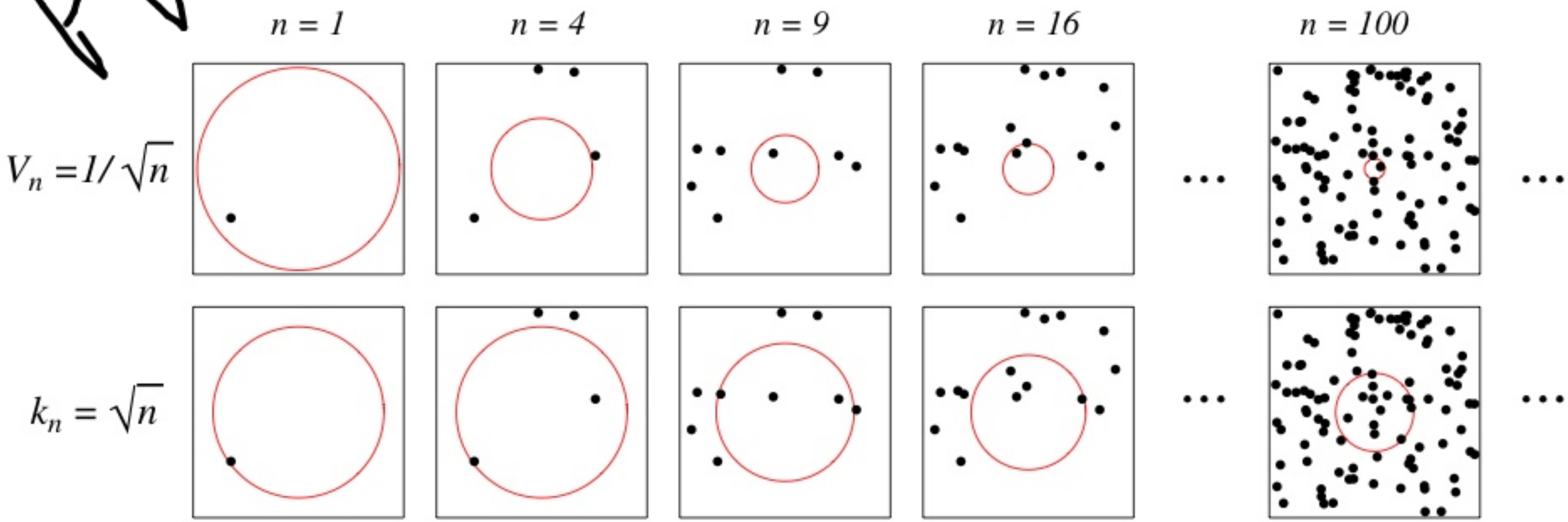
PW




KNN

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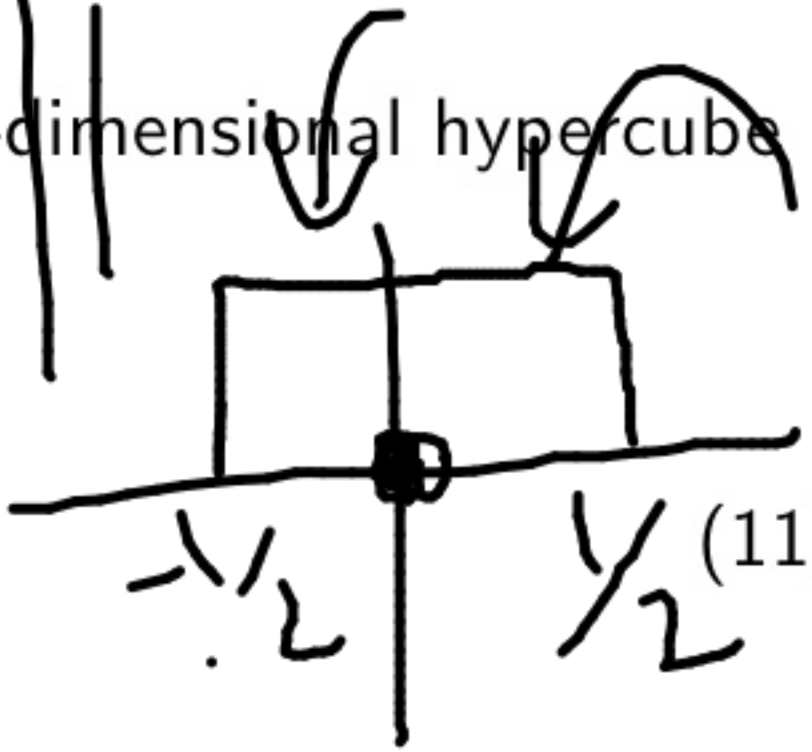


