_n^*
- Γ_n is the set of *polymatroids*, i.e. functions $h : 2^{[n]} \to \mathbb{R}_+$ satisfying

$$\begin{aligned} h(X \cup Y) + h(X \cap Y) \le h(X) + h(Y), \quad & X, Y \subseteq [n] && \text{(submodularity)} \\ h(X) \le h(Y), \quad & X \subseteq Y \subseteq [n] && \text{(monotonicity)} \\ h(\emptyset) = 0 & && \text{(strictness)} \end{aligned}$$

$$\underbrace{\Gamma_n^*}_{\text{entropic functions}} \subset \underbrace{\overline{\Gamma}_n^*}_{\text{topological closure of } \Gamma_n^*} \subset \underbrace{\Gamma_n}_{\text{polymatroids}}$$

Size Bounds for Full Conjunctive Queries

$$\underbrace{\Gamma_n^*}_{\text{entropic functions}} \subset \underbrace{\overline{\Gamma}_n^*}_{\text{topological closure of } \Gamma_n^*} \subset \underbrace{\Gamma_n}_{\text{polymatroids}}$$

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Bound	Entropic Bound	Polymatroid Bound
Definition	$\log\lvert Q\rvert \leq \max_{h \in \overline{\Gamma}_n^* \cap \mathsf{HDC}} h([n])$	$\log\lvert Q\rvert \leq \max_{h \in \Gamma_n \cap \mathsf{HDC}} h([n])$

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Definition	$\log \lvert Q \rvert \leq \max_{h \in \overline{\Gamma}_n^* \cap \mathsf{HDC}} h([n])$	$\log \lvert Q \rvert \leq \max_{h \in \Gamma_n \cap \mathsf{HDC}} h([n])$
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DC	Entropic Bound for DC (Tight [Our work])	Polymatroid Bound for DC (Not tight [Our work])

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Disjunctive Datalog

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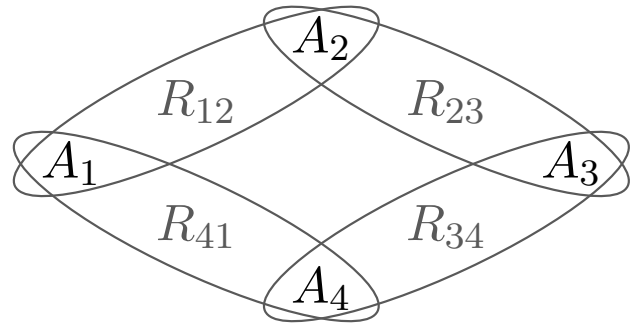

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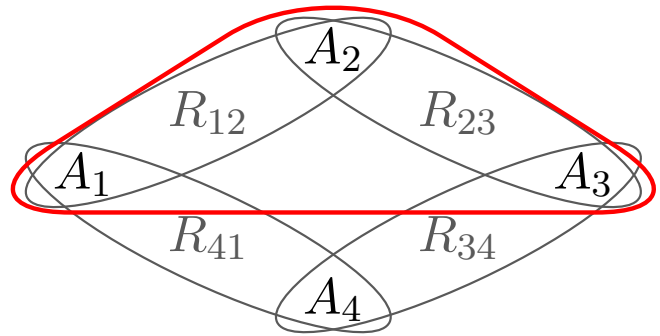

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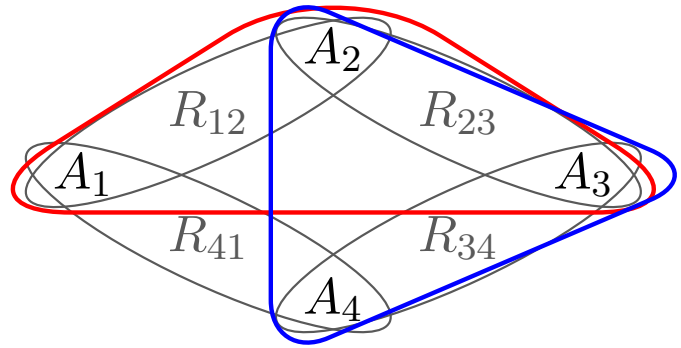

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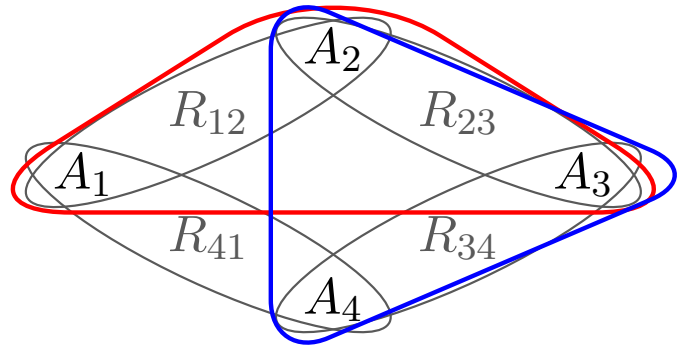

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A_1	A_2
a	1
b	1
b	2

A_2	A_3
1	c
1	d
2	c

A_3	A_4
c	3
d	4
d	5

A_4	A_1
3	b
4	a
4	b

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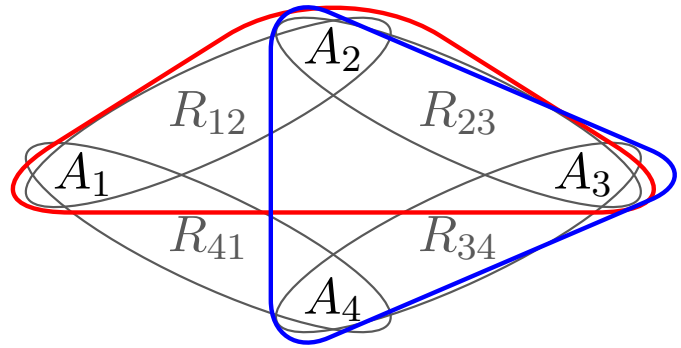

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a	1	d	4
b	1	c	3
b	1	d	4
b	2	c	3

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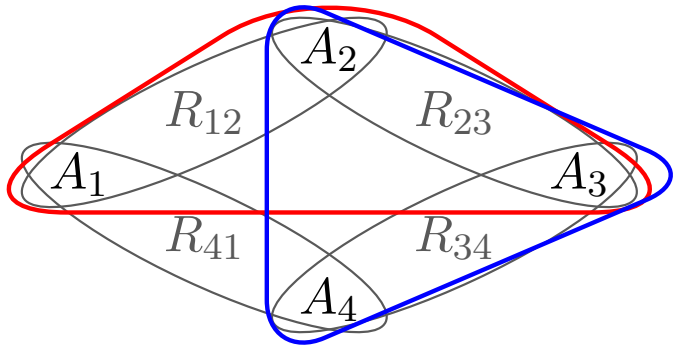

