WDM SWITCHING NETWORKS, REARRANGEABLE AND NONBLOCKING $[W, F]$-CONNECTORS

HUNG Q. NGO*

Abstract. We propose a framework to analyze and compare wavelength division multiplexed (WDM) switching networks qualitatively and quantitatively. The framework not only help analyze and compare the complexity of WDM switching networks, but also explain interesting properties of different designs. Then, several important problems arising from this idea are addressed, and complexity bounds are derived. We also give several applications of the proposed model, including explicit constructions of non-blocking WDM switching fabrics.

Key words. Wavelength division multiplexing, switching networks, $[w, f]$-connectors.

AMS subject classifications. 05C75, 05C90, 05C35

1. Introduction.

With the advances of dense wavelength division multiplexing (DWDM) technology [21, 32, 38], the number of wavelengths in a wavelength division multiplexed (WDM) network increases to hundreds or more per fiber, and each wavelength operates at 10Gbps (OC-192) or higher [17–19]. While raw bandwidth has increased by more than four orders of magnitude over the last decade or so, capacity of switches has only been up by a factor of ten. Switching speed is the bottleneck at the core of the optical network infrastructure [37]. Consequently, a challenge is to design cost-effective WDM cross-connects (WXC) that can scale in size beyond a hundred of inputs and outputs, and at the same time, switch fast (e.g., tens of nanoseconds or less).

The notion of "cost-effectiveness" is difficult to capture. One can analyze and compare WDM switches both qualitatively and quantitatively.

Qualitatively, we need to know if a design is strictly nonblocking (SNB), rearrangeably nonblocking (RNB), and/or wide-sense nonblocking (WSNB) under different request models [20, 24, 25, 31, 33, 34, 39, 41, 42] and different traffic patterns (unicast [24, 25, 42], multicast [22, 26, 43]). A design can also be blocking as long as its blocking probability is below a certain threshold [15, 31]. There are various other qualitative features such as small cross-talk [39], small number of limited-range wavelength converters [25, 42], or fault-tolerant [4]. Presumably each new design is guided by a particular qualitative feature. For example, one might come up with an RNB design under one request model, which may or may not be SNB under another request model. One might also have an intuitively good design, and hence need to know what qualitative feature the design possesses. This question is challenging in general. We will see later that the graph models introduced in this paper help, in several ways, answer these types of questions.

Quantitatively, comparing different designs, or asking how close to be optimal a new design is, are very important questions. This is a multi-dimensional problem, as there are many factors effecting the "cost" of a switch. Some factors such as actual cost in dollars are business matters. Other factors include: the numbers of different types of switching components, such as (de)multiplexors (MUX/DEMUX), full and limited wavelength converters (FWC and LWC), semiconductor optical amplifiers (SOA), optical add-drop multiplexors (OADM), directional couplers (DC), etc; or signal and switch quality parameters, such as cross-talk, power consumption and attenuation, integratability and scalability, blocking probabilities, etc.

*Computer Science and Engineering, 201 Bell Hall, State University of New York at Buffalo, Amherst, NY 14260, USA. (hungngo@cse.buffalo.edu).

It should be apparent that we cannot hope to have a cost model that fits all needs. However, one can devise cost models which give good approximated measures on how "complex" a construction is. The notion of complexity should roughly capture as many practical parameters as possible.

In this paper, we outline an intriguing approach to model switch complexity which not only helps analyze WDM switches quantitatively and qualitatively, but also suggests interesting generalizations of classical switching network theory [2, 29]. Then, we address several important problems arising from the framework.

We consider two dominant request models in this paper. The following phenomena are samples of what our model suggests:

(a) Designing WXCs in the so-called $(\lambda, F, \lambda', F')$-*request model* is basically the same as designing a circuit switch. Hence, many old ideas on circuit switching can be readily reused. (Section 4.1.)
(b) Two SNB switches in two models are equivalent topologically, even though one request model is much less restrictive than the other. (Section 5.1.)
(c) There is an inherent tradeoff between a WXC's "depth" (which is proportional to signal attenuation, cross-talk) and its "size" (which approximates the WXC's complexity). (Section 5.2.)
(d) Different designs of WXCs which make used of different optical components can now be viewed in a unified manner. We can tell if two different-looking designs are equivalent topologically, for example. (Section 7.)

We will also derive several complexity bounds and give a generic construction which can be used to construct RNB switches (Section 6).

The framework proposed here gives rise to interesting mathematical and networking problems, many of which are generalized versions of the well-studied circuit switching problems. We address several of these problems in the second part of the paper.

The rest of the paper is organized as follows. Section 2 introduces basic settings of WXCs, request models, and nonblocking concepts. Section 3 motivates the graph models which will be rigorously defined in Section 4. Section 5 addresses several key complexity problems arising from the framework. Section 6 explicitly constructs graphs with low complexity. The ideas in this section can be used to construct WXCs of low cost. Section 7 discusses several applications of our framework. Lastly, Section 8 concludes the paper with a few remarks and discussions on future works.

2. WDM cross-connects, request models, and nonblockingness. A general WDM cross-connect (WXC) consists of f input fibers each of which can carry a set $\Lambda = \{\lambda_1, \dots, \lambda_w\}$ of w wavelengths, and f' output fibers each of which can carry a set $\Lambda' = \{\lambda'_1, \dots, \lambda'_{w'}\}$ of w' wavelengths, where $fw = f'w'$. (See Figure 2.1.) This setting is referred to as the *het-*

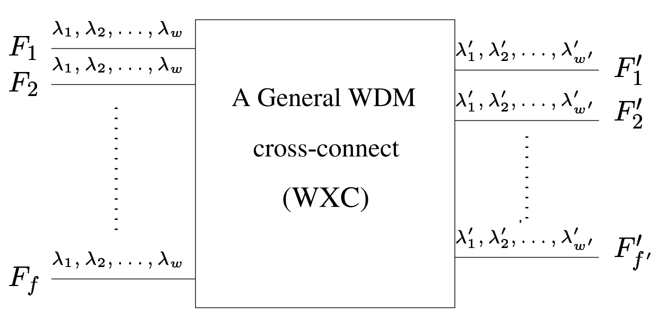


FIG. 2.1. *Heterogeneous WDM Cross-Connect*

erogeneous case [34], which is needed to connect subnetworks from different manufacturers. Henceforth, let $n = fw = f'w'$, unless specified otherwise.

Let $\mathcal{F} = \{F_1, \dots, F_f\}$ and $\mathcal{F}' = \{F'_1, \dots, F'_{f'}\}$ denote the set of input and output fibers, respectively. There are two common types of request models [24, 41]. In the (λ, F, F')-*request model*, a connection request is of the form (λ, F, F'), where $\lambda \in \Lambda, F \in \mathcal{F}$, and $F' \in \mathcal{F}'$. The request asks to establish a connection from wavelength λ in input fiber F to any free wavelength in output fiber F'. In the $(\lambda, F, \lambda', F')$-*request model*, the difference is that the output wavelength λ' in F' is also specified.

In the next sections, we will define the concepts of SNB, WSNB, and RNB for both request models. We will be somewhat informal in our definitions. However, the idea should be clear to readers who have been exposed to switching theory [2, 13, 23, 29].

Consider a WXC with a few connections already established. Under the (λ, F, F')-model, a new request (λ, F, F') is said to be *valid* if and only if λ is a free wavelength in fiber F, and there are at most $w' - 1$ existing connections to F'. Under the $(\lambda, F, \lambda', F')$-model, a new request $(\lambda, F, \lambda', F')$ is *valid* if and only if λ is free in F and λ' is free in F'.

A *request frame* under the (λ, F, F') model is a set of requests such that no two requests are from the same wavelength in the same input fiber, and that there are at most w' requests to any output fiber. A *request frame* under the $(\lambda, F, \lambda', F')$-model is a set of requests such that no two requests are **from** the same input wavelength/fiber pair, nor **to** the same output wavelength/fiber pair.

The following definitions hold for both request models. A request frame is *realizable* by a WXC if all requests in the frame can be routed simultaneously. A WXC is *rearrangeably nonblocking* if and only if any request frame is realizable by the WXC. A WXC is *strictly nonblocking* if and only if a new valid request can always be routed through the WXC without disturbing existing connections. A WXC is *widesense nonblocking* if and only if a new valid request can always be routed through the WXC without disturbing existing connections, provided that new requests are routed according to some routing algorithm. When the routing algorithm is known, we say that the WXC is WSNB with respect to the algorithm.

Henceforth, for any positive integer p, let $[p]$ denote the set $\{1, \dots, p\}$ and S_p denote the set of all permutations on $[p]$. Graph theoretic terminologies and notations we use here are fairly standard (see [40], for instance).

3. Motivations. Main known results on the constructions of (different types of) nonblocking WXCs can be found in [20, 22, 24–26, 31, 33, 34, 39, 41, 42]. (Note that we are not discussing multicast switching in this paper.) The constructions from these references made use of various different types of optical components, such as arrayed waveguide grating routers (AWGR) and LWCs in [24], SOAs and LWCs in [42], OADMs and FWCs in [41], wavelength selective cross-connects (WSC), wavelength interchangers (WI) in [33, 34], directional couplers (DC) in [39]. It is clear that the task of comparing different designs is not easy. Different designs make use of different optical switching components which oftentimes are trade-offs. For instance, the designs in [42] made use of SOAs and LWCs which have lower wavelength conversion cost than those in [24]. On the other hand, the ones in [24] preferred AWGRs over SOAs since AWGRs consume virtually no power.