- We can derive an analytic expression for $k_n$:
  - Define a **windowing function**:

$$\varphi(\mathbf{u}) = \begin{cases} 1 & |u_j| \leq 1/2 \qquad j = 1, \ldots, d \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

  - This windowing function $\varphi$ defines a unit hypercube centered at the origin.
  - Hence, $\varphi((\mathbf{x} - \mathbf{x}_i)/h_n)$ is equal to unity if $\mathbf{x}_i$ falls within the hypercube of volume $V_n$ centered at $\mathbf{x}$, and is zero otherwise.

- The number of samples in this hypercube is therefore given by

$$k_n = \sum_{i=1}^{n} \varphi\left(\frac{\mathbf{x} - \mathbf{x}_i}{h_n}\right) \quad . \tag{13}$$

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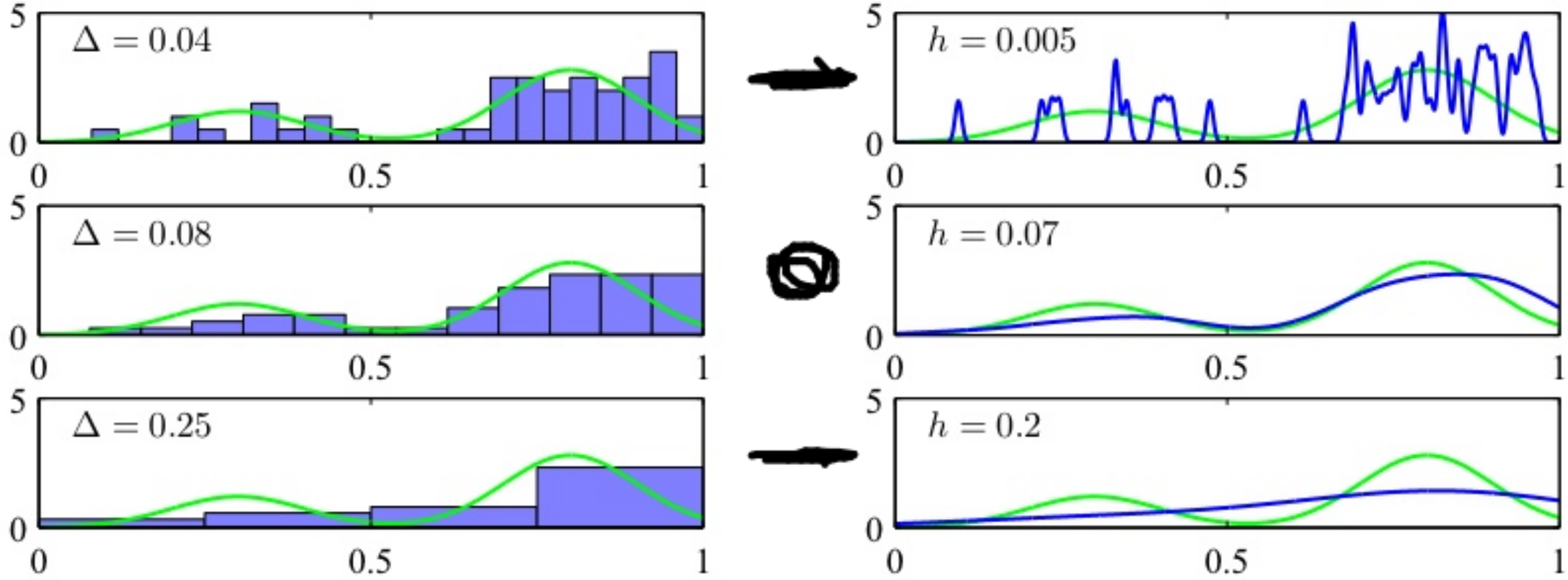
- Hence, the windowing function $\varphi$, in this context called a **Parzen window**, tells us how to **weight** all of the samples in $\mathcal{D}$ to determine $p(\mathbf{x})$ at a particular $\mathbf{x}$.

