Multirate Rearrangeable Clos Networks and a Generalized Edge Coloring Problem on Bipartite Graphs

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Abstract

Chung and Ross (SIAM J. Comput., **20**, 1991) conjectured that the minimum number $m(n,r)$ of middle-state switches for the symmetric 3-stage Clos network $C(n, m(n,r), r)$ to be rearrangeable in the multirate enviroment is at most $2n-1$. This problem is equivalent to a generalized version of the bipartite graph edge coloring problem. The best bounds known so far on the function $m(n,r)$ is $11n/9 \leq m(n,r) \leq 41n/16 + O(1)$, for $n, r \geq 2$, derived by Du-Gao-Hwang-Kim (SIAM J. Comput., **28**, 1999). In this paper, we make several contributions. Firstly, we give evidence to show that even a stronger result might hold. In particular, we give a coloring algorithm to show that $m(n,r) \leq \lceil (r+1)n/2 \rceil$, which implies $m(n,2) \leq \lceil 3n/2 \rceil$ - stronger than the conjectured value of $2n-1$. Secondly, we derive that $m(2,r) = 3$ by an elegant argument. Lastly, we improve both the best upper and lower bounds given above: $\lceil 5n/4 \rceil \leq m(n,r) \leq 2n-1+\lceil (r-1)/2 \rceil$, where the upper bound is an improvement over $41n/16$ when r is relatively small compared to n. We also conjecture that $m(n,r) \leq \lfloor 2n(1-1/2^r) \rfloor$.

1 Introduction

The Clos network has been widely used for data communications and parallel computing systems. Quite a lot of research efforts [1–3, 5, 6, 9–11, 13–17, 21] have been put on investigating the non-blocking properties and rearrangeability of the Clos network. The 3-stage Clos network was paid special attention to since it can be expanded in a "straightforward" way to multistage Clos network. Recently, Ngo [18] observed that the 3-stage Clos network is "equivalent" to the wavelength division multiplexed (WDM) split cross-connects [19, 20], giving new applications to the classic Clos networks. Let us first formally introduce some related concepts.

The Clos network $C(n_1, r_1, m, n_2, r_2)$ is a 3-stage interconnection network, where the first stage consists of r_1 crossbars of size $n_1 \times m$, the last stage has r_2 crossbars of dimension $m \times n_2$, and the middle stage has m crossbars of dimension $r_1 \times r_2$ (see Figure 1). Each input switch I_i ($i = 1, \dots, r_1$) is connected to each middle switch M_j ($j = 1, \dots, m$). Similarly, the middle stage and the last stage are fully connected. When $n_1 = n_2 = n$ and $r_1 = r_2 = r$, the network is called the *symmetric* 3*-stage Clos network*, denoted by $C(n, m, r)$. Any switch is assumed to be non-blocking, i.e. any inlet can be connected to any outlet as long as there's no conflict on the outlet. A switch of dimension $p \times q$ could be thought of as a crossbar of size $p \times q$ with pq cross-points. Having too many cross-points is expensive and we would like to design a huge switch using smaller switches with fewer number of cross-points than when a brute-force design is used. The inlets (outlets) of the input (output) switches are the *inputs* (*outputs*) of the network. Inputs and outputs are referred to as *external links*, while links between switches are referred to as *internal links*.

In the multirate environment, a *connection request* is a triple (i, j, w) where i is an inlet, j an outlet, and w the weight. A *request frame* is a collection of requests such that the total weight of all requests in the frame involving a fixed inlet or outlet does not exceed unity. To discuss routing it is convenient to assume that all links are directed from left to right. Thus a *path* from an inlet to any outlet always consists of the sequence: an inlet link → an input switch → a link → a center switch → a link → an output switch → an outlet link. Furthermore, since the crossbars are assumed to be nonblocking, a request (i, j, w) is *routable* if and only if there exists a path from i to j such that every link on this path has unused capacity at least $1-w$ before carrying out this request. A request frame is routable if there exists a set of paths, one for each request, such that for every link the sum of weights of all requests going through it does not exceed unity. The Clos network $C(n, m, r)$ is said to be *multirate rearrangeable* (or just rearrangeable as in this paper we only consider the multirate environment) if *every* request frame is routable.

