



Note

Super link-connectivity of iterated line digraphs

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Abstract

Many interconnection networks can be constructed with line digraph iterations. A digraph has super link-connectivity d if it has link-connectivity d and every link-cut of cardinality d consists of either all out-links coming from a node, or all in-links ending at a node, excluding loop. In this paper, we show that the link-digraph iteration preserves super link-connectivity.

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1. Introduction

Consider a digraph. A link-cut is *natural* if it consists of either all out-links other than loop at a node or all in-links other than loop at a node. A digraph has *super link-connectivity* d if it has link-connectivity d and every link-cut of cardinality d is natural.

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Consider a digraph $G = (V, E)$. The line digraph of G is defined by

$$L(G) = (E, \{(u, v), (v, w) \mid (u, v), (v, w) \in E\}),$$

that is, $L(G)$ takes the link set E of G as its node set and there exists a link from a node x to another node y in $L(G)$ if x ends at the starting node of y in G .

The line digraph iteration preserves the link-connectivity, that is, the line digraph of a d -link-connected digraph is still d -link-connected. Can the line digraph iteration also preserve the super link-connectivity? In this paper, we will give a positive answer that the line digraph of a super d -link-connected digraph must be super d -link-connected.

This result has many applications in interconnection networks since many interconnection networks are constructed with line digraph iterations [3,16,7,19,10,20,9,11].

The result also yields a corollary on super connectivity defined as follows: Clearly, all ending nodes of out-links at any node form a node-cut. All starting nodes of in-edges at any node also form a node-cut. Those node-cuts are called *natural* node-cuts. A digraph has *super connectivity* d if it has connectivity d and every node-cut of cardinality d is natural.

The super connectivity is an important issue studied in interconnection networks [6,22,2,18].

Suppose G has super connectivity d . Could this imply that $L(G)$ has super connectivity d ? The answer is NO. A counterexample is as shown in Fig. 1. However, as a corollary, we will show that if G has super link-connectivity d , then $L^k(G)$ for $k \geq 1$ also have super connectivity d where $L^k(G) = L(L^{k-1}(G))$.

2. Main result

Note that a natural node-cut of $L(G)$ may not be a natural link-cut of G . However, we have the following result.

Lemma 1. G has super link-connectivity d if and only if $L(G)$ has super connectivity d .

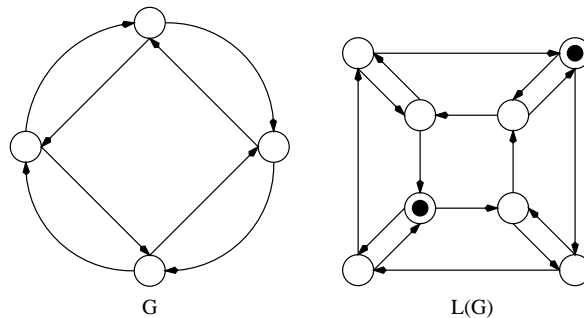


Fig. 1. A counterexample.

Proof. First, assume G has super link-connectivity d . Then $L(G)$ has connectivity d . Thus, every node has out-degree and in-degree at least d . Consider a node-cut C of $L(G)$, with cardinality d . C is a link-cut of G and hence is natural. Without loss of generality, assume that C consists of all out-links other than loop at a node v . If v has a loop (v, v) , then the loop (v, v) has out-degree d and C is exactly the set of ending nodes of those d out-links. Hence, C is a natural node-cut of $L(G)$. If v has no loop, then every in-link at v as a node in $L(G)$ has out-degree d and C is exactly the set of ending nodes of those d out-links. Hence, C is a natural node-cut of $L(G)$.

Conversely, assume $L(G)$ has super connectivity d . Then G has link-connectivity d . Consider a link-cut C of G , with cardinality d . C is a node-cut of $L(G)$ and hence is natural in $L(G)$. Without loss of generality, assume that C is the set of ending nodes of all out-links at a node (u, v) in $L(G)$. Note that C does not contain loop in G since C is a minimum link-cut of G . Therefore, C consists of all out-links at v in G , that is, C is natural. \square

It is quite interesting that the super link-connectivity is preserved by line digraph operations while the super connectivity is not.

Theorem 1. *If G has super link-connectivity d , then $L(G)$ has super link-connectivity d .*

Proof. Consider a minimum link-cut C of $L(G)$. Suppose C breaks the node set of $L(G)$ into two parts A and B such that no link other than those in C is from A to B . Let

$$U = \{(u, v) \mid ((u, v), (v, w)) \in C\},$$

$$W = \{(v, w) \mid ((u, v), (v, w)) \in C\},$$

$$V = \{v \mid ((u, v), (v, w)) \in C\}.$$

We next show several claims.