# Example



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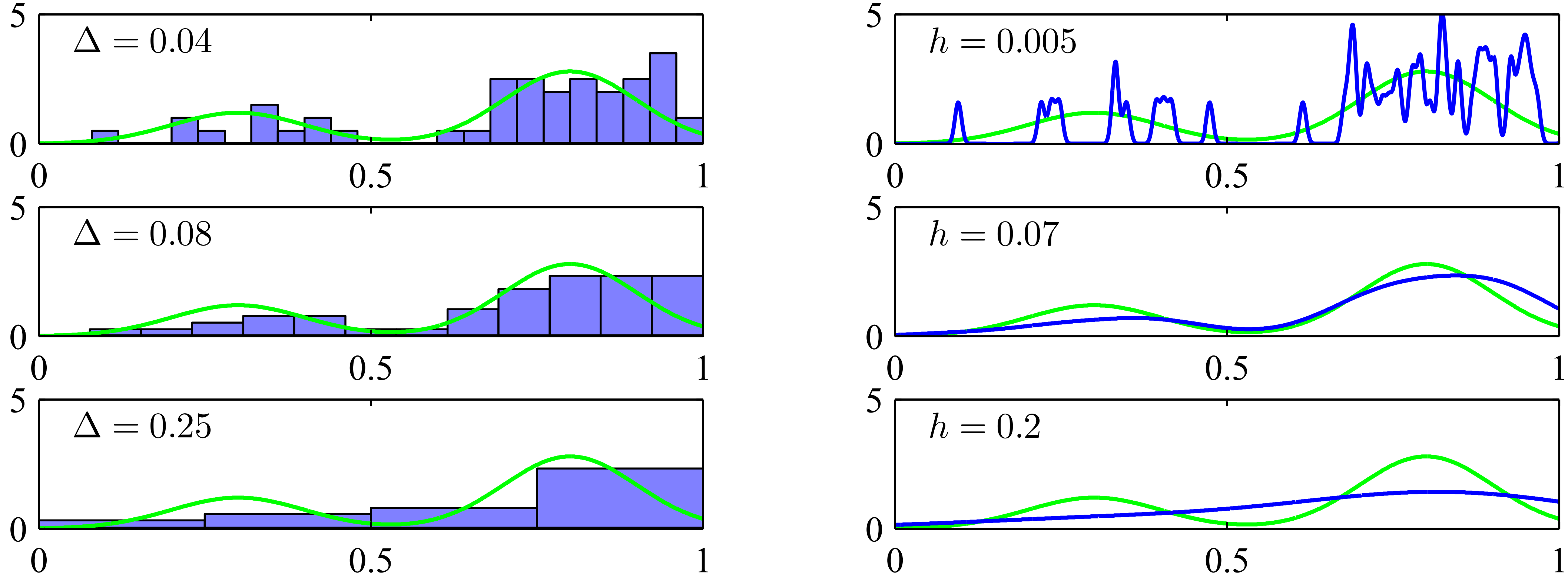


- But, what undesirable traits from histograms are inherited by Parzen window density estimates of the form we've just defined?