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b	2	c

A_2	A_3	A_4
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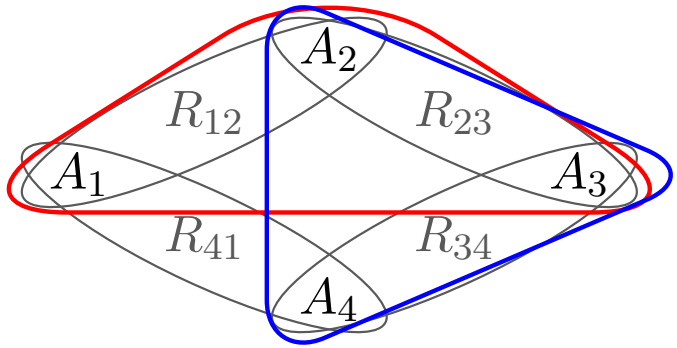

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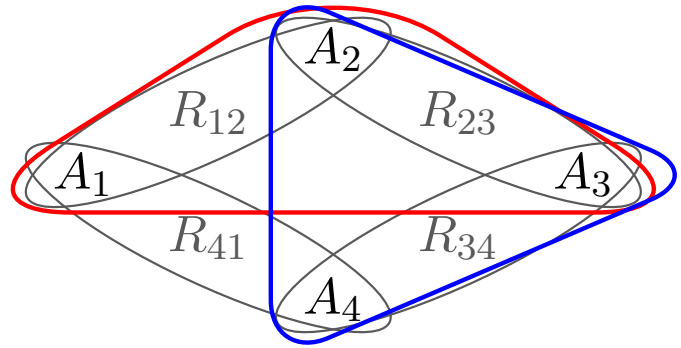

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Model size is $\max(|T_{123}|, |T_{234}|) = 3$

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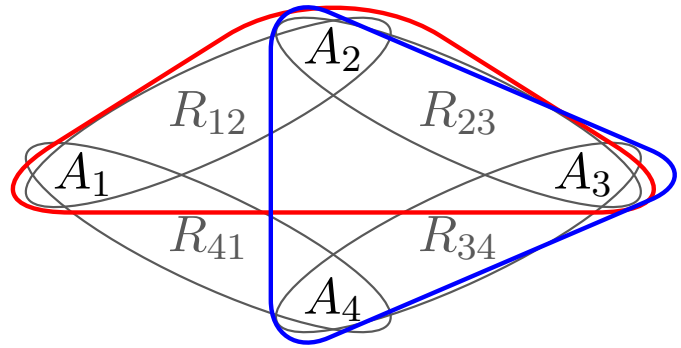

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A_1	A_2	A_3
b	1	c
b	2	c

A_2	A_3	A_4
1	d	4
2	c	3
2	d	4

Output size is the *minimum* over all models

Disjunctive Datalog

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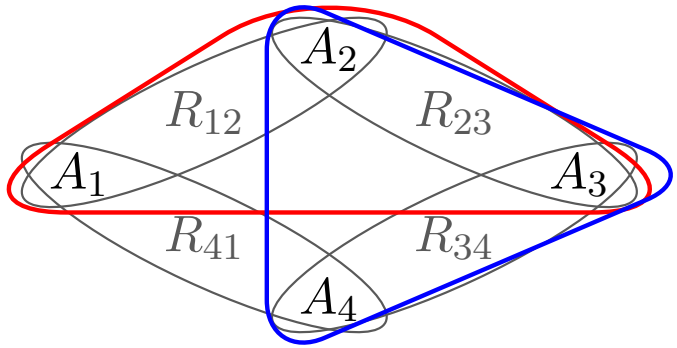

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a	1	d	4
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b	1	d	4
b	2	c	3

A_1	A_2	A_3
b	1	c
b	2	c

A_2	A_3	A_4
1	d	4

A minimum-sized model of size 2

Disjunctive Datalog

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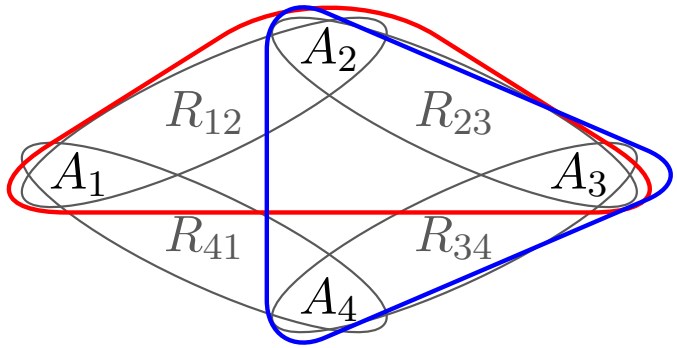

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2	c

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d	4
d	5

A_4	A_1
3	b
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a	1	d	4
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b	1	d	4
b	2	c	3

A_1	A_2	A_3
b	1	c
b	2	c

A_2	A_3	A_4
1	d	4

A minimum-sized model of size 2

$\Rightarrow$ Output size is 2

Disjunctive Datalog: Output Size

$$P: \quad \bigvee_{B \in \mathcal{B}} T_B(\mathbf{A}_B) \text{ :- } \bigwedge_{F \in \mathcal{E}} R_F(\mathbf{A}_F)$$

$$|P(\mathbf{D})| \stackrel{\text{def}}{=} \min_{\mathbf{T}: \mathbf{T} \models P} \max_{B \in \mathcal{B}} |T_B|$$

Disjunctive Datalog: Output Size

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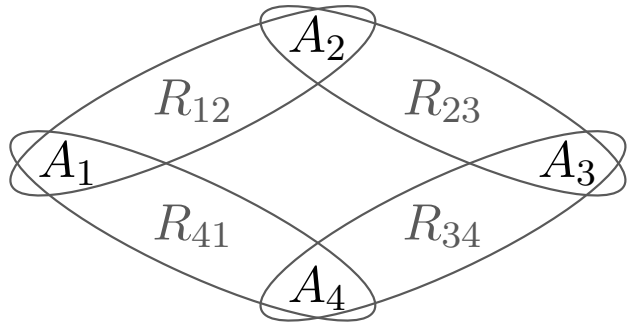

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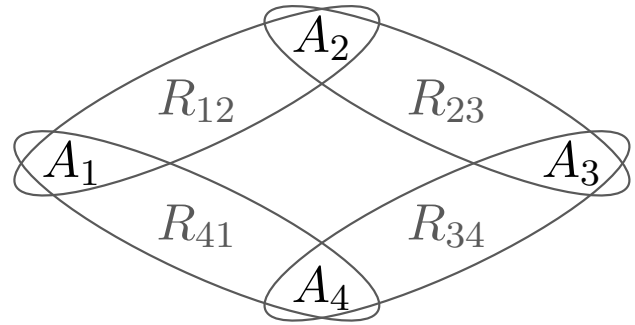

$$\mathbf{D}: \quad |R_{12}| \le N, \quad |R_{23}| \le N, \quad |R_{34}| \le N, \quad |R_{41}| \le N.$$

Disjunctive Datalog: Output Size

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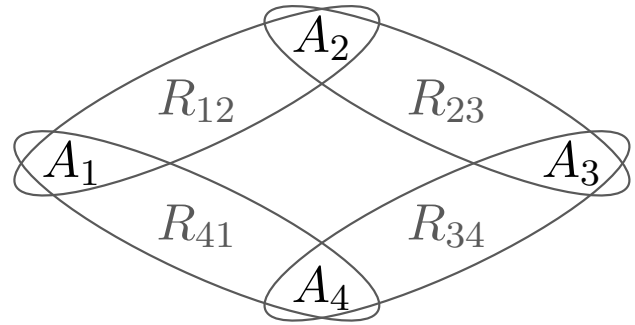