We now propose an approach to uniformly model all designs by graphs, and then discuss switch complexity from the graphs' standpoint.

We classify optical switching components into fibers and other switching components. For any switch design, we apply the following procedure to construct a directed acyclic graph (DAG) from the design: (a) replace each fiber by a set of vertices $\Lambda \cup \Lambda'$, which represents all possible wavelengths which can be carried on the fiber; (b) the edges of the DAG are defined according to the capacity of switching components in the design. The edges connect wavelengths (i.e. vertices) on the inputs of each switching component to the wavelengths on the outputs in accordance with the functionality of the switching component.

We shall be somewhat brief on this construction. However, the reader will undoubtedly see the basic idea. As an example, Figure 3.1 shows how to turn an AWGR, an FWC, and a MUX into edges. Figure 3.2 shows a complete construction of the DAG from the design on the left.

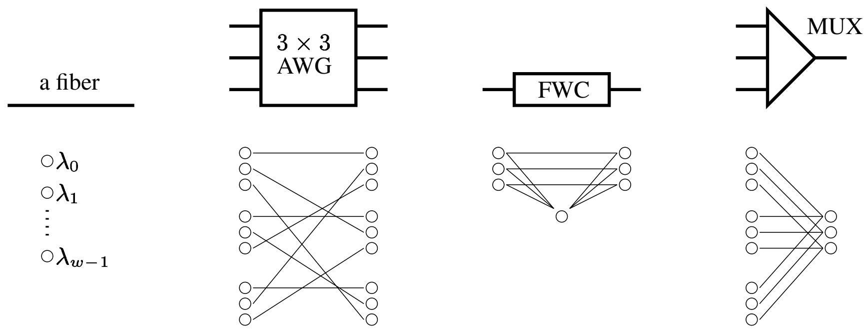


FIG. 3.1. *Turning optical components into parts of a graph. A fiber is replaced by a set of vertices representing the wavelengths it can carry. Other components define edges connecting input wavelengths to output wavelengths. For the AWGR, MUX, and FWC, we illustrate with $w = 3$. Edges are directed from left to right.*

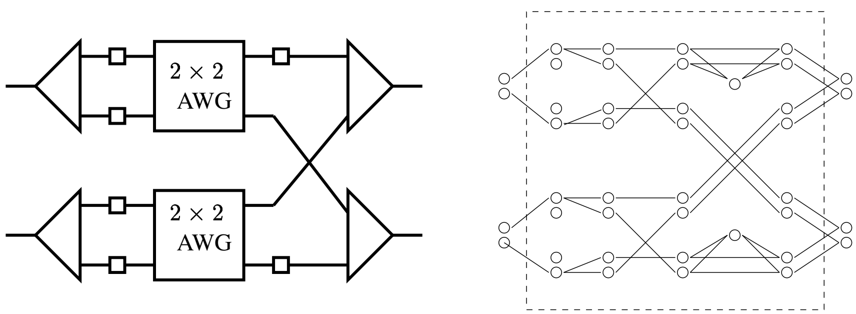


FIG. 3.2. *A WDM switch design and its corresponding DAG.*

The key point is that **a set of compatible routes from input wavelengths to output wavelengths correspond to a set of vertex disjoint paths from the inputs to the outputs of the DAG.**

There are two main parameters of the DAG, which capture the notion of "switch complexity" discussed earlier. The number of edges of the DAG, called the *size* of the DAG, is roughly proportional to the total cost of various components in the design. For example, an FWC corresponds to $3w$ edges while an WI [34] corresponds to w^2 edges; a $w \times w$ AWGR corresponds to w^2 edges, while a $w \times w$ WDM crossbar corresponds to w^4 edges; etc. As WIs and WDM crossbars are more expensive than FWCs and AWGRs, this model makes sense. Other components follow the same trend.

The reader might have noticed that different components contribute different "weights" to the total cost, hence summing up the number of edges may not give the "right" cost. To answer this doubt, we make three points. Firstly, as argued earlier one cannot hope to have a perfect model which fits all needs, and part of the notion of cost is a business matter. Our first aim is at a more theoretical level. Secondly, this is the first step toward a good cost model. One certainly can envision weighted graphs as the next step. Thirdly, we surely can and should still use more traditional cost functions such as the direct counts of the number of each components and compare them individually.

The second measure on the DAG is its *depth*, i.e. the length of a longest path from any input to any output. as signals passing through different components of a design, they lose some power. The depth of the DAG hence reflects power loss, or in some cases even the signal delay. Again, different components impose different power loss factors. Hence, other information need to be taken into account to estimate power loss. However, it is clear that network depth is an important measure.

Last but not least, this DAG model provides a nice bridge between classical switching theory and WDM switching theory. As we shall see in later sections, this model helps us tremendously in answering qualitative questions about a particular construction. For example, if an wf-input wf-output DAG must have size $\Omega(f^2w^2)$ to be strictly nonblocking, then we know for certain that a construction of cost $o(f^2w^2)$ (reflected by the DAG's size) cannot be strictly nonblocking (for sufficiently large values of fw.)

4. Rigorous settings. An (n_1, n_2)*-network* is a directed acyclic graph (DAG) $\mathcal{N} = (V, E; A, B)$, where V is the set of vertices, E is the set of edges, A is a set of n_1 nodes called *inputs*, and B, disjoint from A, is a set of n_2 nodes called *outputs*. The vertices in $V - A \cup B$ are *internal* vertices. The in-degrees of the inputs and the out-degrees of the outputs are 0. The *size* of a network is its number of edges. The *depth* of a network is the maximum length of a path from an input to an output. An n*-network* is an (n, n)-network.

An n-network is meant to represent the DAG from last section under the $(\lambda, F, \lambda', F')$-request model. (Recall $n = wf = w'f'$.) Later on, we shall define $[w, f]$-networks which represent the DAG under the (λ, F, F')-request model.

4.1. The $(\lambda, F, \lambda', F')$-request model. Given an n-network $\mathcal{N} = (V, E; A, B)$, a pair $D = (a, b)$ in $A \times B$ is called a *request* (or *demand*) for $\mathcal{N}$. A set $\mathcal{D}$ of requests is called a *request frame* iff no two requests share an input nor an output. A request $D = (a, b)$ is *compatible* with a request frame $\mathcal{D}$ iff $\mathcal{D} \cup \{D\}$ is also a request frame. A *route* R for a request $D = (a, b)$ is a (directed) path from a to b. We also say R *realizes* D. A *state* of $\mathcal{N}$ is a set $\mathcal{R}$ of vertex disjoint routes. Each state of $\mathcal{N}$ realizes a request frame, one route per request in the frame. A request frame $\mathcal{D}$ is *realizable* iff there is a network state realizing it.

A *rearrangeable (RNB) n-connector* (or just n-connector for short) is an n-network in which the request frame $\mathcal{D} = \{(a, \pi(a)) \mid a \in A\}$ is realizable, for any one-to-one correspondence $\pi : A \to B$.

A *strictly nonblocking (SNB) n-connector* is an n-network $\mathcal{N}$ in which given any network state $\mathcal{R}$ realizing a request set $\mathcal{D}$, and given a new request D compatible with $\mathcal{D}$, there exists a route R such that $\mathcal{R} \cup \{R\}$ is a network state realizing $\mathcal{D} \cup \{D\}$.

As requests come and go, a strategy to pick new routes for new requests is called a *routing algorithm*. An n-network $\mathcal{N}$ is called a *widesense nonblocking (WSNB) n-connector* with respect to a routing algorithm $\mathbf{A}$ if $\mathbf{A}$ can always pick a new route for a new request compatible with the current network state. We can also replace $\mathbf{A}$ by a class of algorithms $\mathcal{A}$. In general, an n-network $\mathcal{N}$ is WSNB iff it is WSNB with respect to *some* algorithm.

We often consider two classes of functions on each network type: (a) the minimum size of a network, and (b) the minimum size of a network with a given depth. The main theme of research on classical switching networks is to investigate the trade-off between size and depth [23, 29].

Let $rc(n)$, $wc(n)$, and $sc(n)$ denote the minimum size of an RNB, WSNB, and SNB n-connector, respectively. Let $rc(n, k)$, $wc(n, k)$, and $sc(n, k)$ denote the minimum size of an

RNB, WSNB, and SNB n-connector with depth k, respectively. Note that $rc(n) \leq wc(n) \leq sc(n)$, and $rc(n,k) \leq wc(n,k) \leq sc(n,k)$. These classes of functions are well studied in the context of circuit switching networks (see, e.g., [23, 29] for nice surveys).

Two key conclusions arise from this formulation:

- Studying WDM switches under the $(\lambda, F, \lambda', F')$-request model is in a sense the same as studying classical switching networks. A lot of results can be readily reused. For example, using our DAG construction, it is easy to see that all of the constructions (under this request model) in [24,33,34,42] made use of various forms of the Clos network [5, 27], Banyan, Butterfly and Base Line networks [11, 12], or Cantor network [3], etc. In fact, under this request model, we do not know of any design which is not topologically isomorphic to some of classical circuit design.
- The situation under the (λ, F, F')-model is different, however. The RNB design in [24], and the designs presented in this paper require several new themes. Particularly, this is because the (λ, F, F')-model is not equivalent to the classical switching case, as we shall see in the next section.

4.2. The (λ, F, F')-request model. In this request model, each pair (λ, F) with $\lambda \in \Lambda$, $F \in \mathcal{F}$ can still be thought of as an "input" to our graphs as in the previous request model. However, on the output side we do have to indicate the number f of fibers and the number of wavelengths w on each fiber.