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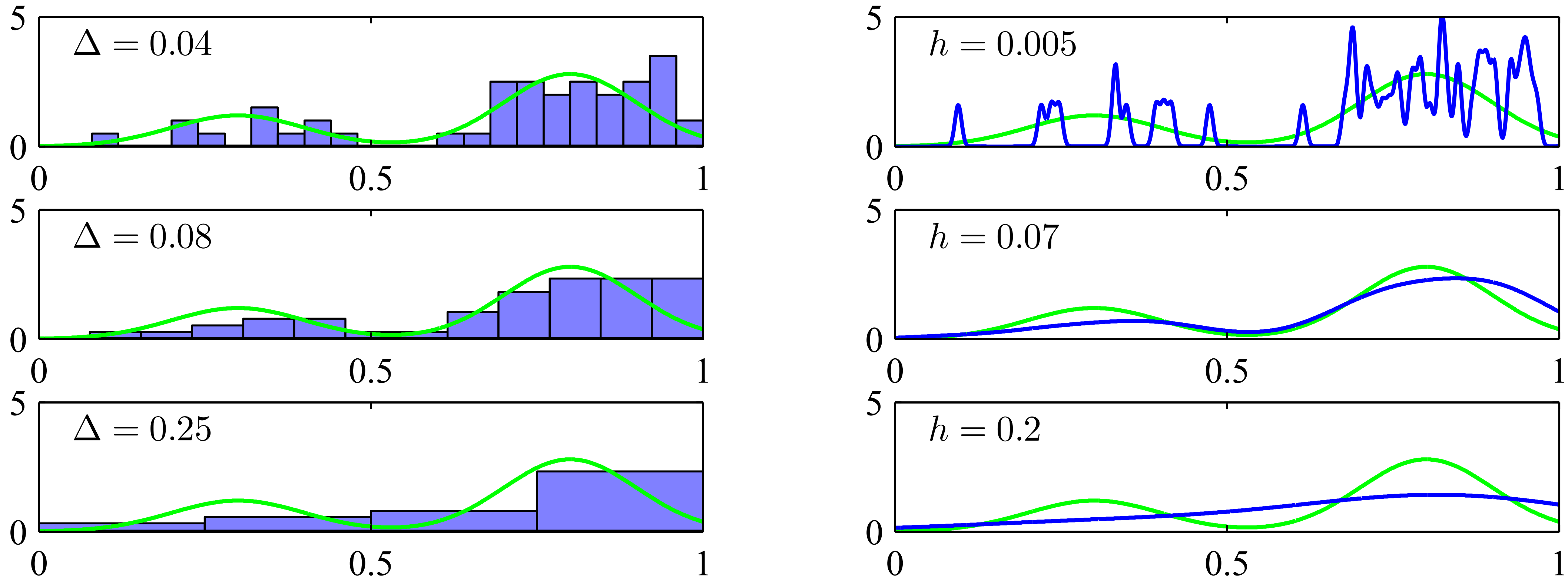


- But, what undesirable traits from histograms are inherited by Parzen window density estimates of the form we've just defined?
- Discontinuities...
- Dependence on the bandwidth.

# Generalizing the Kernel Function

- What if we allow a more general class of windowing functions rather than the hypercube?
- If we think of the windowing function as an interpolator, rather than considering the window function about $\mathbf{x}$ only, we can visualize it as a kernel sitting on each data sample $\mathbf{x}_i$ in $\mathcal{D}$.