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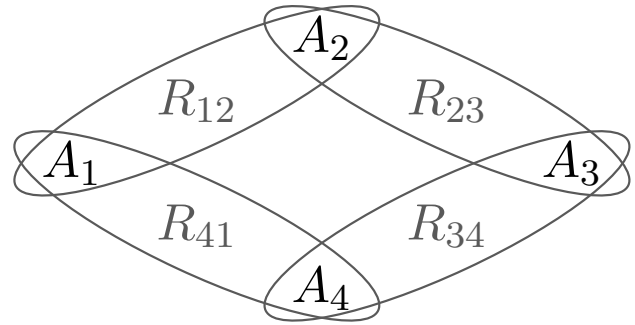

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$|P(\mathbf{D})| \le N^{3/2}$, for all $\mathbf{D}$

Disjunctive Datalog: Output Size

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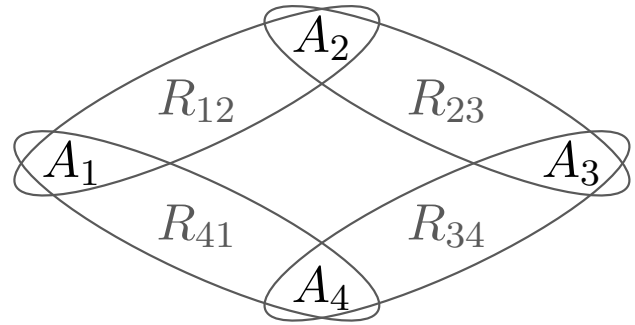

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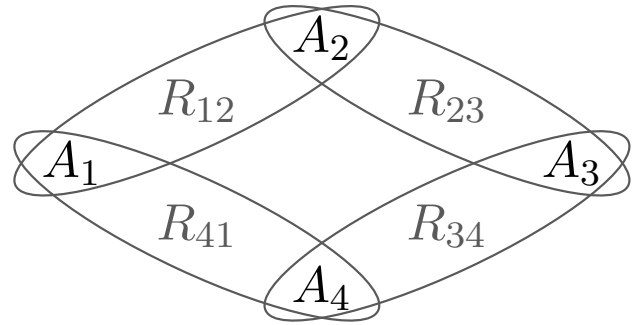

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Disjunctive Datalog: Size Bounds

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- HDC is the set of functions $h : 2^{[n]} \to \mathbb{R}_+$ satisfying the degree constraints
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$\log |P(\mathbf{D})|$

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Theorem

PANDA solves any disjunctive datalog rule P in time within the polymatroid bound of P.

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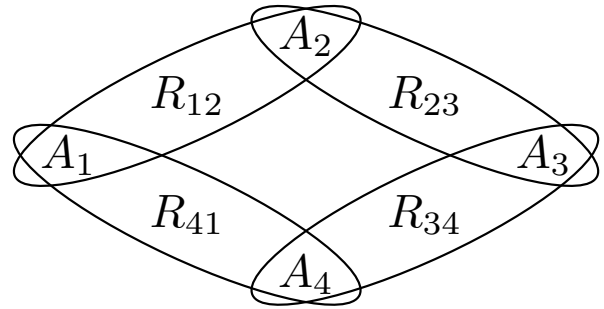

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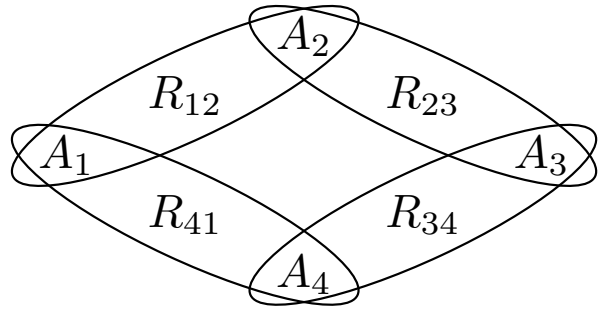

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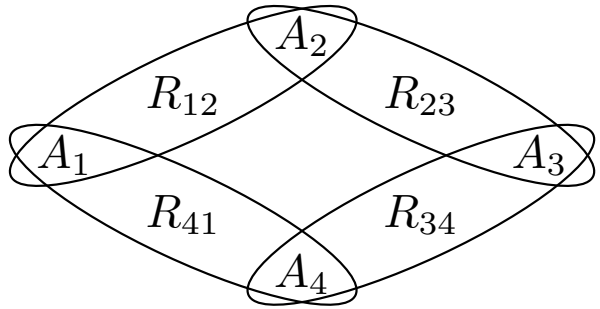

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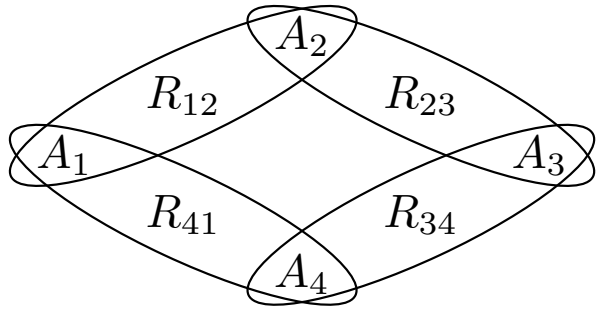

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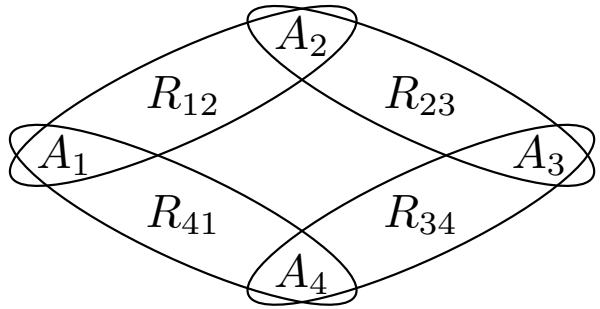

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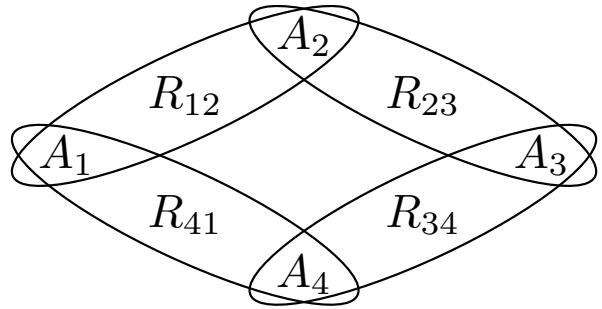