Set $n = wf$. A $[w, f]$-network is an n-network $\mathcal{N} = (V, E; A, B)$ in which the set B of outputs is further partitioned into f subsets $B_1, \dots, B_f$ of size w each. Each set B_i represents an output fiber in the WDM switch. We implicitly assume the existence of the partition in a $[w, f]$-network, in order to simplify notations. (There is a slightly subtle point to be noticed here. The inputs are not distinguishable in this request model, while we do care which fiber an output wavelength is from. The parameters w and f in the above sentence and henceforth should be thought of as w' and f' in the original discussion.)

Given a $[w, f]$-network $\mathcal{N}$, a pair $D = (a, k) \in A \times [f]$ is called a (connection) *request* for $\mathcal{N}$. The number k is called the *output fiber number* of D. A set $\mathcal{D}$ of requests is called a *request frame* iff no two requests share an input, and for any $k \in [f]$, we have $|\{a \mid (a,k) \in \mathcal{D}\}| \leq w$. A request $D = (a, k)$ is *compatible* with a request frame $\mathcal{D}$ iff $\mathcal{D} \cup \{D\}$ is also a request frame.

A *route* R for a request $D = (a, k)$ is a path from a to some vertex b in B_k. We also say R *realizes* D. A *state* of $\mathcal{N}$ is a set $\mathcal{R}$ of vertex disjoint routes. Each state of $\mathcal{N}$ realizes a request frame. A request frame $\mathcal{D}$ is *realizable* iff there is a network state realizing it.

We are interested in (WSNB, SNB, RNB) connectors under this request model. A *(rearrangeable)* $[w, f]$*-connector* is a $[w, f]$-network in which the request frame

$$\mathcal{D} = \{(a, \sigma(a)) \mid a \in A\}$$

is realizable for any mapping $\sigma : A \to [f]$ such that

$$|\{a \mid \sigma(a) = k\}| = w, \ \forall k \in [f].$$

A *strictly nonblocking (SNB)* $[w, f]$*-connector* is an $[w, f]$-network $\mathcal{N}$ in which given any network state $\mathcal{R}$ realizing a request set $\mathcal{D}$, and given a new request D compatible with $\mathcal{D}$, there exists a route R such that $\mathcal{R} \cup \{R\}$ realizes $\mathcal{D} \cup \{D\}$. As requests come and go, a strategy to pick new routes for new requests is called a *routing algorithm*. An $[w, f]$-network $\mathcal{N}$ is called a *widesense nonblocking (WSNB)* $[w, f]$*-connector* with respect to a routing algorithm $\mathbf{A}$ if $\mathbf{A}$ can always pick a new route for a new request compatible with the current network state. We can also replace $\mathbf{A}$ by a class of algorithms $\mathcal{A}$. In general, an $[w, f]$-network $\mathcal{N}$ is WSNB iff it is WSNB with respect to *some* algorithm.

The different $[w, f]$-networks are generalized versions of the corresponding n-networks:

PROPOSITION 4.1. *A network is an SNB, WSNB, RNB* $[1, f]$*-connector if and only if it is an SNB, WSNB, RNB* f*-connector, respectively.*

Let $\overline{rc}(w, f)$, $\overline{wc}(w, f)$, and $\overline{sc}(w, f)$ denote the minimum sizes of an RNB, WSNB, and SNB $[w, f]$-connector, respectively. Similarly, for a fixed depth k, we define $\overline{rc}(w, f, k)$, $\overline{wc}(w, f, k)$, and $\overline{sc}(w, f, k)$. These functions have not been studied before. Some trivial bounds can be summarized as follows:

PROPOSITION 4.2. *Let* $n = wf$*, then*

(i) $\overline{rc}(w, f) \le \overline{wc}(w, f) \le \overline{sc}(w, f)$

(ii) $\overline{rc}(w, f, k) \le \overline{wc}(w, f, k) \le \overline{sc}(w, f, k)$

(iii) a RNB, WSNB, SNB n*-connector is also a RNB, WSNB, SNB* $[w, f]$*-connector, respectively. Consequently,* $\overline{rc}(\cdot) \le rc(\cdot)$*,* $\overline{wc}(\cdot) \le wc(\cdot)$*, and* $\overline{sc}(\cdot) \le sc(\cdot)$*, where the dots on the left hand sides can be replaced by* (w, f) *or* (w, f, k)*, and the dots on the right hand sides by* (n) *or* (n, k)*, correspondingly.*

5. Complexity bounds.

5.1. Strictly nonblocking $[w, f]$-connectors.

We study SNB $[w, f]$-connectors in this section. For $f = 1$, it is easy to see that $\overline{sc}(w, 1, k) = w + k - 1$. We assume $f \ge 2$ from here on.

Intuitively, an optimal SNB $[w, f]$-connector might have (strictly) smaller size than an optimal SNB wf-connector, since the $(\lambda, F, \lambda', F')$-request model is more restrictive than the (λ, F, F')-request model. However, the following theorem shows a somewhat surprising result that we can do no better than an SNB wf-connector when $f \ge 2$. This theorem explains rigorously why the authors in [24] could not construct SNB designs under the (λ, F, F')-model with lower cost than the ones under the other model!

THEOREM 5.1. *Let* $n = wf$*, where* n, w, f *are positive integers, and* $f \ge 2$*. An* n*-network* $\mathcal{N} = (V, E; A, B)$ *is a strictly nonblocking* n*-connector iff it is a strictly nonblocking* $[w, f]$*-connector.*

Proof. An SNB n-connector is also an SNB $[w, f]$-connector, no matter how the fiber partitioning is done. For the converse, let $\mathcal{N}$ be an SNB $[w, f]$-connector. Let $B = B_1 \cup \cdots \cup B_f$ be the partition of B. (Recall, by definition, that $|B_i| = w, \forall i \in [f]$ and that $|A| = wf$.)

Consider a state $\mathcal{R}$ of this network. We shall show that if a is a free input and b is a free output, then there exists a route R from a to b such that $\mathcal{R} \cup \{R\}$ is a network state.

Let X be the set of free inputs and Y the set of free outputs. Note that $a \in X$, $b \in Y$, and $|X| = |Y|$.

Suppose $b \in B_k$, for some $k \in [f]$. Without loss of generality, we assume that there is no free output in any B_j for $j \ne k$. This can be accomplished by creating as many requests of the form (x, j) as possible, where $x \in X - \{a\}$ and $j \ne k$, until there is no more free outputs at the B_j with $j \ne k$. Then, let $\mathcal{R}$ be the new network state (which satisfies all new requests and also contains the old network state). An (a, b)-route compatible with $\mathcal{R}$ is certainly compatible with the old network state.

We can now assume $Y \subseteq B_k$. Create $|X|$ requests of the form (x, k), one for each $x \in X$. Since $\mathcal{N}$ is a $[w, f]$-connector, there is a route R_x for each x in X satisfying the following: (i) R_x starts from x and ends at some vertex in B_k, and (ii) $\mathcal{R} \cup \{R_x \mid x \in X\}$ is a network state.

If R_a is an (a, b)-route, then we are done. Moreover, if $|X| = |Y| = 1$, then R_a must be an (a, b)-route. Consequently, we can assume the following:

- $|X| = |Y| \ge 2$.
- R_a goes from a to some vertex $y \in B_k - \{b\}$.

- There is some $x \in X - \{a\}$ such that R_x ends at b.
- There is some vertex $a' \notin X$ and a route

$$R' = (a', v_1, \dots, v_p, b') \in \mathcal{R}$$

which goes from a' to a vertex $b' \in B_j$, where $j \neq k$. (The route $R' \in \mathcal{R}$ exists since we assumed that the vertices in $B_j, j \neq k$, are all busy.)

To this end, let

$$\mathcal{R}' = \mathcal{R} \cup \{R_t \mid t \in X\} - \{R_a, R_x, R'\}.$$

We shall show that there exists an (a, b)-route R for which $\mathcal{R}' \cup \{R', R\}$ is a state. The route R is then the route we are looking for, because $\mathcal{R} \subseteq \mathcal{R}' \cup \{R'\}$.

We first claim that there exists an (x, y)-route R_{xy} compatible with $\mathcal{R}'$. Consider the state $\mathcal{R}' \cup \{R_a\}$. The request (a', k) is valid (i.e. compatible with the request frame realized by $\mathcal{R}' \cup \{R_a\}$), and b is the only free output in B_k; hence, there is an (a', b)-route $R_{a'b}$ such that $\mathcal{R}' \cup \{R_a, R_{a'b}\}$ is a state. Now, in the state $\mathcal{R}' \cup \{R_{a'b}\}$ the request (x, k) is valid, and y is the only free output in B_k. Hence, there is an (x, y)-route R_{xy} such that $\mathcal{R}' \cup \{R_{a'b}, R_{xy}\}$ is a state. Consequently, there is an (x, y)-route R_{xy} compatible with $\mathcal{R}'$ as claimed.

Now, consider two cases as follows.

Case 1: among all the (x, y)-routes which are compatible with $\mathcal{R}'$ there is some R_{xy} which is vertex also disjoint from R'. Then, in the state $\mathcal{R}' \cup \{R', R_{xy}\}$ the request (a, k) is valid, and b is the only free output in B_k. Hence, there is an (a, b)-path compatible with $\mathcal{R}' \cup \{R'\}$ as desired.