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- What if we allow a more general class of windowing functions rather than the hypercube?
- If we think of the windowing function as an interpolator, rather than considering the window function about $\mathbf{x}$ only, we can visualize it as a kernel sitting on each data sample $\mathbf{x}_i$ in $\mathcal{D}$.
- And, if we require the following two conditions on the kernel function $\varphi$, then we can be assured that the resulting density $p_n(\mathbf{x})$ will be proper: non-negative and integrate to 1.

$$\varphi(\mathbf{x}) \geq 0 \tag{15}$$

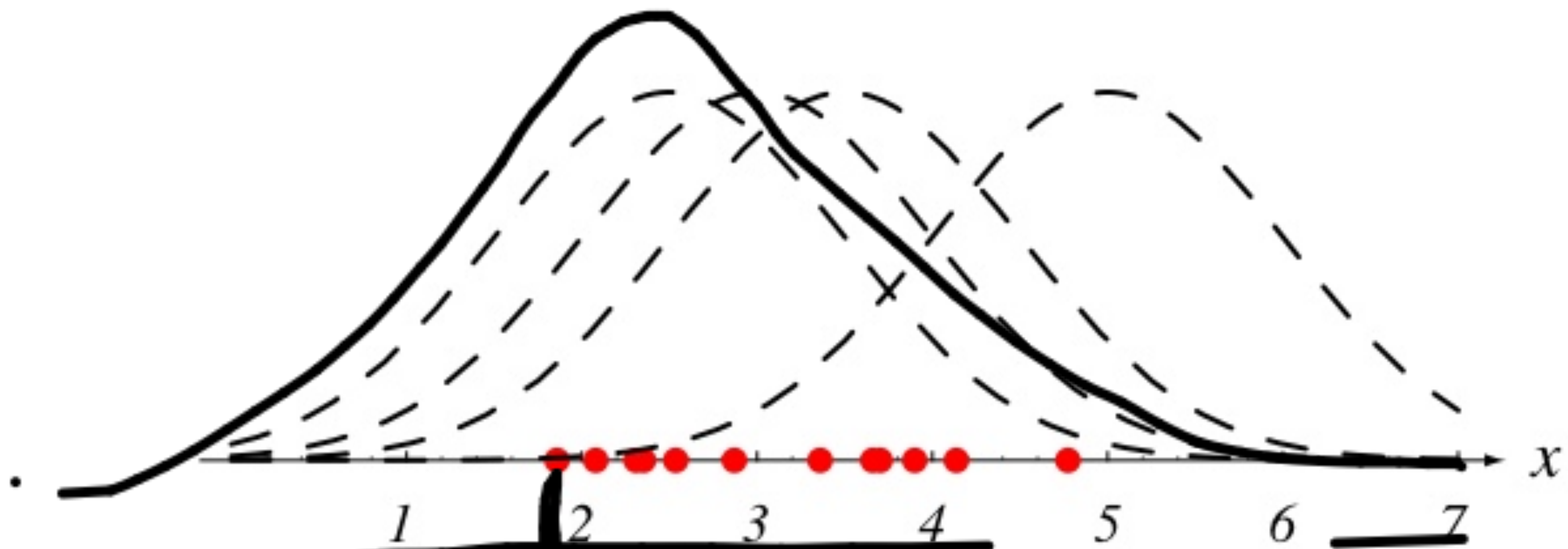
$$\int \varphi(\mathbf{u}) d\mathbf{u} = 1 \tag{16}$$

- For our previous case of $V_n = h_n^d$, then it follows $p_n(\mathbf{x})$ will also satisfy these conditions.

# Example: A Univariate Guassian Kernel

- A popular choice of the kernel is the Gaussian kernel:

$$\varphi_h(u) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}u^2\right] \tag{17}$$




- The resulting density is given by:

$$p(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_n\sqrt{2\pi}} \exp\left[-\frac{1}{2h_n^2}(\mathbf{x}-\mathbf{x}_i)^2\right] \tag{18}$$

- It will give us smoother estimates without the discontinuites from the hypercube kernel.