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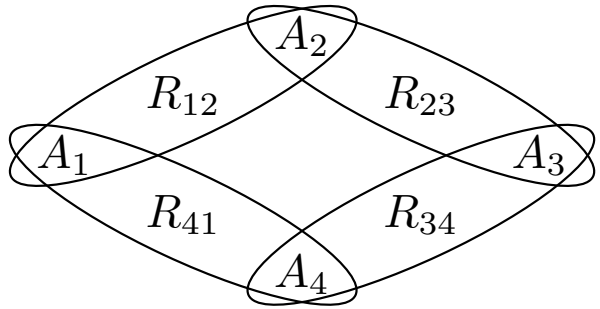

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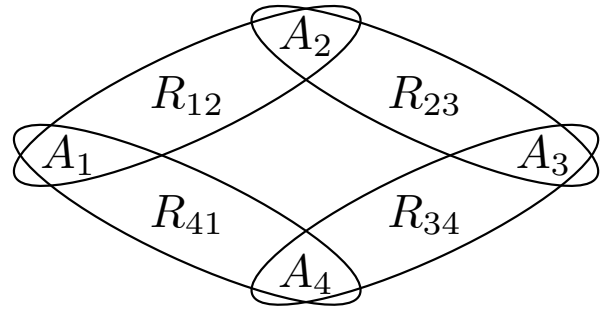

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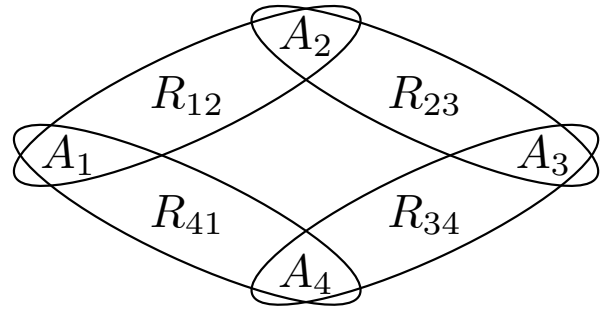

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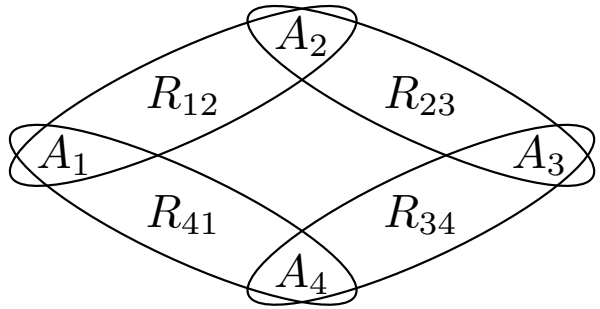

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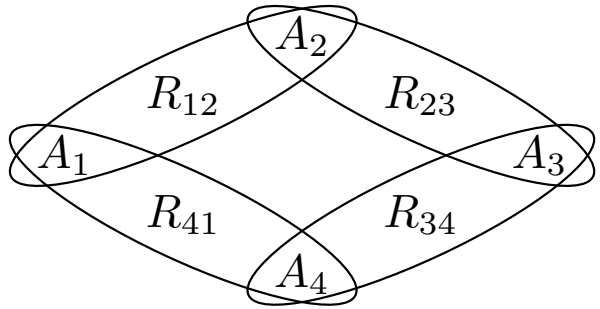

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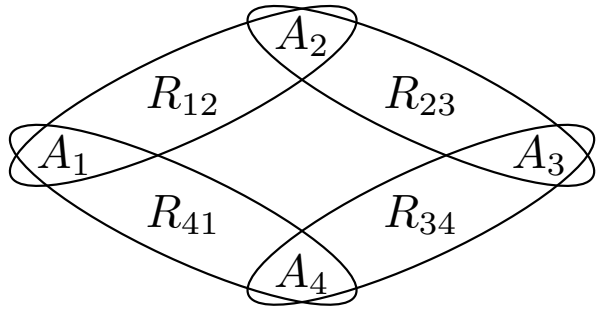
$$h(A_1A_2) + h(A_2A_3) + h(A_3A_4) \quad \to \quad \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big)$$
$$h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) \quad \to \quad \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big)$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
$$R_{34}(A_3, A_4), R_{41}(A_4, A_1).$$



$$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \le N \quad \Rightarrow \quad |P| \le N^{3/2}$$

$$\log|P| \quad \le \quad \min(h(A_1A_2A_3), h(A_2A_3A_4)) \quad \le \quad \frac{1}{2}\big(h(A_1A_2A_3) + h(A_2A_3A_4)\big)$$
$$\le \quad \frac{1}{2}\big(h(A_1A_2) + h(A_2A_3) + h(A_3A_4)\big) \quad \le \quad \frac{3}{2}\log N$$

$$h(A_1A_2) + h(A_2A_3) + h(A_3A_4) \quad \to \quad \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big)$$
$$h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) \quad \to \quad \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big)$$
$$h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3)$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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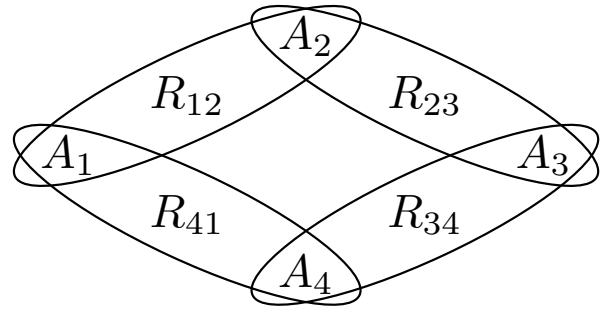

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$$\begin{aligned}
h(A_1A_2) + h(A_2A_3) + h(A_3A_4) \quad &\to \quad \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big) \\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) \quad &\to \quad \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big) \\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3) \quad &\to \quad \big(h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)\big)
\end{aligned}$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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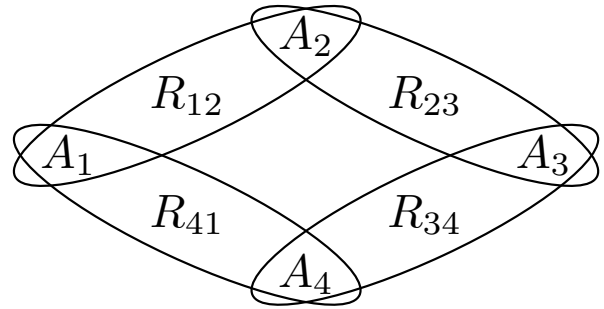

$$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \le N \quad \Rightarrow \quad \boxed{|P| \le N^{3/2}}$$

$$\log|P| \quad \le \quad \min(h(A_1A_2A_3), h(A_2A_3A_4)) \quad \le \quad \frac{1}{2}\big(h(A_1A_2A_3) + h(A_2A_3A_4)\big)$$
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$$\begin{aligned}
h(A_1A_2) + h(A_2A_3) + h(A_3A_4) \quad &\to \quad \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big) \\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) \quad &\to \quad \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big) \\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3) \quad &\to \quad \big(h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)\big) \\
h(A_1A_2) + h(A_2A_3A_4) + h(A_3)
\end{aligned}$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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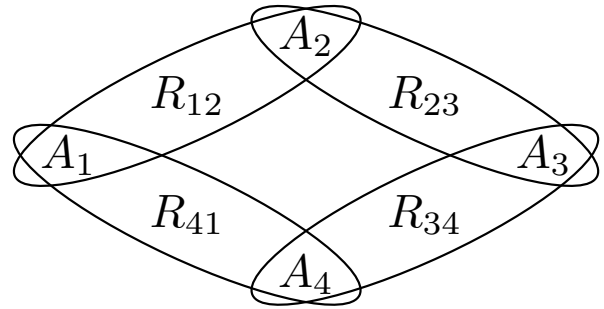