Let $m(n,r)$ denote the minimum value of m such that $C(n, m, r)$ is multirate rearrangeable for $n, r \geq 2$. (The cases where either n or r are 1 are trivial, hence we only consider $n, r \geq 2$ from here on.) Our problem is to find $m(n,r)$, or at least some good bounds for this function.

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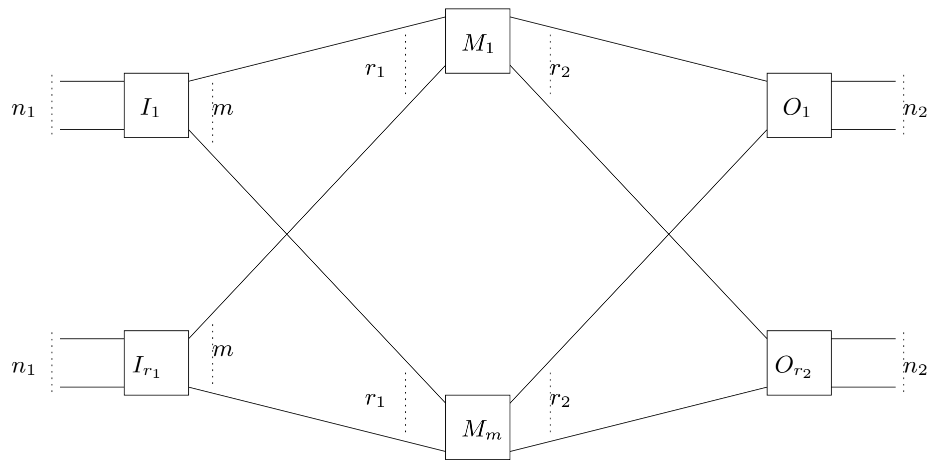

Figure 1: The 3-stage Clos network $C(n_1, r_1, m, n_2, r_2)$

The problem appears to be difficult. Let us first review some previous works on this problem. Melen and Turner (1989, [16]) initiated the research on multirate switching networks. In 1991, Chung and Ross [3] conjectured that $m(n,r) \le 2n-1$ and untill now no one has been able to prove or disprove the conjecture. The best bounds known so far on the function $m(n,r)$ was obtained by Du-Gao-Hwang-Kim (1999, [5]):

$$11n/9 \le m(n,r) \le 41n/16 + O(1).$$

Lin et al. (1999, [14]) comfirmed Chung-Ross conjecture for a restricted discrete bandwidth case where each connection has a weight chosen from a set $\{1 \ge w_1 > \cdots > w_h > 1/2 \ge w_{h+1} > \cdots > w_k\}$ which satisfies the condidion that w_i is an integer multiple of w_{i+1} for $i = h+1, \ldots, k-1$. Hu et al. (2001, [10]) studied the monotone routing strategy and showed that under this strategy

$$\text{(1.1)} \qquad m(n,r) \le 2n+1 \text{ for } n = 2,3,4$$

$$\text{(1.2)} \qquad m(n,r) \le 2n+3 \text{ for } n = 5,6.$$

Ngo (2002, [17]) proposed the grouping algorithm which shows that $m(n,r) \le 2n-1+r$, and that $m(n,r) \le 2n + \frac{n-1}{2^k}$ whenever $r \le \frac{n}{2^k-1}$.

In this paper, we give evidence to show that a stronger version of Chung-Ross conjecture might hold. In particular, we show that $m(n,r) \le \left\lceil \frac{(r+1)n}{2} \right\rceil$, which implies $m(n,2) \le \left\lceil \frac{3n}{2} \right\rceil$. This is stronger than the conjectured value of $2n-1$. We conjecture that

$$m(n,r) \le \left\lfloor 2n\left(1 - \frac{1}{2^r}\right) \right\rfloor, \;\; n, r \ge 2.$$

We believe that the new conjectured upper bound is also the correct value for $m(n,r)$. Secondly, we verify that Chung and Ross were right on target when $n = 2$, i.e. $m(2,r) = 3$, by a new elegant argument. Lastly, we give better upper and lower bounds for the general case:

$$\left\lceil \frac{5n}{4} \right\rceil \le m(n,r) \le 2n - 1 + \left\lceil \frac{r-1}{2} \right\rceil.$$

All these is done in the context of a generalized version of the edge-coloring problem on weighted bipartite graphs, to be introduced in the next section. These weighted graphs have maximum degree n in the weighted sense.