Case 2: every (x, y)-route compatible with $\mathcal{R}'$ intersects R' at some point. Let R_{xy} be such an (x, y)-route whose last intersection vertex on $(v_1, \dots, v_p)$ has the largest index, say v_j, for some $j \in [p]$. Then, R_{xy} is composed of two parts: the part from x to v_j and the part from v_j to y.

Let $R_{a'y}$ be an (a', y)-path consisting of the part $(a', v_1, \dots, v_j)$ of R' and the (v_j, y)-part of R_{xy}. Then, certainly $R_{a'y}$ is compatible with $\mathcal{R}'$. In the state $\mathcal{R}' \cup \{R_{a'y}\}$ the request (a, k) is valid, and b is the only free output in B_k. Hence, there is an (a, b)-path R_{ab} compatible with $\mathcal{R}' \cup \{R_{a'y}\}$.

If R_{ab} does not intersect R', then we are done. Otherwise, R_{ab} must intersect R' at some $v_{j'}$ for which $j' > j$. Similar to the previous paragraph, we can form an (a', b)-path $R_{a'b}$ compatible with $\mathcal{R}'$ consisting of $(a', v_1, \dots, v_{j'})$ and the part of R_{ab} from $v_{j'}$ to b. Now, consider the state $\mathcal{R}' \cup \{R_{a'b}\}$ in which y is the only free vertex in B_k. The request (x, k) is valid, hence there is some (x, y)-path compatible with $\mathcal{R}' \cup \{R_{a'b}\}$. This (x, y)-path must then intersect R' (since we are in case 2) at some vertex after $v_{j'}$ (for compatibility with $R_{a'b}$), contradicting our choice of R_{xy} earlier. □

COROLLARY 5.2. *The following hold for $f \geq 2$:*

(i) $\overline{sc}(w, f, 1) = w^2 f^2$.
(ii) $\overline{sc}(w, f, k) = \Omega\left((wf)^{1+1/(k-1)}\right)$ *and* $\overline{sc}(w, f, k) = O((wf)^{1+1/\lfloor \frac{k+1}{2} \rfloor})$.
(iii) $\overline{sc}(w, f) = \Theta(wf \lg(wf))$.

Proof. Let $n = wf$, then $\overline{sc}(w, f, k) = sc(n, k)$ by Theorem 5.1. The first equality is obvious. The fact that $sc(n, k) = O(n^{1+1/\lfloor \frac{k+1}{2} \rfloor})$ can be seen from the constructions in [3,5,27]. The lower bound $\Omega\left((wf)^{1+1/(k-1)}\right)$ was shown in [8]. That $sc(n) = \Theta(n \lg n)$ can be found in [1,36]. The reader is referred to the surveys [23,29] for more details on what is known about these functions. □

5.2. Rearrangeable $[w, f]$-connectors. In this section, we first devise lower bounds for the optimal size of RNB $[w, f]$-connectors and connectors of a fixed depth. The upper bounds follow from explicit constructions presented in Section 6.

An idea of Pippenger [28] can be used to show the following theorem.

THEOREM 5.3. *Every rearrangeable $[w, f]$-connector must have size at least*

$$\frac{45}{7} wf \log_6 f + O(f) - O(f \lg w).$$

In particular, $\overline{rc}(w, f) = \Omega(wf \lg f)$.

Proof. The proof is completely similar to that of Pippenger's theorem and thus will not be repeated here. The only difference is that, the number of valid request frames is no longer $n!$ as in the case of an n-connector. In our case, the total number of different request frames for $\mathcal{N}$ is the multinomial coefficient

$$\binom{wf}{\underbrace{w, \dots, w}_{f \text{ times}}} = \frac{(wf)!}{(w!)^f} \geq \frac{\sqrt{2\pi wf}(wf/e)^{wf}}{e^{\frac{f}{12w}}(2\pi w)^{f/2}(w/e)^{wf}} = (2\pi w)^{1-f/2} e^{-\frac{f}{12w}} f^{wf+1/2}, \quad (5.1)$$

where the inequality follows from Stirling's approximation [35]. □

The bound $\Omega(wf \lg f)$ implies that for $w \leq f$, $[w, f]$-connectors must have size at least $\Omega(wf \lg(wf))$, which is asymptotically no better than a wf-connector. This confirms our intuition that for small values of w, $[w, f]$-connectors are almost the same as wf-connectors.

Fortunately, in WDM networks it is often the case that $w \geq f$, i.e. the number of wavelengths per fiber (in the hundreds) is often much larger than the number of fiber (in the tens). The next section shows that we can construct $[w, f]$-connectors that are asymptotically less expansive than all known constructions of wf-connectors.

We next give lower bounds for fixed depth $[w, f]$-connectors.

THEOREM 5.4. *The optimal size of a depth-*1 $[w, f]$*-connector is* $wf(wf - w + 1)$*, namely* $\overline{rc}(w, f, 1) = wf(wf - w + 1)$.

Proof. In the next section, we shall construct depth-1 $[w, f]$-connectors of size $wf(wf - w + 1)$, which proves the upper bound $\overline{rc}(w, f, 1) \leq wf(wf - w + 1)$.

For the lower bound, let $\mathcal{N} = (A \cup B, E; A, B)$ be a depth-1 $[w, f]$-connector. Then, $\mathcal{N}$ is a (directed) bipartite graph where $|A| = wf$, $|B| = wf$, and B has a partition into $B_1 \cup \dots \cup B_f$, such that $|B_i| = w, \forall i$. The network $\mathcal{N}$ is a $[w, f]$-connector iff for every partition of A into $A_1, \dots, A_f$ with $|A_i| = w, \forall i$, there exist f complete matchings from each A_i to each B_i.

It follows that each vertex $b \in B$ must have a neighbor in every w-subset of A. Consequently, each vertex $b \in B$ must be of degree at least $|A| - w + 1$. Hence, the number of edges of $\mathcal{N}$ is at least $|B|(|A| - w + 1) = wf(wf - w + 1)$. □

For $k \geq 2$, we can use an idea by by Pippenger and Yao [30] on n-shifters to find a lower bound for depth-k $[w, f]$-connectors. The proof is similar and left as an exercise.

THEOREM 5.5. *Let $k \geq 2$ be an integer, a depth-k $[w, f]$-connector must have size at least $kwf^{1+1/k}$. Specifically,* $\overline{rc}(w, f, k) = \Omega(kwf^{1+1/k})$. Noting that the function $kwf^{1+1/k}$ is minimized at $k = \ln f$, we get the result $\overline{rc}(w, f) = \Omega(wf \lg f)$ from the previous theorem (with a worse constant than $45/7$):

COROLLARY 5.6. *For $k \geq 2$, $\overline{rc}_2(w, f) \geq ewf \ln f$, where e is the base of the natural log.*

6. Explicit Constructions. For any network $\mathcal{N}$, let $A(\mathcal{N})$ and $B(\mathcal{N})$ denote the set of inputs and outputs of $\mathcal{N}$, respectively. For any $[w, f]$-network $\mathcal{N}$, we shall always use $B_1(\mathcal{N}), \dots, B_f(\mathcal{N})$ to denote the partition of $B(\mathcal{N})$. An important fact to notice is that all presumably "theoretic" constructions presented in this section can easily be converted to practical constructions. We shall not elaborate on this point due to space limitation.

6.1. Atomic networks. Let $\mathcal{B}(x, y) = (A \cup B; E)$ denote the complete $x \times y$ directed bipartite graph, i.e. $|A| = x, |B| = y$, and $E = A \times B$. The (x, y)-network $\mathcal{B}(x, y)$ is called an *(x, y)-crossbar*. When $x = y$, we use the shorter notation $\mathcal{B}(x)$, and call it the *x-crossbar*. For any positive integer m, let $\mathcal{M}(m) = (A \cup B; E)$ denote a perfect matching of size m from A into B. (Therefore, $|A| = |B| = m$.) An *(n, m)-concentrator* is an (n, m)-network where $n \geq m$, such that for any subset S of m inputs there exists a set of m vertex disjoint paths connecting S to the outputs.

6.2. Union-networks and optimal depth-1 connectors. Let $\mathcal{N}_1, \dots, \mathcal{N}_f$ be (wf, w)-networks, with input sets $A_1, \dots, A_f$, and output sets $B_1, \dots, B_f$, respectively. For each $i = 1, \dots, f-1$, let $\phi_i : A_i \to A_{i+1}$ be some one-to-one mapping. A *left union* or *$\triangleleft$-union* of $\mathcal{N}_1, \dots, \mathcal{N}_f$ is a $[w, f]$-network $\mathcal{N}$ constructed by identifying each vertex $a \in A_1$ with all vertices $\phi_1(a), \phi_2 \circ \phi_1(a), \dots, \phi_{f-1} \circ \cdots \circ \phi_1(a)$ (to become an input of $\mathcal{N}$), and let $B_1, \dots, B_f$ be, naturally, the partition of the outputs of $\mathcal{N}$ (see Figure 6.1.) We denote $\mathcal{N}$ as $\mathcal{N} = \triangleleft(\mathcal{N}_1, \dots, \mathcal{N}_f)$.