Claim 1. $|C| = d$.

Proof. By Lemma 1, $L(G)$ has super connectivity d and hence link-connectivity at least d . This means $|C| \geq d$. On the other hand, since G has super link-connectivity d , there exists a node v of G such that either all out-links at v other than loop or all in-link at v other than loop form a minimum link-cut D of G , with cardinality d . Without loss of generality, assume the former occurs. If v has a loop, then this loop has out-degree d in $L(G)$. If v has no loop, then every in-link of v has out-degree d in $L(G)$. This means that in any case, $L(G)$ has link-connectivity at most d . Hence, $|C| = d$. \square

Claim 2. $A = U$ or $B = W$.

Proof. For contradiction, suppose $A - U \neq \emptyset$ and $B - W \neq \emptyset$. Define

$$X = \{x \mid (x, w) \in B\},$$

$$Y = V(G) - X,$$

where $V(G)$ is the node set of G . Note that $\{y \mid (u, y) \in A - U\} \subseteq Y$. Thus, $X \neq \emptyset$ and $Y \neq \emptyset$. Moreover, every link (y, x) from a node y in Y to a node x in X must belong to A (since $(y, x) \in B$ implies $y \in X$) and hence belongs to U . Therefore, U is a link-cut of G .

Similarly, we can show that W is a link-cut of G . Note that $|U| \leq |C| = d$ and $|W| \leq |C| = d$. Since G has super link-connectivity d , both U and W are natural link-cuts of cardinality d . In particular, U and W do not contain any loop. Hence, we have $|C| = |U| = |W|$. It follows that any two links U cannot share the same ending node. Therefore, U must consist of out-links at a node x and W must consist of in-links at a node y (Fig. 2). It also follows that $|V| = |C|$.

Choose $v \in V$. Note that U and W do not contain any loop. Then every out-link at v , not in W , must belong to $A - U$. Thus, any path from v to y not passing link (v, y) must pass some link in $U - \{(x, v)\}$. (Otherwise, the path will go from a link in $A - U$ to a link in B . This produces a link from A to B in $L(G)$, not in C , a contradiction.) This means that $(U - \{(x, v)\}) \cup \{(v, y)\}$ is also a link-cut of G , with cardinality d , which is not natural, contradicting the super link-connectivity of G . \square

Claim 3. C is natural.

Proof. First, we show $|V| = 1$. For contradiction, suppose $|V| \geq 2$. Note that each node in G has at least d out-links other than loop and at least d in-links other than loop. Moreover, $|U| \leq |C| = d$ and $|W| \leq |C| = d$. Thus, each node in V must have an in-link not in U and an out-link not in W . Such an in-link must belong to $B - W$ and such an

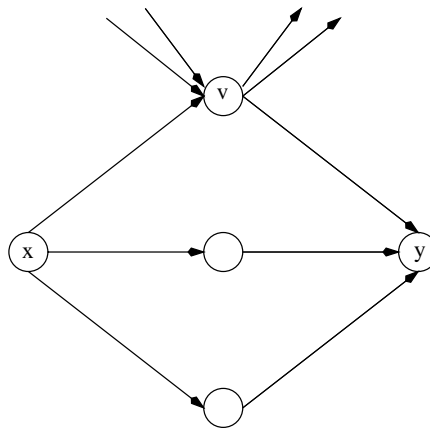


Fig. 2. $|C| = |U| = |W| = |V|$.

out-link must belong to $A - U$. This means that $A - U \neq \emptyset$ and $B - W \neq \emptyset$, contradicting to Claim 2.

Now, assume $V = \{v\}$. Without loss of generality, assume also $A = U$. Note that if v has an out-link, other than loop, not in W , then it must belong to $A - U$. Therefore, all out-links at v , other than loop, belong to W . Since v has at least d out-links other than loop and $|W| \leq d$, we have $|W| = d$ and that W is exactly the set of all out-links, other than loop, at v . It follows that $|U| = 1$. Hence, if v has no loop, then C is natural. If v has a loop, then this loop must belong to U . In fact, the loop being in $A - U$ would introduce a link from A to B , but not in C , and the loop being in $B - W$ would introduce a link from A to B , but not in C , a contradiction. Thus, the loop (v, v) not being in W implies it being in U . Since $|U| = 1$, v can have only one loop. Hence, C is exactly the set of all out-links, other than loop, at the node (v, v) in $L(G)$, that is, C is natural. \square

By Claims 1 and 3, every minimum link-cut of $L(G)$ is a natural link-cut of cardinality d . Therefore, $L(G)$ has super link-connectivity d .