As a side note: Ngo [18] showed that the 3-stage Clos network is equivalent to the WDM split cross-connects [19, 20] under this multirate environment, hence the results in this paper also apply to the split cross-connects. Each rate can be thought of as the bandwidth fraction of a wavelength obtained from time division multiplexing.

2 A Generalized Bipartite Graph Edge Coloring Problem

Given a request frame $\mathcal{F}$, define a weighted bipartite multi-graph $G_{\mathcal{F}} = (I, O; E)$ where I (respectively O) contains all the input (respectively output) switches. There is an edge with weight w between vertices X, Y of G for each request (x, y, w) where x (respectively y) is an inlet (respectively outlet) of X (respectively Y). $C(n, m, r)$ is rearrangeable iff for all $\mathcal{F}$ the edges of $G_{\mathcal{F}}$ can be m-colored such that at every vertex, the total weight of edges of the same color incident to this vertex is at most unity. To see this, just associate each color with a center switch.

We now formally define the equivalent bipartite graph edge-coloring problem. Throughout this paper we assume $n, r \ge 2$ are integers. Let $\mathcal{B}_r^n$ be the collection of edge-weighted $r \times r$ bipartite multi-graphs $G = (A, B; E)$ ($|A| = |B| = r$) with weight function $w : E \to (0, 1]$ satisfying the condition that for every $v \in V(G) = A \cup B$, the set $I(v)$ of edges incident to v can be partitioned into n groups $g(v,i)$, $1 \le i \le n$, such that

$$\sum_{e \in g(v,i)} w(e) \le 1, \forall i = 1, \ldots, n. \tag{2.3}$$

We shall refer to condition (2.3) as the *grouping condition*. The grouping condition simply refers to the fact that the total weight of all requests from an inlet **or** to an outlet is at most unity.

A k-edge-coloring of $G \in \mathcal{B}_r^n$ is a coloring $l : E(G) \to C$, where C is a set of k colors, such that for every $v \in V(G)$ and every color $c \in C$

$$\sum_{\substack{e \in I(v) \\ l(e)=c}} w(e) \le 1. \tag{2.4}$$

Let $m(n,r)$ be the minimum integer k such that every $G \in \mathcal{B}_r^n$ is k-edge-colorable. Our job is to find good bounds for $m(n,r)$, or the exact value if possible. Notice that when all the weights are 1, this problem reduces to the edge coloring of a bipartite graph with maximum degree at most n. Thus, $m(n,r) = n$ when the weights are all unity. This can be shown as a trivial consequence of P. Hall's matching condition, or of König's Line Coloring Theorem [12].

3 A new lower bound

THEOREM 3.1. *For integers $n, r \ge 2$, we have $m(n,r) \ge m(n,2)$. Furthermore,*

$$m(n,2) \ge \left\lceil \frac{5n}{4} \right\rceil, \textit{when } n \textit{ is even},$$

and

$$m(n,2) \ge \left\lceil \frac{5n-1}{4} \right\rceil, \textit{when } n \textit{ is odd}.$$

Proof. The natural approach to find a lower bound k for $m(n,r)$ is to find a particular graph $G \in \mathcal{B}_r^n$ which requires at least k colors. The fact that $m(n,r) \ge m(n,2)$ is trivial. To show the inequality for even n, consider the following graph $G \in \mathcal{B}_r^2$:

- $G = (\{1,2\}, \{1',2'\}; E)$.
- There are n edges from 1 to $1'$ with weight 0.6.
- There are n edges from 1 to $2'$ with weight 0.4.
- There are $n/2$ edges from 2 to $2'$ with weight 1.