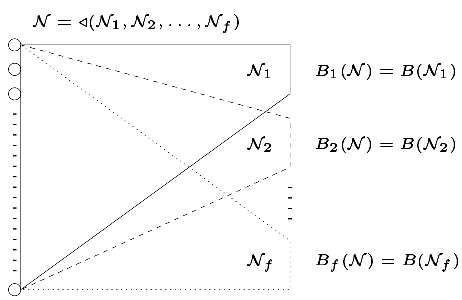


FIG. 6.1. *The left union $\mathcal{N}$ of f (wf, w)-networks is a $[w, f]$-network.*

Let $\mathcal{N}_1, \dots, \mathcal{N}_k$ be (m, n)-networks. An (mk, n)-network $\mathcal{N} = \triangleright(\mathcal{N}_1, \dots, \mathcal{N}_k)$ constructed by identifying outputs of the $\mathcal{N}_i$ in some one to one manner is called a *right union* (or *$\triangleright$-union*) of the $\mathcal{N}_i$. The picture is virtually symmetrical to the left union picture.

The next theorem summarizes a few important properties of the union constructions. The proof is simple and thus omitted. Note that part (iii) completes the proof of Theorem 5.4.

THEOREM 6.1 (Optimal depth-1 construction). *Let w, f be positive integers, then the following hold*

(i) Suppose $\mathcal{N}_1, \dots, \mathcal{N}_f$ are (wf, w)-concentrators, then the network $\triangleleft(\mathcal{N}_1, \dots, \mathcal{N}_f)$ is a $[w, f]$-connector.

(ii) The network $\mathcal{C}_1(w, f) = \triangleright(\mathcal{B}(wf - w, w), \mathcal{M}(w))$, is a depth-1 (wf, w)-concentrator of size $w(wf - w + 1)$.

(iii) Let $\mathcal{S}_1(w, f)$ be a left union of f copies of $\mathcal{C}_1(w, f)$. Then, $\mathcal{S}_1(w, f)$ is a depth-1 $[w, f]$-connector of size $wf(wf - w + 1)$, which is optimal!

6.3. Constructions of product networks and $[w, f]$-connectors of depth two. DEFINITION 6.2 (The $\times\times$-product). *Let $\mathcal{N}_1$ be an m-network, and $\mathcal{N}_2$ be a $[w, f]$-network, define the* ordered product *(for lack of better term) $\mathcal{N} = \mathcal{N}_1 \times \times \mathcal{N}_2$ as follows. We shall "connect"*

wf copies of $\mathcal{N}_1$, denoted by $\mathcal{N}_1^{(1)}, \dots, \mathcal{N}_1^{(wf)}$, to m copies $\mathcal{N}_2^{(1)}, \dots, \mathcal{N}_2^{(m)}$ of $\mathcal{N}_2$. For each $i \in \{1, \dots, wf\}$ and $j \in \{1, \dots, m\}$, we identify the jth output of $\mathcal{N}_1^{(i)}$ with the ith input of $\mathcal{N}_2^{(j)}$. The output partition for $\mathcal{N}$ is defined by

$$B_k(\mathcal{N}) = \bigcup_{j=1}^{m} B_k(\mathcal{N}_2^{(j)}).$$

Naturally, $A(\mathcal{N}) = \cup_{i=1}^{wf} A(\mathcal{N}_1^{(i)})$. Figure 6.2 illustrates the construction.

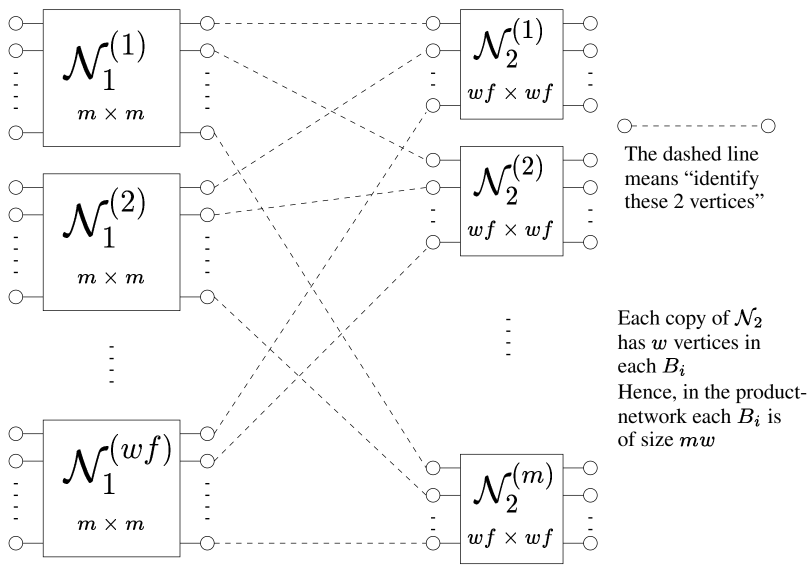


FIG. 6.2. *Product of two networks: $\mathcal{N}_1$ is an m-network and $\mathcal{N}_2$ is a $[w, f]$-network.*

The following proposition summarizes a few trivial properties of the product network.

PROPOSITION 6.3. *Let $\mathcal{N}_1$ be an m-network of size s_1 and depth d_1, and $\mathcal{N}_2$ a $[w, f]$-network of size s_2 and depth d_2. Then, the network $\mathcal{N} = \mathcal{N}_1 \times \times \mathcal{N}_2$ is an $[mw, f]$-network of size $s = wfs_1 + ms_2$, and depth $d = d_1 + d_2$.* Before proving a crucial property of this construction, we need a simple yet important lemma.

LEMMA 6.4. *Let $G = (X \cup Y; E)$ be a bipartite multi-graph where the degree of each vertex $x \in X$ is m and the degree of each vertex $y \in Y$ is mw. Then, there is an edge coloring for G with exactly m colors such that vertices in X are incident to different colors, and vertices in Y are incident to exactly w edges of each color.*

Proof. Split each vertex $y \in Y$ into w copies $y^{(1)}, \dots, y^{(w)}$ such that each copy has degree m. The resulting graph is an m-regular bipartite graph, which can be m-edge-colored, by König's line coloring theorem [16]. This induces a coloring of G as desired. □The following lemma is the point of the ordered-product construction.

LEMMA 6.5. *If $\mathcal{N}_2$ is a rearrangeable $[w, f]$-connector and $\mathcal{N}_1$ is a rearrangeable m-connector, then $\mathcal{N} = \mathcal{N}_1 \times \times \mathcal{N}_2$ is a rearrangeable $[mw, f]$-connector.*

Proof. Consider a request frame $\mathcal{D}$ for $\mathcal{N}$. We use $(a^{(i)}, k)$ to denote a request $(a, k) \in \mathcal{D}$ if $a \in A(\mathcal{N}_1^{(i)})$. This is to signify the fact that the request was from the ath input of $\mathcal{N}_1^{(i)}$ to B_k. By definition of a request frame, $|\{(a^{(i)}, k) \mid (a^{(i)}, k) \in \mathcal{D}\}| = m$ for a fixed i. We shall find vertex disjoint routes realizing requests in $\mathcal{D}$.

Construct a bipartite graph $G = (X \cup Y; E)$ where $X = \{\mathcal{N}_1^{(1)}, \dots, \mathcal{N}_1^{(wf)}\}$ is the set of all copies of $\mathcal{N}_1$, and $Y = \{B_1, \dots, B_f\}$. There is (a copy of) an edge of G between $\mathcal{N}_1^{(i)}$ and B_k for each request $(a^{(i)}, k)$ Clearly G is a bipartite graph satisfying the conditions of Lemma 6.4.

As each edge of G represents a request $D \in \mathcal{D}$, Lemma 6.4 implies that there is an m-coloring of all the requests such that, for a fixed i, requests of the form $(a^{(i)}, k)$ get different colors. Moreover, for a fixed k, requests of the form $(a^{(i)}, k)$ can be partitioned into m classes, where each class consists of exactly w requests of the same color.

Let $C = \{1, \dots, m\}$ be the set of colors. Let $c(a, k)$ denote the color of request $(a, k) \in \mathcal{D}$. Without loss of generality, we number the m outputs of $\mathcal{N}_1^{(i)}$ with numbers from 1 to m, i.e. $B(\mathcal{N}_1^{(i)}) = C$, for all $i \in \{1, \dots, wf\}$.

Fix an $i \in \{1, \dots, wf\}$. As the m requests coming out of $\mathcal{N}_1^{(i)}$ have different colors, the correspondence $a^{(i)} \leftrightarrow c(a^{(i)}, k)$, where $(a^{(i)}, k) \in \mathcal{D}$, is a one-to-one correspondence between the inputs and the outputs of $\mathcal{N}_1^{(i)}$. Hence, for the m requests $(a^{(i)}, k)$, there exist m vertex disjoint routes $R_1(a^{(i)}, k)$ connecting input $a^{(i)}$ to the output numbered $c(a^{(i)}, k)$ of $\mathcal{N}_1^{(i)}$.

Fix a $j \in \{1, \dots, m\}$. The ith input of $\mathcal{N}_2^{(j)}$ is the jth output of $\mathcal{N}_1^{(i)}$, which is the end point of some route $R_1(a^{(i)}, k)$ for which $c(a^{(i)}, k) = j$. Let $k(i, j)$ be the number k such that the request $(a^{(i)}, k) \in \mathcal{D}$ has color $c(a^{(i)}, k) = j$. Then, for the fixed j and any $k \in \{1, \dots, f\}$, Lemma 6.4 ensures that there are exactly w of the $k(i, j)$ with value k, namely $|\{i \,:\, k(i, j) = k\}| = w$. Thus, $\mathcal{D}' = \{(i, k(i, j)) \mid 1 \le i \le wf\}$ is a valid request frame for the rearrangeable $[w, f]$-connector $\mathcal{N}_2^{(j)}$. Consequently, we can find vertex disjoint routes $R_2(i, k(i, j))$ connecting input i to some output in $B_{k(i,j)}$ of $\mathcal{N}_2^{(j)}$.