Corollary 1. *If G has super link-connectivity d , then $L^k(G)$ for $k \geq 1$ has super connectivity d .*

The counterexample in Fig. 1 tells us that in general, a digraph having super connectivity d may not have super link-connectivity d . However, Theorem 1 tells us that this is true for a special family of digraphs—line digraphs.

3. Applications

When an interconnection network contains possible node-faults there are two fault-tolerance measures in the literature [1,13–15].

The first one is the connectivity. The second one is the probability of the remaining network being connected when nodes fail with certain probabilistic distribution. Let F be the family of all node-cuts of a digraph G . By the exclusion–inclusion principal,

$$\begin{aligned} \text{Prob}(G \text{ connected}) &= 1 - \text{Prob}(G \text{ disconnected}) \\ &= 1 - \sum_{c \in F} \text{Prob}(c) + \sum_{c_1, c_2 \in F, c_1 \neq c_2} \text{Prob}(c_1 \cup c_2) - \dots \end{aligned}$$

where $\text{Prob}(c)$ is the probability of all nodes in c fail. When all nodes are independent, $\text{Prob}(c)$ is a product of failure probabilities of nodes in c . Therefore, if every node has the same fault probability of a small number, then $\text{Prob}(G \text{ connected})$ depends mainly on the number of the minimum node-cuts. The number of the minimum natural node-cuts is certainly a lower bound of the number of minimum node-cuts. Therefore, the super-connected digraph reaches maximum fault-tolerance in certain sense.

Given a degree bound d , many constructions have been found in the literature to achieve the maximum connectivity d and near-minimum diameter [21,8], including Kautz digraphs [16], cyclically modified de Bruijn digraphs [8,17], generalized cycles

[10], etc. Do they also have super connectivity? In this section, we study some of them.

Example 1. Kautz digraph $K(d, 1)$ is the complete digraph on $d + 1$ nodes without loop and in general $K(d, D) = L^{D-1}(K(d, 1))$ [16]. We claim that $K(d, 1)$ has super link-connectivity d . Consider a link-cut C of size d in $K(d, 1)$, which breaks the node set of $K(d, 1)$ into two parts A and B such that every link from A to B belongs to C . Note that there are $|A|(|A| - 1)$ links from A to A and each node has d out-links. Therefore, $|A|d - |A|(|A| - 1) = d$. That is, $(|A| - 1)(d - |A|) = 0$. Thus, $|A| = 1$ or $|A| = d$. Since $|A| = d$ implies $|B| = 1$. Therefore, C is a natural link-cut.

Corollary 2. Kautz digraph $K(d, D)$ has super connectivity d for $D \geq 2$.

Example 2. Ferrero and Padró [10] studied a family of digraphs $BGC(p, d, d^k)$ where $BGC(p, d, d^k) = L(BGC(p, d, d^{k-1}))$ for $k \geq 2$ and $BGC(p, d, d)$ is a p -partite digraph $(V_1, V_2, \dots, V_p, E)$ that $|V_1| = |V_2| = \dots = |V_p| = d$ and that a link (u, v) exists if and only if $u \in V_i$ and $v \in V_{i+1}$ for some $1 \leq i \leq p$ ($V_{p+1} = V_1$). We claim that for $d \geq 3$ and $p \geq 2$, $BGC(p, d, d)$ has super link-connectivity d . To show it, consider a link-cut C of cardinality at most d , which breaks the node set into two parts A and B such that every link from A to B belongs to C . Denote $a_i = |A \cap V_i|$ and $b_i = |B \cap V_i|$. Then we must have

$$a_1 b_2 + a_2 b_3 + \dots + a_p b_1 \leq d.$$

First, we show that there exists i such that $a_i = 0$ or $b_i = 0$. For contradiction, suppose such an i does not exist. Then for every i , $a_i > 0$ and $b_i > 0$. Note that $a_i + b_i = d \geq 3$. Therefore,

$$d \geq a_1 b_2 + a_2 b_3 + \dots + a_p b_1 > b_2 + a_2 b_3 + \dots + a_p \geq b_2 + a_2 = d,$$

a contradiction.

Now, suppose, without loss of generality, that $a_i = 0$ for some i . Since $a_1 + \dots + a_p = |A| > 0$, there exists i such that $a_{i-1} \neq 0$ and $a_i = 0$ (denote $a_0 = a_p$). Without loss of generality, assume $a_{p-1} \neq 0$ and $a_p = 0$. Then $b_p = d$. This implies $a_{p-1} = 1$ and $a_1 b_2 + a_2 b_3 + \dots + a_{p-2} b_{p-1} = 0$. This in turn implies $a_{p-2} = a_{p-3} = \dots = a_1 = 0$. Hence, $|A| = 1$. Since $BGC(p, d, d)$ contains no loop, C consists of d out-links at the node in A . This means that C is natural.