The grouping condition is easily seen to be satisfiable. The 0.6-edges in $I(1)$ require n colors. Let k be the number of colors shared by the 0.6-edges and 0.4-edges of $I(1)$. Then, looking from vertex 1 we need at least $n + \frac{n-k}{2}$ colors. While, looking from vertex $2'$ we need at least $\frac{n}{2} + k + \frac{n-k}{2}$ colors. Consequently, the total number of colors needed is at least

$$\begin{aligned} & \max\{n + \frac{n-k}{2}, \frac{n}{2} + k + \frac{n-k}{2}\} \\ \ge\; & \frac{n + \frac{n-k}{2} + \frac{n}{2} + k + \frac{n-k}{2}}{2} \\ =\; & \frac{5n}{4}. \end{aligned}$$

The case when n is odd can be shown similarly. □

4 The exact value of $m(2,r)$

The main result of this section is an algorithm to color all graphs in $\mathcal{B}_r^2$ using at most 3 colors.

THEOREM 4.1. *When $r \ge 2$, we have*

$$m(2,r) = 3.$$

Proof. Theorem 3.1 implies $m(2,r) \ge 3$. We are left to show that every graph $G \in \mathcal{B}_r^2$ is 3-colorable. For $G = (A,B;E) \in \mathcal{B}_r^2$, let $A = B = \{1,2,\ldots,r\}$. The grouping condition indicates that edges incident to each vertex v could be partitioned into two groups $g(v,1)$ and $g(v,2)$ with total weight at each group at most 1. For $i,j \in \{1,2\}$ and $a \in A, b \in B$, let

$$w_{ij}(a,b) = \sum_{\substack{e=(a,b)\in E \\ e \in g(a,i)\cap g(b,j)}} w(e). \tag{4.5}$$

In words, $w_{ij}(a,b)$ is the total weight of all edges e from $a \in A$ to $b \in B$ where e belongs to group i of vertex a and group j of vertex b. The grouping condition implies that for a fixed $i_0 \in \{1,2\}$ and $a_0 \in A$, we have

$$\sum_{b \in B} \left(w_{i_0 1}(a_0,b) + w_{i_0 2}(a_0,b) \right) \le 1. \tag{4.6}$$

Similarly, for a fixed $j_0 \in \{1,2\}$ and $b_0 \in B$, we get

$$\sum_{a \in A} \left(w_{1 j_0}(a,b_0) + w_{2 j_0}(a,b_0) \right) \le 1. \tag{4.7}$$

Clearly, the number of colors needed to color G does not change if at any vertex $v \in V$, we re-label the groups $g(v,1)$ and $g(v,2)$. (Namely, group 1 becomes group 2 and vice versa.) This re-labelling does change the values $w_{ij}(v,b)$ or $w_{ij}(a,v)$, though. Now, re-label the groups at all vertices of G to maximize the following sum

$$\sum_{\substack{a \in A, \\ b \in B}} \left(w_{11}(a,b) + w_{22}(a,b) \right). \tag{4.8}$$

To this end, we use 3 colors to color all edges of G as follows.

- One color for all edges in

$$\bigcup_{\substack{a\in A,\\ b\in B}} (g(a,1)\cap g(b,1)) \tag{4.9}$$

- Another color for all edges in

$$\bigcup_{\substack{a\in A,\\ b\in B}} (g(a,2)\cap g(b,2)) \tag{4.10}$$

- The last color for all edges in

$$\bigcup_{\substack{a\in A,\\ b\in B}} (g(a,1)\cap g(b,2)) \bigcup \bigcup_{\substack{a\in A,\\ b\in B}} (g(a,2)\cap g(b,1)) \tag{4.11}$$

It's straightforward to verify that all edges belong to one of the three color classes. To show that this is a valid coloring, we shall verify that the total weight of edges at each color class which are incident to the same vertex is at most 1. The total weight of edges of color class (4.9) which are incident to vertex $a \in A$ is

$$\sum_{b\in B} w_{11}(a,b) \le \sum_{b\in B} (w_{11}(a,b) + w_{12}(a,b)) \le 1.$$

The cases of color class (4.9) with a vertex $b \in B$, and of color class (4.10) are done similarly.