$$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \le N \quad \Rightarrow \quad \boxed{|P| \le N^{3/2}}$$

$$\log|P| \quad \le \quad \min(h(A_1A_2A_3), h(A_2A_3A_4)) \quad \le \quad \frac{1}{2}\big(h(A_1A_2A_3) + h(A_2A_3A_4)\big)$$
$$\boxed{\le} \quad \frac{1}{2}\big(h(A_1A_2) + h(A_2A_3) + h(A_3A_4)\big) \quad \le \quad \frac{3}{2}\log N$$

$$\begin{aligned}
h(A_1A_2) + h(A_2A_3) + h(A_3A_4) \quad &\to \quad \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big) \\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) \quad &\to \quad \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big) \\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3) \quad &\to \quad \big(h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)\big) \\
h(A_1A_2) + h(A_2A_3A_4) + h(A_3) \quad &\to \quad \big(h(A_1A_2) \to h(A_1A_2|A_3)\big)
\end{aligned}$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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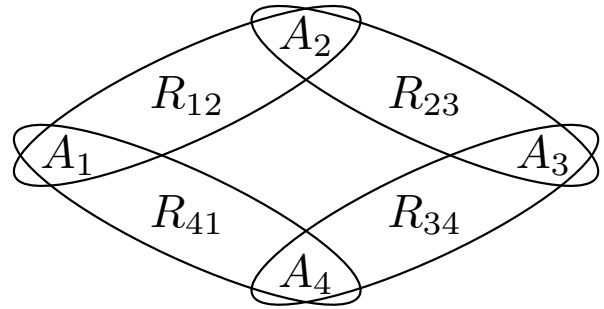

$$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \le N \quad \Rightarrow \quad |P| \le N^{3/2}$$

$$\log|P| \le \min(h(A_1A_2A_3), h(A_2A_3A_4)) \le \frac{1}{2}\big(h(A_1A_2A_3) + h(A_2A_3A_4)\big)$$
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$$\begin{aligned}
h(A_1A_2) + h(A_2A_3) + h(A_3A_4) &\to \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big)\\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) &\to \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big)\\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3) &\to \big(h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)\big)\\
h(A_1A_2) + h(A_2A_3A_4) + h(A_3) &\to \big(h(A_1A_2) \to h(A_1A_2|A_3)\big)\\
h(A_1A_2|A_3) + h(A_2A_3A_4) + h(A_3)
\end{aligned}$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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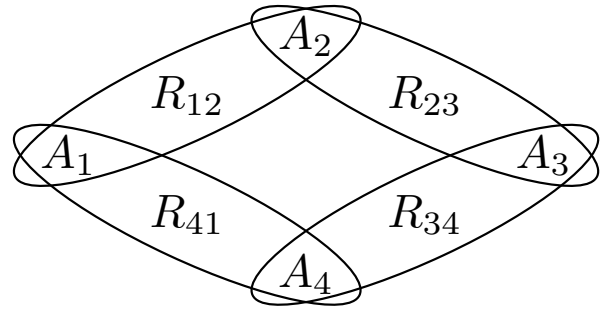

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$$\le \frac{1}{2}\big(h(A_1A_2) + h(A_2A_3) + h(A_3A_4)\big) \le \frac{3}{2}\log N$$

$$\begin{aligned}
h(A_1A_2) + h(A_2A_3) + h(A_3A_4) &\to \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big)\\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) &\to \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big)\\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3) &\to \big(h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)\big)\\
h(A_1A_2) + h(A_2A_3A_4) + h(A_3) &\to \big(h(A_1A_2) \to h(A_1A_2|A_3)\big)\\
h(A_1A_2|A_3) + h(A_2A_3A_4) + h(A_3) &\to \big(h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)\big)
\end{aligned}$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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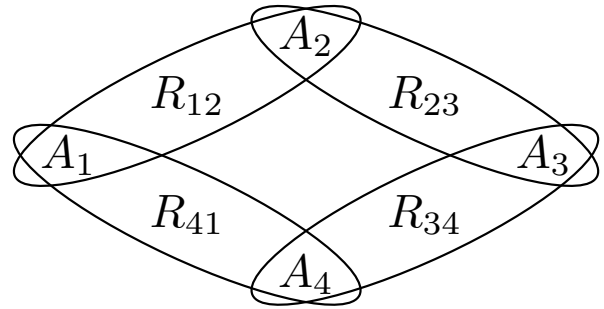

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h(A_1A_2) + h(A_2A_3) + h(A_3A_4) &\to \big(h(A_3A_4) \to h(A_4|A_3) + h(A_3)\big)\\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_3) + h(A_3) &\to \big(h(A_4|A_3) \to h(A_4|A_2A_3)\big)\\
h(A_1A_2) + h(A_2A_3) + h(A_4|A_2A_3) + h(A_3) &\to \big(h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)\big)\\
h(A_1A_2) + h(A_2A_3A_4) + h(A_3) &\to \big(h(A_1A_2) \to h(A_1A_2|A_3)\big)\\
h(A_1A_2|A_3) + h(A_2A_3A_4) + h(A_3) &\to \big(h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)\big)\\
h(A_1A_2A_3) + h(A_2A_3A_4)
\end{aligned}$$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
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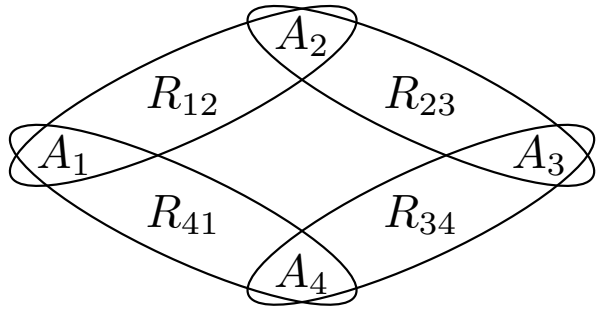

$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \leq N \quad \Rightarrow \quad |P| \leq N^{3/2}$

$h(A_3A_4) \to h(A_4|A_3) + h(A_3)$

$h(A_4|A_3) \to h(A_4|A_2A_3)$

$h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)$

$h(A_1A_2) \to h(A_1A_2|A_3)$

$h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
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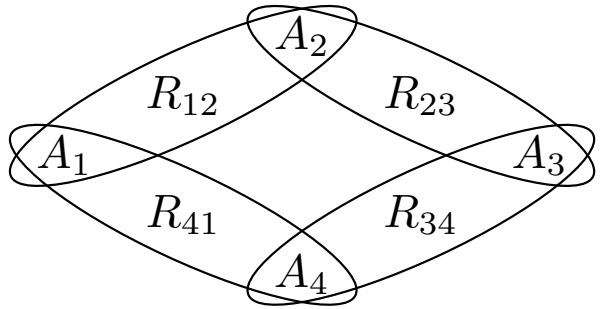

$$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \leq N \quad \Rightarrow \quad \boxed{|P| \leq N^{3/2}}$$