Corollary 3. For $d \geq 3$, $p \geq 2$, and $k \geq 2$, $BGC(p, d, d^k)$ has super connectivity d .

Example 3. Ferrero and Padró [10] also studied a family of digraphs $KGC(p, d, n) = C_p \otimes GK(d, n)$ where C_p is a directed cycle of length p , $GK(d, n)$ is the generalized Kautz digraph with node set $Z_n = \{0, 1, \dots, n-1\}$ and link set $\{(i, -id+k) \mid i \in Z_n, 1 \leq k \leq d\}$, and operation \otimes is defined as follows: Let $G = (V, E)$ and $G' = (V', E')$. Then $G \otimes G'$ has node set $V \times V'$ and link set $\{((u, u'), (v, v')) \mid (u, v) \in E, (u', v') \in E'\}$. It is not hard to prove that $KGC(p, d, d^k(d^p + 1)) = L(KGC(p, d, d^{k-1}(d^p + 1)))$ for $k \geq 1$

and $KGC(p, d, d^p + 1)$ is a p -partite digraph $(V_1, V_2, \dots, V_p, E)$ with $|V_1| = |V_2| = \dots = |V_p| = d^p + 1$ and that each node in V_i has d out-links which reach d consecutive nodes in V_{i+1} , the d^2 out-links of those d consecutive nodes in V_{i+1} reach d^3 consecutive nodes in V_{i+2}, \dots , that is, take each node in V_i as a root, we can find a complete d -nary tree such the second level consists of d consecutive nodes in V_{i+1} , the third level consists of d^2 consecutive nodes in V_{i+2}, \dots , the $(p + 1)$ -level consists of d^p consecutive nodes in $V_{i+p} = V_i$ which are exactly those d^p nodes in V_i other than the root.

Now, we claim that $KGC(p, d, d^p + 1)$ has super link-connectivity d . To show it, consider a link-cut C with cardinality at most d , which breaks the node set into two nonempty parts A and B such that every link from A to B belongs to C . First, we note that there must exist an i such that $A \cap V_i \neq \emptyset$ and $B \cap V_i \neq \emptyset$. In fact, if such an i does not exist, then there must exist $V_i \subseteq A$ and $V_{i+1} \subseteq B$ (note: $V_{p+1} = V_1$). It follows that C contains all $d(d^p + 1)$ links from V_i to V_{i+1} , contradicting $|C| \leq d$.

Without loss of generality, we may assume $A \cap V_1 \neq \emptyset$ and $B \cap V_2 \neq \emptyset$. Construct a digraph $H = (V_1, E_1)$ as follows: $(u, v) \in E_1$ if and only if there exists a path of length p from u to v in $KGC(p, d, d^p + 1)$. From the property of $KGC(p, d, d^p + 1)$ described as above, it is easy to see that H is isomorphic to $K(d^p, 1)$. Note that if there exists a path of length p from u to v in $KGC(p, d, d^p + 1)$, then such a path is unique. This means that each link of H uniquely corresponds to a path of length p between two nodes in V_1 . Let C' be the set of links corresponding to those paths containing a link in C . Then C' is a link-cut of H . Since each link in C can be contained by exactly d^{p-1} such paths (Fig. 3), we have $|C'| \leq d^p$. Note that H has super link-connectivity d^p . Therefore, $|A \cap V_1| = 1$ or $|B \cap V_1| = 1$. First, assume $|A \cap V_1| = 1$. This means that C must break all d^p paths which form a complete d -nary tree rooted at a node in V_1 . This can be done only if C consists of d out-links at the root. Hence, C is natural. Similarly, $|B \cap V_1| = 1$ also implies that C is natural.

Corollary 4. For $d \geq 3$ and $k \geq 1$, $KGC(p, d, d^{p+k} + d^k)$ has super connectivity d .

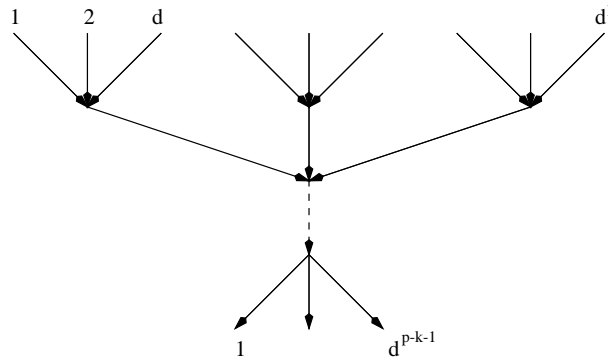


Fig. 3. d^{p-1} paths pass through the same link.