Lastly, the total weight of edges of color class (4.11) which are incident to vertex $a \in A$ is

$$\sum_{b\in B} (w_{12}(a,b) + w_{21}(a,b)). \tag{4.12}$$

If this sum is > 1, then

$$\sum_{b\in B} (w_{11}(a,b) + w_{22}(a,b)) < 1, \tag{4.13}$$

since

$$\begin{aligned} & \sum_{b\in B} (w_{12}(a,b) + w_{21}(a,b)) \\ + & \sum_{b\in B} (w_{11}(a,b) + w_{22}(a,b)) \\ = & \sum_{b\in B} (w_{11}(a,b) + w_{12}(a,b)) \\ + & \sum_{b\in B} (w_{21}(a,b) + w_{22}(a,b)) \\ \le & \; 2. \end{aligned}$$

However, (4.13) and the fact that the sum (4.12) is > 1 imply that re-labelling the two groups $g(a,1)$ and $g(a,2)$ would increase the sum (4.8), contradicting the maximality of (4.8). □

The above result can be extended in a "straightforward" way to show that

COROLLARY 4.1. *(i)* $m(2^k, r) \le 3^k$*, for any positive integer* $k \ge 1$.

(ii) $m(n, r) \le 3^{\lceil \log_2 n \rceil}$.

Basically, for part (i) we can induct on k, and part (ii) follows from (i). This extended result gives good bounds when n is small. In fact, we can also show results such as $m(3, r) \le 6$ by the same idea, with more tedious analysis. Since these results are not generally good, and the arguments, though intuitively simple, are too tedious to present, we omit their proofs here.

5 The new upper bounds

Next, we give a coloring algorithm yielding a general upper bound which is good for small values of r. The new upper bound implies a stronger value than the conjectured value of $2n - 1$ when $r = 2$.

THEOREM 5.1. *When* $n, r \ge 2$*, we have*

$$m(n,r) \le \left\lceil \left(\frac{r+1}{2}\right) n \right\rceil.$$

Proof. Consider $G = (A, B; E) \in \mathcal{B}_r^n$. Recall that for each $v \in V = A \cup B$, we use $I(v)$ to denote the set of edges incident to v, and $g(v,i)$ the set of edges in group i of v. Now, for each vertex $u \in A$ (respectively B) and each vertex $v \in B$ (respectively A), define n sets of edges $S_u(v,i)$ as follows.

$$S_u(v,i) = g(u,i) \cap I(v), \;\; i = 1, \ldots, n. \tag{5.14}$$

In other words, $S_u(v,i)$ is the set of edges in group i of u which are incident to v. Let $w_u(v,i)$ be the total weight of edges in $S_u(v,i)$. (We set $w_u(v,i) = 0$ if $S_u(v,i) = \emptyset$.) Then, the grouping condition on G implies that

$$\sum_{b\in B} w_a(b,i) \le 1, \;\; \forall a \in A, i = 1, \ldots, n \tag{5.15}$$

$$\sum_{a\in A} w_b(a,i) \le 1, \;\; \forall b \in B, i = 1, \ldots, n. \tag{5.16}$$

To this end, for each $u \in A$ (respectively B) and each $v \in B$ (repsectively A), let $L_u(v)$ be the set of group names i, $1 \le i \le n$, for which $w_u(v,i) > 1/2$, and $\bar{L}_u(v)$ be the set of the rest of the indices. More formally,

$$L_u(v) = \{i \mid w_u(v,i) > 1/2, i = 1, \ldots, n\} \tag{5.17}$$

$$\bar{L}_u(v) = \{1, \ldots, n\} - L_u(v). \tag{5.18}$$

Due to (5.15), for each index i and a particular vertex $a \in A$, there can be at most one $b \in B$ where $w_a(b,i) > 1/2$. Hence, for each $a \in A$ we must have

$$\sum_{b \in B} |L_a(b)| \leq n. \tag{5.19}$$

Similarly, due to (5.16), for each $b \in B$ the following holds:

$$\sum_{a \in A} |L_b(a)| \leq n. \tag{5.20}$$

Now, define a weighted bipartite multi-graph $G' = (A, B; E')$ as follows.