$h(A_3A_4) \to h(A_4|A_3) + h(A_3)$ $\qquad$ $R_{34}(A_3, A_4) \to R_{34}^{(l)}(A_3, A_4), R_3^{(h)}(A_3)$

$h(A_4|A_3) \to h(A_4|A_2A_3)$

$h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)$

$h(A_1A_2) \to h(A_1A_2|A_3)$

$h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)$

PANDA: Example

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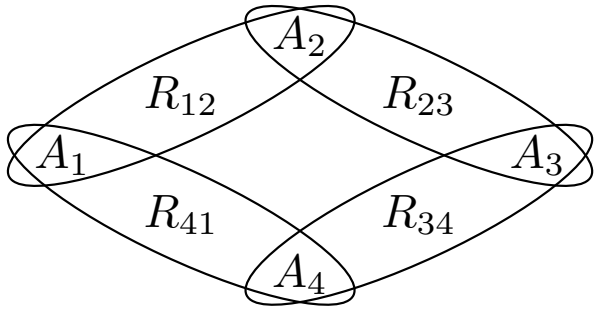

$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \leq N \quad \Rightarrow \quad \boxed{|P| \leq N^{3/2}}$

$h(A_3A_4) \to h(A_4|A_3) + h(A_3)$

$h(A_4|A_3) \to h(A_4|A_2A_3)$

$h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)$

$h(A_1A_2) \to h(A_1A_2|A_3)$

$h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)$

$R_{34}(A_3, A_4) \to R_{34}^{(l)}(A_3, A_4), R_3^{(h)}(A_3)$

$R_{34}^{(l)}(A_3, A_4) \to R_{34}^{(l)}(A_3, A_4)$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
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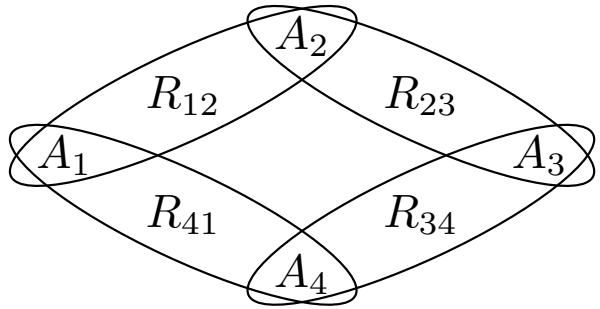

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$h(A_3A_4) \to h(A_4|A_3) + h(A_3)$

$h(A_4|A_3) \to h(A_4|A_2A_3)$

$h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)$

$h(A_1A_2) \to h(A_1A_2|A_3)$

$h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)$

$R_{34}(A_3, A_4) \to R^{(l)}_{34}(A_3, A_4), R^{(h)}_{3}(A_3)$

$R^{(l)}_{34}(A_3, A_4) \to R^{(l)}_{34}(A_3, A_4)$

$R_{23}(A_2, A_3) \bowtie R^{(l)}_{34}(A_3, A_4) \to T_{234}(A_2, A_3, A_4)$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
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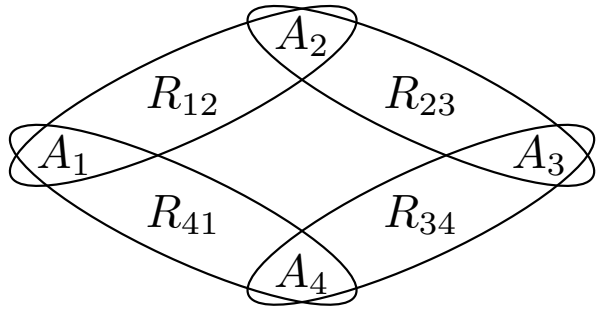

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$h(A_3A_4) \to h(A_4|A_3) + h(A_3)$

$h(A_4|A_3) \to h(A_4|A_2A_3)$

$h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)$

$h(A_1A_2) \to h(A_1A_2|A_3)$

$h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)$

$R_{34}(A_3, A_4) \to R^{(l)}_{34}(A_3, A_4), R^{(h)}_{3}(A_3)$

$R^{(l)}_{34}(A_3, A_4) \to R^{(l)}_{34}(A_3, A_4)$

$R_{23}(A_2, A_3) \bowtie R^{(l)}_{34}(A_3, A_4) \to T_{234}(A_2, A_3, A_4)$

$R_{12}(A_1, A_2) \to R_{12}(A_1, A_2)$

PANDA: Example

$$P: \quad T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3),$$
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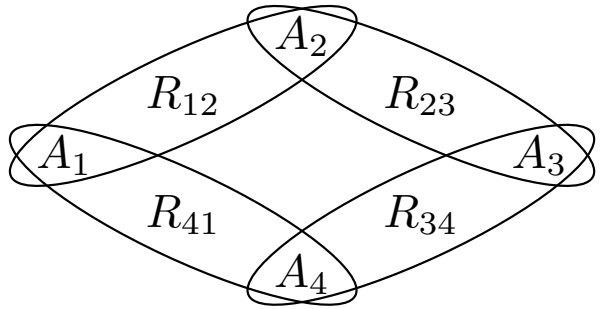

$$|R_{12}|, |R_{23}|, |R_{34}|, |R_{41}| \leq N \quad \Rightarrow \quad \boxed{|P| \leq N^{3/2}}$$

$h(A_3A_4) \to h(A_4|A_3) + h(A_3)$ $\qquad$ $R_{34}(A_3, A_4) \to R_{34}^{(l)}(A_3, A_4), R_3^{(h)}(A_3)$

$h(A_4|A_3) \to h(A_4|A_2A_3)$ $\qquad$ $R_{34}^{(l)}(A_3, A_4) \to R_{34}^{(l)}(A_3, A_4)$

$h(A_2A_3) + h(A_4|A_2A_3) \to h(A_2A_3A_4)$ $\qquad$ $R_{23}(A_2, A_3) \bowtie R_{34}^{(l)}(A_3, A_4) \to T_{234}(A_2, A_3, A_4)$

$h(A_1A_2) \to h(A_1A_2|A_3)$ $\qquad$ $R_{12}(A_1, A_2) \to R_{12}(A_1, A_2)$

$h(A_1A_2|A_3) + h(A_3) \to h(A_1A_2A_3)$ $\qquad$ $R_{12}(A_1, A_2) \bowtie R_3^{(h)}(A_3) \to T_{123}(A_1, A_2, A_3)$

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- Our goals

$$\textit{Any } Q \quad \Rightarrow \quad \tilde{O}\Big(N^{\text{da-}\,\mathsf{subw}(Q)} \times 1 + |\text{output}|\Big)$$

Submodular Width

$$\mathsf{fhtw}(Q) \stackrel{\text{def}}{=} \min_{(T,\chi)} \max_{t \in V(T)} \rho^*(\chi(t))$$