# Effect of the Window Width

## Slide I

- An important question is what effect does the window width $h_n$ have on $p_n(\mathbf{x})$?
- Define $\delta_n(\mathbf{x})$ as

$$\delta_n(\mathbf{x}) = \frac{1}{V_n}\varphi\left(\frac{\mathbf{x}}{h_n}\right) \tag{19}$$

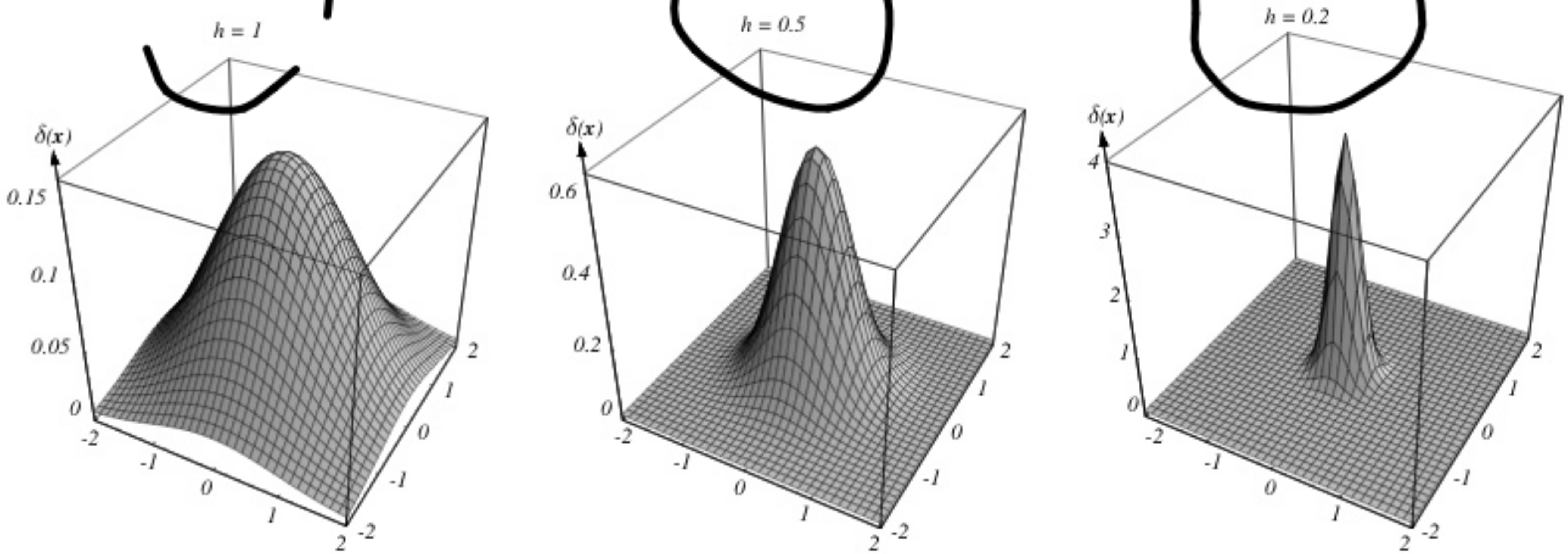
and rewrite $p_n(\mathbf{x})$ as the average

$$p_n(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\delta_n(\mathbf{x}-\mathbf{x}_i) \tag{20}$$

# Effect of the Window Width

## Slide II

- $h_n$ clearly affects both the amplitude and the width of $\delta_n(\mathbf{x})$.