- For each $a \in A$ and $b \in B$, there are n edges between a and b in G', denoted by $e(a,b,i)$, $1 \leq i \leq n$. The weight of $e(a,b,i)$, denoted by $w'(a,b,i)$, is defined below. Note that G' is rn-regular.
- For each $a \in A$ and $b \in B$, if $|L_a(b)| \leq |L_b(a)|$ then

  $$w'(a,b,i) = w_a(b,i), \;\; i = 1, \dots, n.$$

  Otherwise, when $|L_a(b)| > |L_b(a)|$ define

  $$w'(a,b,i) = w_b(a,i), \;\; i = 1, \dots, n.$$

Firstly, we claim that any valid coloring of G' induces a valid coloring of G. The term "valid coloring" here means that the total weight of same color edges which are incident to a particular vertex of G' is at most 1. To see this, given a valid coloring of G' where the edge $e(a,b,i)$ is colored $c(a,b,i)$, say. Then when $|L_a(b)| \leq |L_b(a)|$ we color all edges in the set $S_a(b,i)$ with color $c(a,b,i)$. On the other hand, when $|L_a(b)| > |L_b(a)|$ the set $S_b(a,i)$ gets the color instead.

To this end, let H be the spanning bipartite subgraph of G' obtained from G' by taking only edges whose weights are $> 1/2$. We claim that H has maximum degree at most n. To see this, consider any vertex $a \in A$ of H. We have

$$\begin{aligned} deg_H(a) &= \sum_{b \in B} \min\{|L_a(b)|, |L_b(a)|\} \\ &\leq \sum_{b \in B} L_a(b) \\ &\leq n, \end{aligned}$$

by (5.19). Similarly, $deg_H(b) \leq n$ for all $b \in B$. Add more edges of G' into H so that H is n-regular. This is possible since G' has n parallel edges between any pair $(a,b) \in A \times B$. König's Line Coloring Theorem [12] implies that H is n-edge-colorable. (The actual coloring algorithms can be found in [4,7,8], for instance.) The graph $G' - E(H)$ is $(r-1)n$-regular, hence it is $(r-1)n$-edge-colorable. However, each edge of $G' - E(H)$ has weight at most $1/2$, hence every two colors can be combined into one without violating the condition that the total weight of same color edges at each vertex is at most 1. Consequently, we can color edges of G' with

$$n + \left\lceil \left(\frac{r-1}{2}\right) n \right\rceil = \left\lceil \left(\frac{r+1}{2}\right) n \right\rceil$$

colors. □

Note that this theorem gives the best upper bounds so far for $m(n,r)$ when r is small, as formally put in the following corollary:

COROLLARY 5.1. *When $n \geq 2$, we have*

(i) $m(n,2) \leq \lceil \frac{3n}{2} \rceil$

(ii) $m(n,3) \leq 2n$

(iii) $m(n,4) \leq \lceil \frac{5n}{2} \rceil$

The argument given in Theorem 5.1 can be extended easily to show the following corollary, whose proof we omit.

COROLLARY 5.2. *The general 3-stage Clos network $C(n_1, r_1, m, n_2, r_2)$ is multirate rearrangeable when*

$$m \geq \frac{(r+1)n}{2},$$

where $n = \max\{n_1, n_2\}$, and $r = \max\{r_1, r_2\}$.

Theorem 3.1 and part (i) of corollary 5.1 implies $5n/4 \leq m(n,2) \leq 6n/4$. Given that the number $5/4$ is somewhat "ugly", we conjecture that

CONJECTURE 5.1.

$$m(n,2) = \left\lfloor \frac{3n}{2} \right\rfloor, \; n \geq 2.$$

In fact, recalling $m(2,r) = 3$, it is very tempting to conjecture also that

CONJECTURE 5.2. *The symmetric 3 stage Clos network $C(n,m,r)$ is multirate rearrangeable if there are at least*

$$\left\lfloor \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{r-1}}\right) n \right\rfloor = \left\lfloor 2n \left(1 - \frac{1}{2^r}\right) \right\rfloor$$

middle stage switches. In other words,

$$m(n,r) \leq \left\lfloor 2n \left(1 - \frac{1}{2^r}\right) \right\rfloor.$$

We believe that the upper bound is also the exact value for $m(n,r)$. However, as there is no rigorous evidence yet, we have conjectured a weaker result. Next, we give another upper bound which beats all existing bounds when r is relatively small compared to n.