The concatenation of $R_1(a^{(i)}, k)$ and $R_2(i, k)$ completes a route realizing request $(a^{(i)}, k)$. These routes are vertex disjoint as desired. □

We now illustrate the use of the Lemma 6.5 by a simple construction of depth-2 $[w, f]$-connectors.

THEOREM 6.6 (Depth-2 constructions). *Let w, f be positive integers, then*
- *(i) for $w \le f - 1$, we can construct depth-2 $[w, f]$-connectors of size $wf(w + f)$.*
- *(ii) for $w \ge f$, we can construct depth-2 $[w, f]$-connectors of size $wf(2\sqrt{w(f-1)} + 1)$.*

Proof. We ignore the issue of integrality for the sake of a clean presentation.

Write $w = mx$. By Theorem 6.1, $\mathcal{S}_1(x, f)$ is an $[x, f]$-connector of depth 1 and size $xf(xf - x + 1)$. By Proposition 6.3 and Lemma 6.5, the network $\mathcal{B}(m) \times \times \mathcal{S}_1(x, f)$ is a $[w, f]$-connector of depth-2 and size

$$s(x) = xfm^2 + mxf(xf - x + 1) = wf(w/x + (f - 1)x + 1).$$

Minimizing $s(x)$ as a function of x, with $1 \le x \le w$, we get the desired results. We pick $x = 1$ in case (i) and $x = \sqrt{w/(f-1)}$ in case (ii). □

6.4. Recursive constructions. Toward the constructions of $[w, f]$-connectors, we need a few more definitions and properties.

DEFINITION 6.7. *Let $w_0, w_1, \dots, w_k$ and f be positive integers and G be any $[w_0, f]$-network. Let $\mathcal{N}(w_k, \dots, w_1; G)$ denote the recursively constructed network defined as follows.*

$$\mathcal{N}(\cdot; G) = G$$
$$\mathcal{N}(w_k, \dots, w_1; G) = \mathcal{B}(w_k) \times \times \mathcal{N}(w_{k-1}, \dots, w_1; G).$$

LEMMA 6.8. *Given positive integers $w_0, w_1, \dots, w_k$ and f. Let $w = \prod_{i=0}^k w_i$, and G be any $[w_0, f]$-connector of size $s(G)$ and depth $d(G)$. Then, the network $\mathcal{N} = \mathcal{N}(w_k, \dots, w_1; G)$ is a $[w, f]$-connector of size*

$$s(\mathcal{N}) = w_0 \dots w_k f \cdot (w_1 + \cdots + w_k) + w_1 \dots w_k \cdot s(G).$$

and depth $d(\mathcal{N}) = (k + d(G))$. (We set $s(\mathcal{N}) = s(G)$ when $k = 0$.)

Proof. This follows from Proposition 6.3 and Lemma 6.5 □

PROPOSITION 6.9. *For any positive integers w and f, the following hold*

(i) Let $\mathcal{N}$ be any wf-connector. The $[w, f]$-network $\mathcal{N}'$ obtained by partitioning the outputs of $\mathcal{N}$ arbitrarily into f subsets of size w is a $[w, f]$-connector.

(ii) $\mathcal{B}(f)$ is an f-connector and also a $[1, f]$-connector.

Basically, Proposition 6.9 implies that one can use good $w_0 f$-networks to serve as the network G in Lemma 6.8. A general $[w, f]$-network can then be constructed by decomposing $w = w_0 \dots w_k$ with the right set of divisors $w_0, \dots, w_k$. As $[w, f]$-networks of depth 2 have been constructed, we shall attempt to construct good networks of general depth and networks of a fixed depth at least 3.

Pippenger [27] has constructed a rearrangeable n-network, which we shall call $\mathcal{P}(n)$, of size $6n \log_3 n + O(n)$. He also constructed rearrangeable n-networks of depth $2i + 1$, $i \geq 1$, and size $2(i + 1)n \left(\frac{n}{2}\right)^{1/(i+1)} + O(n)$. An n-connector of depth $2i + 2$ can be constructed by concatenating an n-matching with a depth-$(2i + 1)$ n-connector. Hence, we can construct an n-connector of depth $j \geq 3$ and size $2\lceil j/2 \rceil n \left(\frac{n}{2}\right)^{1/\lceil j/2 \rceil} + O(n)$. We denote this network by $\mathcal{P}_j(n)$. For $j = 2$, [7] a construction of size $O(n^{5/3})$ was given in [7] and [14]. Abusing notation, we shall also use $\mathcal{P}_2(n)$ to denote an n-connector of depth-2 and size $O(n^{5/3})$.

In the following results, we ignore the issue of integrality for the sake of clarity. We first address the general depth case.

THEOREM 6.10. *We can construct rearrangeable $[w, f]$-connectors of size*

$$e \cdot wf \ln w + \frac{6}{\ln 3} wf \ln f + O(fw).$$

Proof. Let $w = xw_1 \dots w_k$. By Lemma 6.8 the network

$$\mathcal{N} = \mathcal{N}(w_k, \dots, w_1; \mathcal{P}(xf))$$

is a $[w, f]$-connector of size

$$\begin{aligned} s(\mathcal{N}) &= wf(w_1 + \cdots + w_k) + 6fx \log_3(fx) + O(fx) \\ &\geq wf \cdot k \cdot (\frac{w}{x})^{1/k} + 6fx \log_3(fx) + O(fx). \end{aligned}$$

The right hand side is minimized at $x = 1$ and $k = \ln w$. Equality can be obtained by setting $w_i = w^{1/k}, \forall i$. □

We now consider the fixed depth case. The networks $\mathcal{P}_j(n)$ are to be used. The following three theorems apply Lemma 6.8 with $G = \mathcal{P}_1, \mathcal{P}_2$, or $\mathcal{P}_j$ with $j \geq 3$. Depending on the relative values between f, w and k, one theorem may be better than the others.

THEOREM 6.11. *Let w, f, and $k \geq 3$ be positive integers.*

(i) If $w < (f - 1)^{k-1}$, then we can construct a $[w, f]$-connector of depth k and size

$$(k - 1) f w^{1 + \frac{1}{k-1}} + wf^2 = O(kwf^2). \qquad (6.1)$$

(ii) If $w \geq (f-1)^{k-1}$, then we can construct a $[w,f]$-connector of depth k and size

$$wf(k-1)[w(f-1)]^{\frac{1}{k}} + w^{1+\frac{1}{k}} f(f-1)^{\frac{1}{k}} + wf\Theta\left(k(wf)^{1+\frac{1}{k}}\right). \tag{6.2}$$

Proof. Write $w = xw_1 \dots w_{k-1}$. Then, Lemma 6.8 implies that

$$\mathcal{N}(w_{k-1},\dots,w_1;\mathcal{S}_1(x,f))$$

is a $[w,f]$-network of depth k and size

$$\begin{aligned} s &= wf(w_1+\cdots+w_{k-1}) + w_1\dots w_{k-1}\cdot xf(xf-x+1) \\ &\geq wf(k-1)(w/x)^{1/(k-1)} + wf(x(f-1)+1). \end{aligned} \tag{6.3}$$

Minimizing the right hand side with respect to x and we get the desired results.

In case (i), equality can be obtained when $w_i = w^{1/(k-1)}, \forall i$, and $x=1$. In case (ii), equality can be obtained when $w_i = w^{1/(k-1)}, \forall i$, and $x = \left(\frac{w}{(f-1)^{k-1}}\right)^{1/k}$. □We omit the proofs of the next two theorems due to the similarity to the above proof.

THEOREM 6.12. *Let w, f, and $k \geq 3$ be positive integers. Then, there are positive real constants c_1, c_2 and c_3 such that*

(i) if $w < c_1 f^{\frac{2}{3}(k-2)}$, then we can construct a $[w,f]$-connector of depth k and size

$$(k-2)f\cdot w^{1+\frac{1}{k-2}} + c_2 wf^{5/3} = O\left(kwf^{5/3}\right). \tag{6.4}$$

(ii) If $w \geq c_1 f^{\frac{2}{3}(k-2)}$, then we can construct a $[w,f]$-connector of depth k and size

$$(k-2)\cdot(wf)^{1+\frac{1}{k-3/2}} + c_3 w^{1+\frac{1}{k-3/2}} f^{\frac{6k-7}{6k-9}} = O\left(k(wf)^{1+\frac{1}{k-3/2}}\right). \tag{6.5}$$

The following result can be improved by finer analysis. We give a somewhat "cleaner" version.