4. Discussion

The line digraph iteration preserves the degree, that is, the line digraph of a d -regular digraph is still d -regular. This is a very important property different from line graph iteration. This property enables the line digraph iteration to become a very useful tool to construct interconnection networks. In this paper, we showed that the line digraph iteration preserves the super connectivity under certain condition. We also established that two families of generalized cycles are super connected. Recently, generalized cycles have been studied extensively [10,4,5,12]. They contain many important interconnection networks as special cases.

References

- [1] J.C. Bermond, N. Homobono, C. Peyrat, Large fault-tolerant interconnection networks, *Graphs Combin.* 5 (2) (1989) 107–123.
- [2] F.T. Boesch, J.F. Wang, Super line-connectivity properties of circulant graphs, *SIAM J. Algebraic Discrete Methods* 7 (1) (1986) 89–98.
- [3] N.G. de Bruijn, A combinatorial problem, *Koninklijke Nederlandse Academie van Wetenschappen Proc. A* 49 (1946) 758–764.
- [4] J.M. Brunat, M. Espona, M.A. Fiol, O. Serra, Cayley digraphs from complete generalized cycles, *European J. Combin.* 20 (1999) 337–349.
- [5] J.M. Brunat, M. Maureso, M. Mora, Endo-circulant digraphs: connectivity and generalized cycles, *Discrete Math.* 197/198 (1999) 157–167.
- [6] F. Cao, D.-Z. Du, S. Han, D. Kim, T. Yu, Line digraph iterations and diameter vulnerability, *Taiwanese J. Math.* 3 (1999) 281–290.
- [7] F. Cao, D.-Z. Du, D.F. Hsu, L. Hwang, W. Wu, Super line-connectivity of consecutive- d digraphs, *Discrete Math.* 183 (1998) 27–38.
- [8] D.-Z. Du, F.K. Hwang, Generalized de Bruijn digraphs, *Networks* 18 (1988) 27–33.
- [9] D.-Z. Du, Y.D. Lyuu, D.F. Hsu, Line digraph iterations and connectivity analysis of de Bruijn and Kautz graphs, *IEEE Trans. Comput.* 42 (1993) 612–616.
- [10] D. Ferrero, C. Padró, Disjoint paths of bounded length in large generalized cycles, *Discrete Math.* 197/198 (1999) 285–298.
- [11] M.A. Fiol, J.L.A. Yebra, I. Alegre, Line digraph iterations and the (d, k) digraph problem, *IEEE Trans. Comput.* C-33 (1984) 400–403.
- [12] J. Gómez, C. Padró, S. Perennes, Large generalized cycles, *Discrete Appl. Math.* 89 (1998) 107–123.
- [13] N. Homobono, C. Peyrat, Connectivity of Imase and Itoh digraphs, *IEEE Trans. Comput.* 37 (11) (1988) 1459–1461.
- [14] M. Imase, M. Itoh, A design for directed graph with minimum diameter, *IEEE Trans. Comput.* C-32 (1983) 782–784.
- [15] M. Imase, I. Soneoka, K. Okada, A fault tolerant processor interconnection network, *Systems Comput. Japan* 17:8 (1986) 21–30.
- [16] W.H. Kautz, Bounds on directed (d, k) graphs, *Theory of cellular logic networks and machines*, AFCRL-68-0668 Final Report, 1968, pp. 20–28.
- [17] V.P. Kumar, S.M. Reddy, A class of graphs for fault-tolerant processor interconnections, *IEEE 1984 Int. Conf. on Distributed Computing Systems*, 1984, pp. 448–460.
- [18] Q.L. Li, Q. Li, Super edge connectivity properties of connected edge symmetric graphs, *Networks* 33 (1999) 157–159.
- [19] C. Padró, P. Morrillo, Diameter-vulnerability of iterated line digraphs, *Discrete Math.* 149 (1996) 189–204.

- [20] C. Padró, P. Morrillo, E. Llobet, Diameter-vulnerability of large bipartite digraphs, *Discrete Appl. Math.* 64 (1996) 239–248.
- [21] S.M. Reddy, J.G. Kuhl, S.H. Hosseini, H. Lee, On digraphs with minimum diameter and maximum connectivity, *Proc. 20th Ann. Allerton Conf. 1982*, pp. 1018–1026.
- [22] T. Soneoka, Super edge-connectivity of dense digraphs and graphs, *Discrete Appl. Math.* 37/38 (1992) 511–523.