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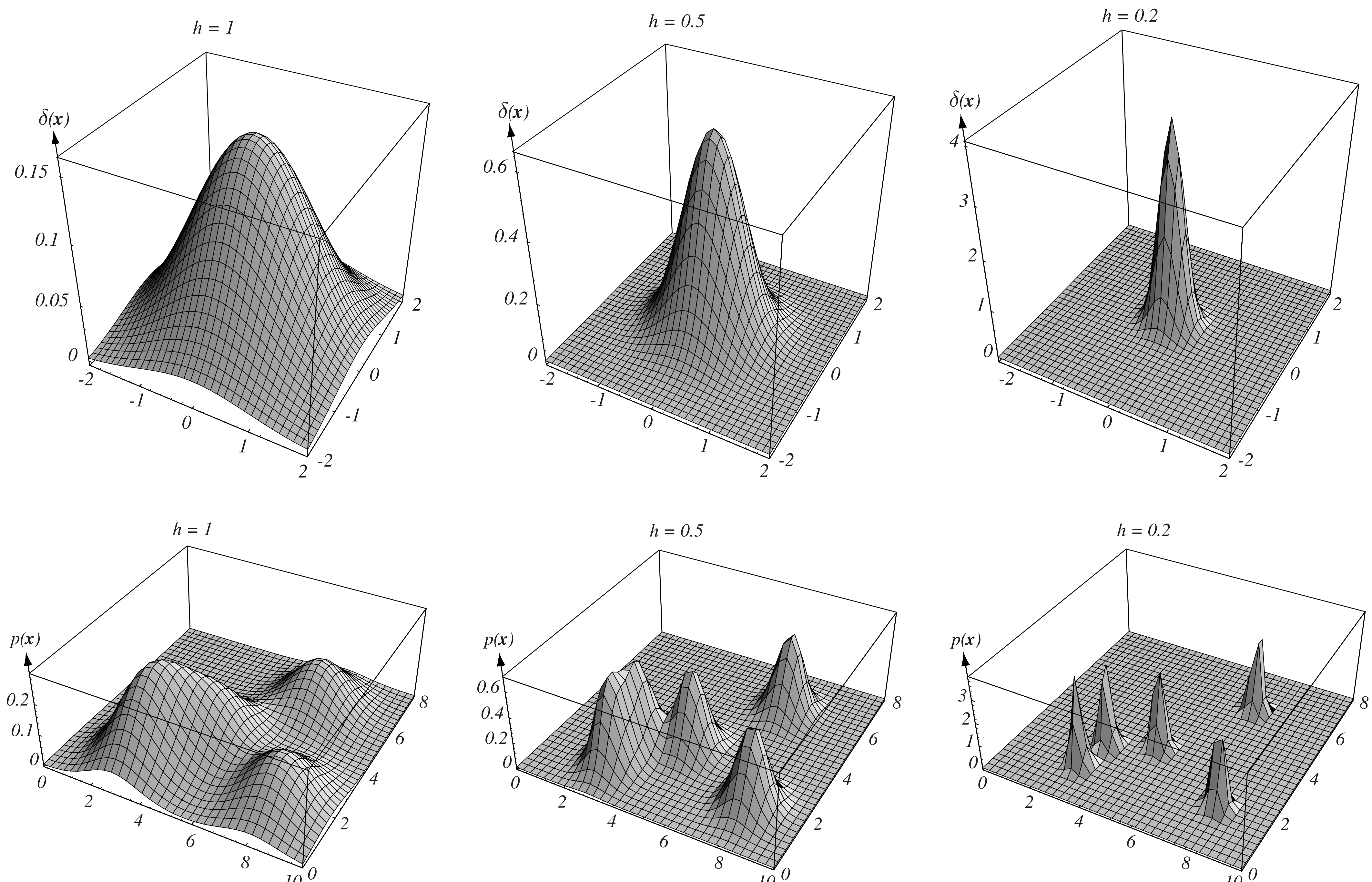

# Effect of Window Width (And, hence, Volume $V_n$)

- But, for any value of $h_n$, the distribution is normalized:

$$\int \delta(\mathbf{x} - \mathbf{x}_i) d\mathbf{x} = \int \frac{1}{V_n} \varphi\left(\frac{\mathbf{x} - \mathbf{x}_i}{h_n}\right) d\mathbf{x} = \int \varphi(\mathbf{u}) d\mathbf{u} = 1 \tag{21}$$