THEOREM 6.13. *Let w, f, and $k \geq 4$ be positive integers.*

(i) If $w < f$, then we can construct a $[w,f]$-connector of depth k and size

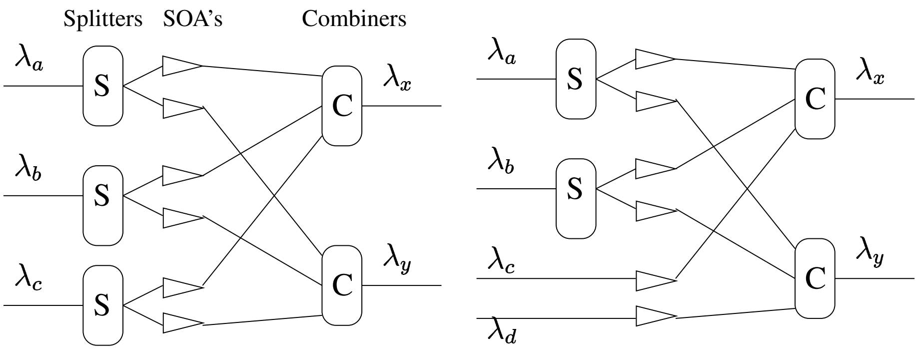
$$O\left(kw^{1+\frac{1}{(k+1)}} f^{1+\frac{2}{(k+1)}}\right). \tag{6.6}$$

(ii) If $w \geq f$, then we can construct a $[w,f]$-connector of depth k and size

$$O\left(k(wf)^{1+\frac{3}{2(k+1)}}\right). \tag{6.7}$$

7. Applications of our framework. In this section, we outline several practical applications coming from our theoretical formulation presented earlier. We will present only representative results. The reader should be able to see the main line of thoughts, nevertheless.

The applications fall into two main categories: **(a)** explicit constructions of WXCs, and **(b)** complexity comparisons of known constructions.



(a) A realization of $\mathcal{B}(3, 2)$ (b) A realization of $\mathcal{C}_1(2, 2)$

FIG. 7.1. *Sample realizations of atomic networks.*

7.1. Explicit constructions of WXCs. The ideas for explicit constructions come from the physical realizations of atomic networks (Section 6.1), the union of networks (Section 6.2), the $\times\times$-product of networks, and the recursive construction (Section 6.4).

There are several ways to physically realize an (x, y)-crossbar $\mathcal{B}(x, y)$. Figure 7.1(a) shows one possibility and it is self-explanatory. The one thing to notice is that each fiber is aimed to carry one wavelength only. Another possibility to realize $\mathcal{B}(x, y)$ is to use a combination of of an AWGR of dimension $\max\{x, y\}$ and the same number of LWC's, as was done in [24]. The advantage of using AWGR over SOA is that AWGR consume virtually no power.

Our second atomic component is a depth-1 (wf, w)-concentrator $\mathcal{C}_1(w, f)$, which can be constructed by taking the left-union of a $\mathcal{B}(wf - w, w)$ and a perfect matching $\mathcal{M}(w)$ as illustrated in Figure 7.1(b).

Given the aforementioned two atomic networks, we readily have a construction of a one-stage rearrangeably non-blocking WXC as shown in Figure 7.2. There is one column of tunable input – fixed output LWC's at the end to ensure no wavelength conflict. This one column of LWC's is needed in all realizations of the theoretical constructions shown in Section 6.

This construction has cost a little higher than that of Theorem 6.1 because of the LWCs, which are of total cost $w^2 f$. Asymptotically, however, $w^2 f \ll wf(wf - w + 1)$, hence we did not create too much of a gap between the theoretical construction and the physical realization. Another interesting point to notice is that the WXC RNB-1 construction of [24] is a special case of this idea. The only difference is that they use AWGR's and LWC's to realize the crossbars.

The same idea idea can be used to realize the generic product network of Lemma 6.5. The proof of the lemma gives also an efficient routing algorithm. (Efficient algorithms for bipartite graph edge-coloring can be found in [6, 9, 10].) Thus, we readily have constructions of a depth-2 RNB WXC as in Theorem 6.6, and several different recursive constructions as reported in Theorems 6.10, 6.11, 6.12, and 6.13. Depending on the relationship between w and f, we pick the best one to use.

7.2. Complexity comparisons of known constructions. Table 7.1 summarizes the costs of various recent constructions (including some of the ones in this paper). The costs are assessed in terms of architecture depths and sizes. From the table, we see the following:

- For the $(\lambda, F, \lambda', F')$-request model (SNB-1, RNB-1), the various constructions have costs asymptotically the same as those of their circuit switching counterparts.

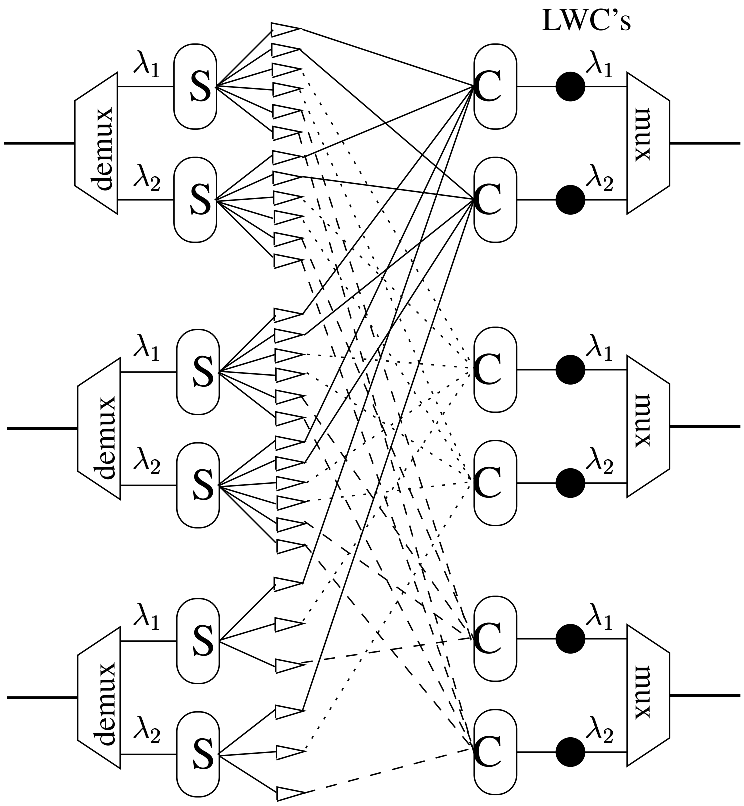


FIG. 7.2. *A realization of the one-stage construction $\mathcal{S}_1(2,3)$. The LWC's are wavelength converters with tunable inputs and a fixed output.*

This was expected, since our formulation in Section 4.1 has indicated that the WXC under this request model and the corresponding circuit switches are equivalent topologically.
(This is not to say that there is nothing to study in this request model. Our cost model does not capture precisely more practical criteria such as cross-talks, attenuation, and wavelength conversion costs.)

- For the (λ, F, F')-request model (SNB-1, RNB-1), there is much room for improvement.
  In the SNB case, our construction of Corollary 5.2 is already better then the existing construction CBC. However, since SNB in this request model is the same as the other, we cannot expect much more improvement.
  In the RNB case, our constructions of Theorems 6.10, 6.11, 6.12, and 6.13 are better than all known constructions. However, there are still gaps between the constructions and theoretical lower bounds of Theorems 5.3 and 5.5. We expect both the lower bounds and the constructions can be improved until they are asymptotically equal.

8. Conclusions and Future Works. There are several benefits of the proposed graph models: they help analyze the switches qualitatively and quantitatively, they can be used to compare switch complexity, they give rise to interesting mathematical problems relating to many areas such as classical switching theory, graph theory, algebraic graph theory. Some

TABLE 7.1

Cost comparisons of different constructions under the two request models. SNB/RNB-1 refers to the (λ, F, F')-request model. SNB/RNB-2 refers to the other request model.

	Type	depth	size	cond
[41]/CBC	SNB-1	$2\lg f$	$\Theta(wf(w+\lg f)\lg f)$	-
Corollary 5.2(ii)	SNB-1	k	$O((wf)^{1+1/\lfloor \frac{k+1}{2} \rfloor})$	-
Corollary 5.2(iii)	SNB-1	$\lg(wf)$	$\Theta(wf\lg(wf))$	-
[41]/WI-Cantor	RNB-1	$2\lg f$	$\Theta(wf(w+\lg^2 f))$	-
[41]/WI-Beneš	RNB-1	$2\lg f$	$\Theta(wf(w+\lg f))$	-
[24]/WXC-RNB-1	RNB-1	2	$2(wf)^{3/2}$	$f < w$
Theorem 6.6(i)	RNB-1	2	$wf(w+f)$	$f > w$
Theorem 6.6(ii)	RNB-1	2	$2(wf)^{3/2}$	$f \le w$
Theorem 6.10	RNB-1	$\lg(wf)$	$ewf\ln w + \frac{6}{\ln 3}wf\ln f + O(wf)$	-
Theorem 6.11	RNB-1	$k \ge 3$	$(k-1)fw^{1+\frac{1}{k-1}} + wf^2$	$w < (f-1)^{k-1}$
Theorem 6.11	RNB-1	$k \ge 3$	$\Theta\left(k(wf)^{1+\frac{1}{k}}\right)$	$w \ge (f-1)^{k-1}$
Theorem 6.12	RNB-1	$k \ge 3$	$(k-2)f \cdot w^{1+\frac{1}{k-2}} + c_2wf^{5/3}$	$w < c_1 f^{\frac{2}{3}(k-2)}$
Theorem 6.12	RNB-1	$k \ge 3$	$O\left(k(wf)^{1+\frac{1}{k-3/2}}\right)$	$w \ge c_1 f^{\frac{2}{3}(k-2)}$
Theorem 6.13	RNB-1	$k \ge 4$	$O\left(kw^{1+\frac{1}{(k+1)}} f^{1+\frac{2}{(k+1)}}\right)$	$w < f$
Theorem 6.13	RNB-1	$k \ge 4$	$O\left(k(wf)^{1+\frac{3}{2(k+1)}}\right)$	$w < f$
[33, 34]	SNB-2	3	$2w^2(2f+w)$	-
[24]/WXC-SNB-2	SNB-2	3	$4\sqrt{2}(wf)^{3/2}$	$f < w$
[42]/2S/P/N	SNB-2	3	$4(wf)^{3/2}$	$f < w$
[42]/3S/P/N	SNB-2	4	$4\sqrt{2}(wf)^{3/2}$	-
[24]WXC-RNB-2	RNB-2	3	$2(wf)^{3/2}$	$f < w$
[42]/2S/P/R	RNB-2	3	$2(wf)^{3/2}$	$f < w$
[42]/3S/P/R	RNB-2	4	$2\sqrt{2}(wf)^{3/2}$	-
[42]/B/P/R	RNB-2	$\Theta(\lg(wf))$	$\Theta(wf\lg wf)$	$f < w$

of these points were not discussed in the paper. It would be interesting, for instance, to investigate the use of expanders for constructing $[w, f]$-connectors.