Submodular Width

$$\begin{aligned}\mathsf{fhtw}(Q) &\overset{\text{def}}{=} \min_{(T,\chi)} \max_{t\in V(T)} \rho^*(\chi(t)) \\ &= \min_{(T,\chi)} \max_{t\in V(T)} \max_{h\in \mathsf{ED}\cap\Gamma_n} h(\chi(t))\end{aligned}$$

Submodular Width

$$
\begin{aligned}
\mathsf{fhtw}(Q) &\overset{\text{def}}{=} \min_{(T,\chi)} \max_{t \in V(T)} \rho^*(\chi(t)) \\
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\end{aligned}
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\end{aligned}
$$

$$\mathsf{subw}(Q) \quad \leq \quad \mathsf{fhtw}(Q)$$

Submodular Width: Example

$$\mathsf{fhtw}(Q) \stackrel{\text{def}}{=} \min_{(T,\chi)} \max_{t \in V(T)} \rho^*(\chi(t)), \qquad \mathsf{subw}(Q) \stackrel{\text{def}}{=} \max_{h \in \mathsf{ED} \cap \Gamma_n} \min_{(T,\chi)} \max_{t \in V(T)} h(\chi(t))$$

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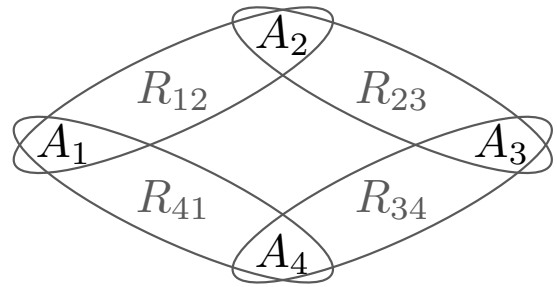

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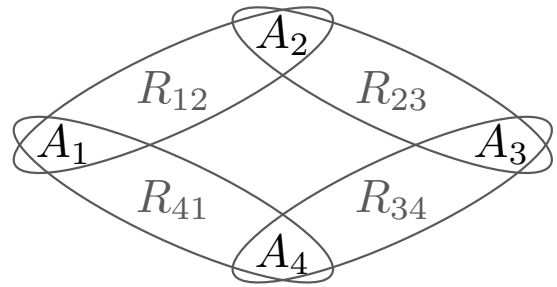

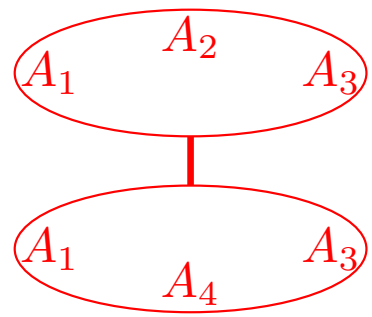

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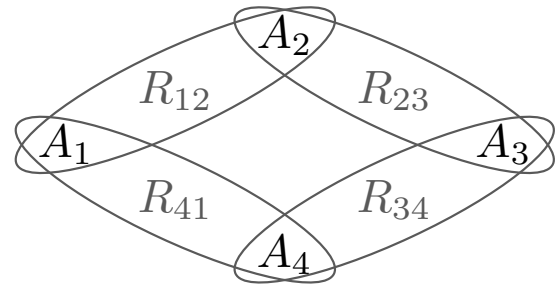

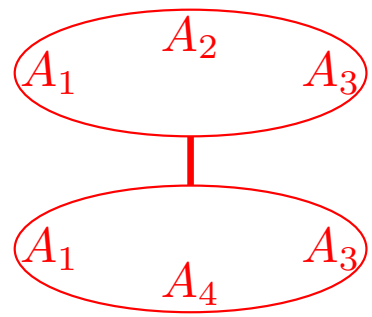

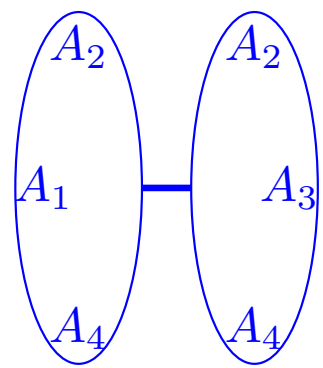

Submodular Width: Example

$$\mathsf{fhtw}(Q) \stackrel{\text{def}}{=} \min_{(T,\chi)} \max_{t \in V(T)} \rho^*(\chi(t)), \qquad \mathsf{subw}(Q) \stackrel{\text{def}}{=} \max_{h \in \mathsf{ED} \cap \Gamma_n} \min_{(T,\chi)} \max_{t \in V(T)} h(\chi(t))$$

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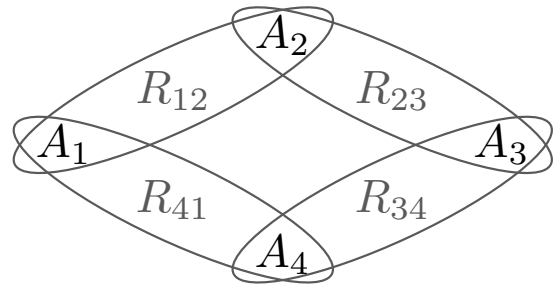
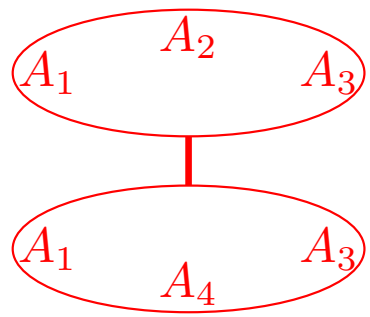
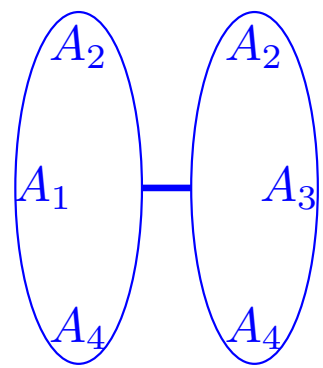
$$\mathsf{fhtw}(Q) \quad = \quad 2$$







Submodular Width: Example

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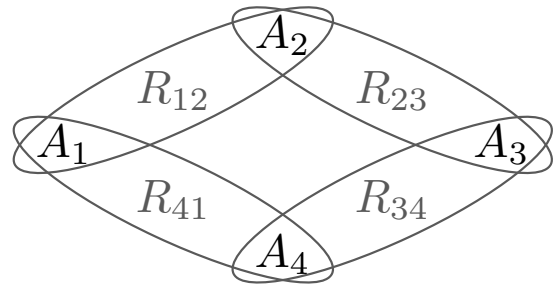
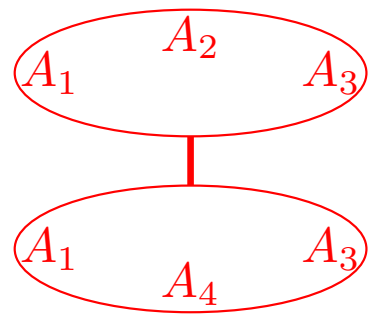
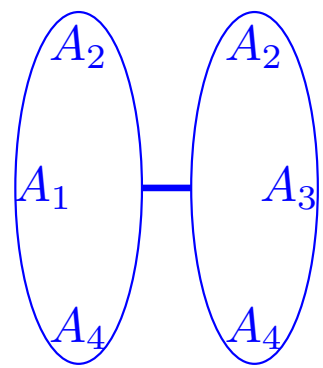
$$\begin{aligned} \mathsf{fhtw}(Q) &= 2 \\ \mathsf{subw}(Q) &= 3/2 \end{aligned}$$