THEOREM 5.2. *When $n, r \geq 2$, we have*

$$m(n,r) \leq 2n - 1 + \left\lceil \frac{r-1}{2} \right\rceil . \tag{5.21}$$

Proof. Consider $G = (A, B; E) \in \mathcal{B}_r^n$. Suppose e and e' are two edges connecting two vertices $a \in A$ and $b \in B$, with $w(e) + w(e') \leq 1$. Create a new graph G' from G by collapsing e and e' into one edge with weight $w(e) + w(e')$. Then, a valid coloring of G' induces a valid coloring of G.

Now, for every pair $(a, b) \in A \times B$, as long as there are two edges e and e' between a and b for which $w(e) + w(e') \leq 1$, collapse e and e' into one as described. After this procedure is finished, between any pair a and b there is at most one edge with weight $\leq 1/2$, the rest have weights $> 1/2$. Let H be the resulting graph. Call the edges of H with weight $> 1/2$ *heavy*, and the rest of the edges *light*. Since the total weight of edges incident to each vertex of G is at most n, every vertex of H is incident to at most $2n - 1$ heavy edges. In other words, the heavy degree of any vertex of H is at most $2n - 1$.

We claim that the light degree of any vertex of H is at most $r - 1$. To see this, consider $a \in A$. If the heavy degree of A is $2n - 1$, then no light edge incident to a can share the same neighbor as a heavy edge of a. Suppose for the contrary, that there is a heavy edge e and a light edge e' both of which connect a and b. Then, the total weight of the other $2n-2$ heavy edges of a except e is $> n-1$, hence $w(e) + w(e') < 1$, as the total weight associated with a is at most n. Consequently, e and e' must have been collapsed by our procedure. Thus, the light degree of a is at most $r - 1$. Now, if the heavy degree of a is at most $2n-2$, then there is also a vertex $b \in B$ with heavy degree at most $2n-2$. If there was no light edge between a and b, then the light degree of a is at most $r-1$. If there was one light edge between a and b, re-label this light edge "heavy", which does not change the fact that the maximum heavy degree of H is at most $2n - 1$. Again, the light degree of a is now at most $r - 1$.

König's Line Coloring Theorem [12] implies that we can use at most $2n - 1$ colors to color the heavy edges of H, and at most $r - 1$ colors to color the light edges of H. As the light edges have weights $\leq 1/2$, every two colors of $r - 1$ colors can be combined into one, for a total of at most $2n - 1 + \lceil (r-1)/2 \rceil$ colors as desired. (Again, the actual coloring algorithms can be found in [4, 7, 8].) □

As we have mentioned, the new bound is good when r is relatively small. This is formally put in the following corollary:

COROLLARY 5.3. *When $r \leq \frac{n}{2^{k-1}} + 1$, we have*

$$m(n,r) \leq 2n - 1 + \left\lceil \frac{n}{2^k} \right\rceil .$$

For example, if $r \leq n+1$, the Clos network $C(n, m, r)$ is multirate rearrangeable with at most $\lceil 5n/2 \rceil - 1$ middle-stage switches; When $r \leq n/4 + 1$, we only need about $17n/8 - 1$ middle-stage switches; and so on ... The argument given in Theorem 5.2 generalizes straightforwardly to the general Clos network case. Hence, we get the following result:

COROLLARY 5.4. *The general 3-stage Clos network $C(n_1, r_1, m, n_2, r_2)$ is multirate rearrangeable when*

$$m \geq 2n - 1 + \left\lceil \frac{r-1}{2} \right\rceil ,$$

where $n = \max\{n_1, n_2\}$, and $r = \max\{r_1, r_2\}$.

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