We have addressed several important problems arising from this framework, including studying optimal networks and their constructions, the trade-off between network depth and size, and the equivalence of networks under different request models. Some practical applications have also been pointed out.

Many problems remain open. In particular, we have not touched upon the wide-sense nonblocking case much. The multicast switch complexity was not considered. The asymptotic bounds of various complexity functions are still not optimal.

REFERENCES

[1] L. A. BASSALYGO AND M. S. PINSKER, *The complexity of an optimal non-blocking commutation scheme without reorganization*, Problemy Peredači Informacii, 9 (1973), pp. 84–87.

[2] V. E. BENEŠ, *Mathematical theory of connecting networks and telephone traffic*, Academic Press, New York, 1965. Mathematics in Science and Engineering, Vol. 17.

[3] D. G. CANTOR, *On non-blocking switching networks*, Networks, 1 (1971/72), pp. 367–377.

[4] FENG CAO AND DING-ZHU DU, *Fault-tolerant routing and multicasting in butterfly networks*, in SAC '99: Proceedings of the 1999 ACM symposium on Applied computing, ACM Press, 1999, pp. 455–460.

[5] C. CLOS, *A study of non-blocking switching networks*, Bell System Tech. J., 32 (1953), pp. 406–424.

[6] RICHARD COLE AND JOHN HOPCROFT, *On edge coloring bipartite graphs*, SIAM J. Comput., 11 (1982), pp. 540–546.

[7] PAUL FELDMAN, JOEL FRIEDMAN, AND NICHOLAS PIPPENGER, *Wide-sense nonblocking networks*, SIAM J. Discrete Math., 1 (1988), pp. 158–173.

[8] J. FRIEDMAN, *A lower bound on strictly nonblocking networks*, Combinatorica, 8 (1988), pp. 185–188.

[9] HAROLD N. GABOW, *Using Euler partitions to edge color bipartite multigraphs*, Internat. J. Comput. Information Sci., 5 (1976), pp. 345–355.

[10] HAROLD N. GABOW AND ODED KARIV, *Algorithms for edge coloring bipartite graphs and multigraphs*, SIAM J. Comput., 11 (1982), pp. 117–129.

[11] H. HINTON, *An Introduction to Photonic Switching Fabrics*, Plenum Press, New York, NY, 1993.

[12] J. H. HUI, *Switching and traffic theory for integrated broadband networks*, Kluwer Academic Publishers, Boston/Dordrecht/London, 1990.

[13] FRANK K. HWANG, *The mathematical theory of nonblocking switching networks*, World Scientific Publishing Co. Inc., River Edge, NJ, 1998.

[14] F. K. HWANG AND G. W. RICHARDS, *A two-stage network with dual partial concentrators*, Networks, 23 (1993), pp. 53–58.

[15] XIAOHONG JIANG, HONG SHEN, MD. MAMUN UR RASHID KHANDKER, AND SUSUMU HORIGUCHI, *Blocking behaviors of crosstalk-free optical banyan networks on vertical stacking*, IEEE/ACM Transactions on Networking, 11 (2003), pp. 982–993.

[16] D. KÖNIG, *Über graphen und ihre anwendung auf determinantentheorie und mengenlehre*, Math. Ann., 77 (1916), pp. 453–465.

[17] LUCENT TECHNOLOGIES PRESS RELEASE, *Lucent Technologies unveils untra-high-capacity optical system; Time Warner Telecom first to announce it will deploy the system*, 2001. http://www.lucent.com/press/0101/010117.nsa.html.

[18] ———, *Lucent Technologies engineer and scientists set new fiber optic transmission record*, 2002. http://www.lucent.com/press/0302/020322.bla.html.

[19] LUCENT TECHNOLOGIES WEBSITE, *What is dense wave division multiplexing (DWDM)*, 2002. http://www.bell-labs.com/technology/lightwave/dwdm.html.

[20] G. MAIER AND A. PATTAVINA, *Design of photonic rearrangeable networks with zero first-order switching-element-crosstalk*, IEEE Trans. Comm., 49 (2001), pp. 1248–1279.

[21] B. MUKHERJEE, *Optical Communication Networks*, McGraw-Hill, New York, NY, 1997.

[22] HUNG Q. NGO, *Multiwavelength distribution networks*, in Proceedings of the 2004 Workshop on High Performance Switching and Routing (HPSR 2004, Phoenix, Arizona), IEEE, 2004, pp. 186–190.

[23] HUNG Q. NGO AND DING-ZHU DU, *Notes on the complexity of switching networks*, in Advances in Switching Networks, D.-Z. Du and Hung Q. Ngo, eds., Kluwer Academic Publishers, 2001, pp. 307–367.

[24] HUNG Q. NGO, DAZHEN PAN, AND CHUNMING QIAO, *Nonblocking WDM switches based on arrayed waveguide grating and limited wavelength conversion*, in Proceedings of the 23rd Conference of the IEEE Communications Society (INFOCOM'2004, HongKong), IEEE, 2004.

[25] HUNG Q. NGO, DAZHEN PAN, AND YUANYUAN YANG, *Optical switching networks with minimum number of limited range wavelength converters*, in Proceedings of the 24rd Conference of the IEEE Communications Society (INFOCOM'2005, Miami, U.S.A.), IEEE, 2005.

[26] DAZHEN PAN, VISHAL ANAND, AND HUNG Q. NGO, *Cost-effective constructions for nonblocking wdm multicast switching networks*, in Proceedings of the 2004 International Conference on Communications (ICC'2004, Paris, France), IEEE, 2004.

[27] NICHOLAS PIPPENGER, *On rearrangeable and nonblocking switching networks*, J. Comput. System Sci., 17 (1978), pp. 145–162.

[28] ———, *A new lower bound for the number of switches in rearrangeable networks*, SIAM J. Algebraic Discrete Methods, 1 (1980), pp. 164–167.

[29] ———, *Communication networks*, in Handbook of theoretical computer science, Vol. A, Elsevier, Amsterdam, 1990, pp. 805–833.

[30] NICHOLAS PIPPENGER AND ANDREW C. C. YAO, *Rearrangeable networks with limited depth*, SIAM J. Algebraic Discrete Methods, 3 (1982), pp. 411–417.

[31] JEYASHANKHER RAMAMIRTHAM AND JONATHAN S. TURNER, *Design of wavelength converting switches for optical burst switching*, in Proceedings of the 21st Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM), vol. 2, IEEE, 2002, pp. 1162–1171.

[32] RAJIV RAMASWAMI AND KUMAR SIVARAJAN, *Optical Networks: A Practical Perspective (Second Edition)*, Morgan Kaufmann Publishers, San Francisco, CA, 2001.

[33] APRIL RASALA AND GORDON WILFONG, *Strictly non-blocking WDM cross-connects*, in Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'2000, San Francisco, CA),

New York, 2000, ACM, pp. 606–615.

[34] ———, *Strictly non-blocking WDM cross-connects for heterogeneous networks*, in Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing (STOC'2000, Portland, OR), New York, 2000, ACM, pp. 513–524.

[35] HERBERT ROBBINS, *A remark on Stirling's formula*, Amer. Math. Monthly, 62 (1955), pp. 26–29.

[36] C. E. SHANNON, *Memory requirements in a telephone exchange*, Bell System Tech. J., 29 (1950), pp. 343–349.

[37] AMIT SINGHAL AND RAJ JAIN, *Terabit switching: a survey of techniques and current products*, Computer communications, 25 (2002), pp. 547–556.

[38] THOMAS E. STERN AND KIRSHNA BALA, *Multiwavelength Optical Networks: A Layered Approach*, Prentice Hall PTR, Upper Saddle River, NJ, 1999.

[39] M.M. VAEZ AND C.-T. LEA, *Strictly nonblocking directional-coupler-based switching networks under crosstalk constraint*, IEEE Trans. Comm., 48 (2000), pp. 316–323.

[40] DOUGLAS B. WEST, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

[41] G. WILFONG, B. MIKKELSEN, C. DOERR, AND M. ZIRNGIBL, *WDM cross-connect architectures with reduced complexity*, Journal of Lightwave Technology, 17 (1999), pp. 1732–1741.

[42] Y. YANG AND J. WANG, *Designing WDM optical interconnects with full connectivity by using limited wavelength conversion*, IEEE Trans. Comput., 53 (2004), pp. 1547–1556.

[43] YUANYUAN YANG, JIANCHAO WANG, AND CHUNMING QIAO, *Nonblocking WDM multicast switching networks*, IEEE Trans. Para. and Dist. Sys., 11 (2000), pp. 1274–1287.