Submodular Width: Example

$$\mathsf{fhtw}(Q) \stackrel{\text{def}}{=} \min_{(T,\chi)} \max_{t \in V(T)} \rho^*(\chi(t)), \qquad \mathsf{subw}(Q) \stackrel{\text{def}}{=} \max_{h \in \mathsf{ED} \cap \Gamma_n} \min_{(T,\chi)} \max_{t \in V(T)} h(\chi(t))$$

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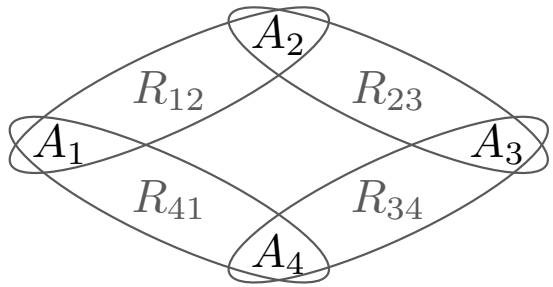
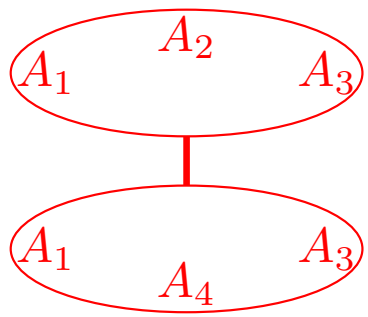
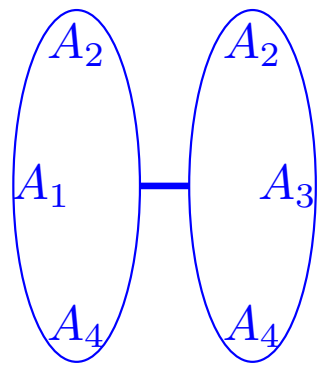


$$\begin{aligned} \mathsf{fhtw}(Q) &= 2 \\ \mathsf{subw}(Q) &= 3/2 \end{aligned}$$

$$\min(\max(h(A_1A_2A_3), h(A_3A_4A_1)), \\ \max(h(A_4A_1A_2), h(A_2A_3A_4))) \leq 3/2$$





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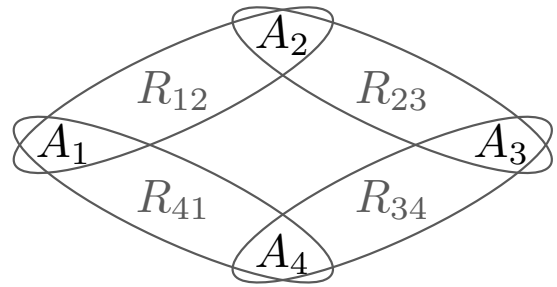
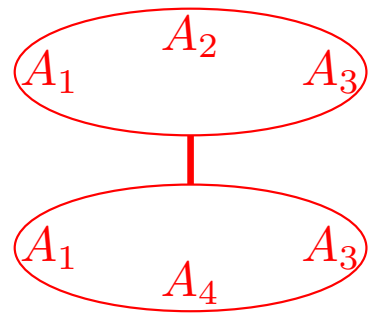
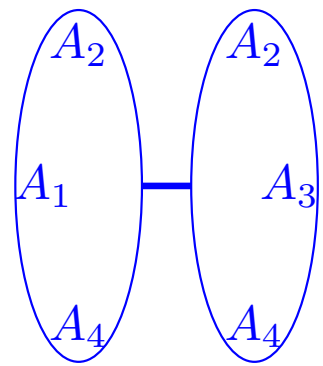


$$\begin{aligned} \mathsf{fhtw}(Q) &= 2 \\ \mathsf{subw}(Q) &= 3/2 \end{aligned}$$

$$\min(\max(h(A_1A_2A_3), h(A_3A_4A_1)), \\ \max(h(A_4A_1A_2), h(A_2A_3A_4))) \le 3/2$$



$$\begin{aligned} \min(h(A_1A_2A_3), h(A_4A_1A_2)) &\le 3/2 \\ \min(h(A_1A_2A_3), h(A_2A_3A_4)) &\le 3/2 \\ \min(h(A_3A_4A_1), h(A_4A_1A_2)) &\le 3/2 \\ \min(h(A_3A_4A_1), h(A_2A_3A_4)) &\le 3/2 \end{aligned}$$



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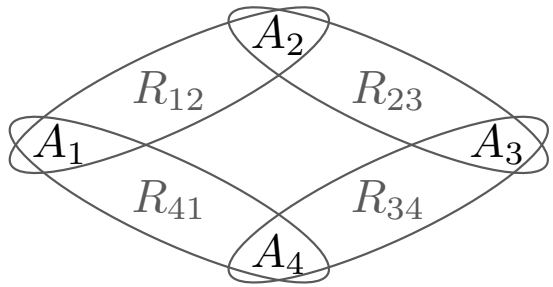
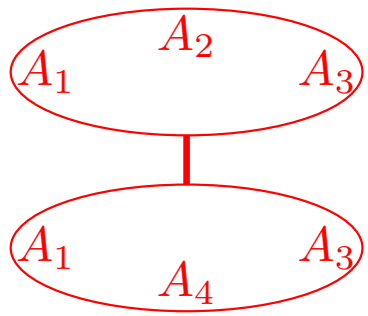
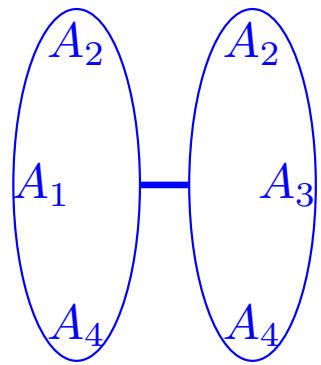


$$T_{123}(A_1, A_2, A_3) \vee T_{412}(A_4, A_1, A_2) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3), R_{34}(A_3, A_4), R_{41}(A_4, A_1).$$



$$T_{123}(A_1, A_2, A_3) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3), R_{34}(A_3, A_4), R_{41}(A_4, A_1).$$

$$T_{341}(A_3, A_4, A_1) \vee T_{412}(A_4, A_1, A_2) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3), R_{34}(A_3, A_4), R_{41}(A_4, A_1).$$



$$T_{341}(A_3, A_4, A_1) \vee T_{234}(A_2, A_3, A_4) \text{ :-}$$
$$R_{12}(A_1, A_2), R_{23}(A_2, A_3), R_{34}(A_3, A_4), R_{41}(A_4, A_1).$$

Summary of Bounds