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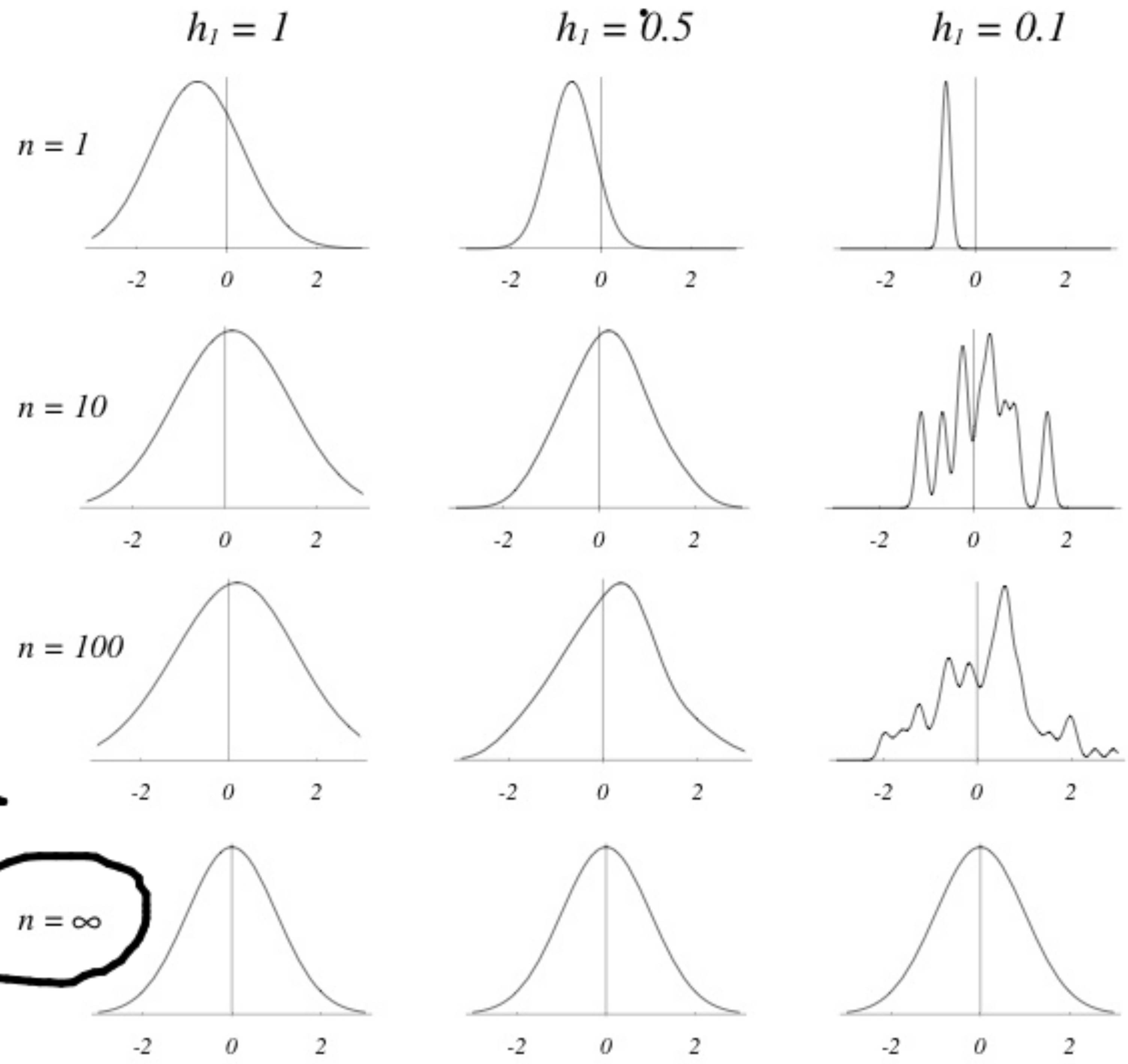
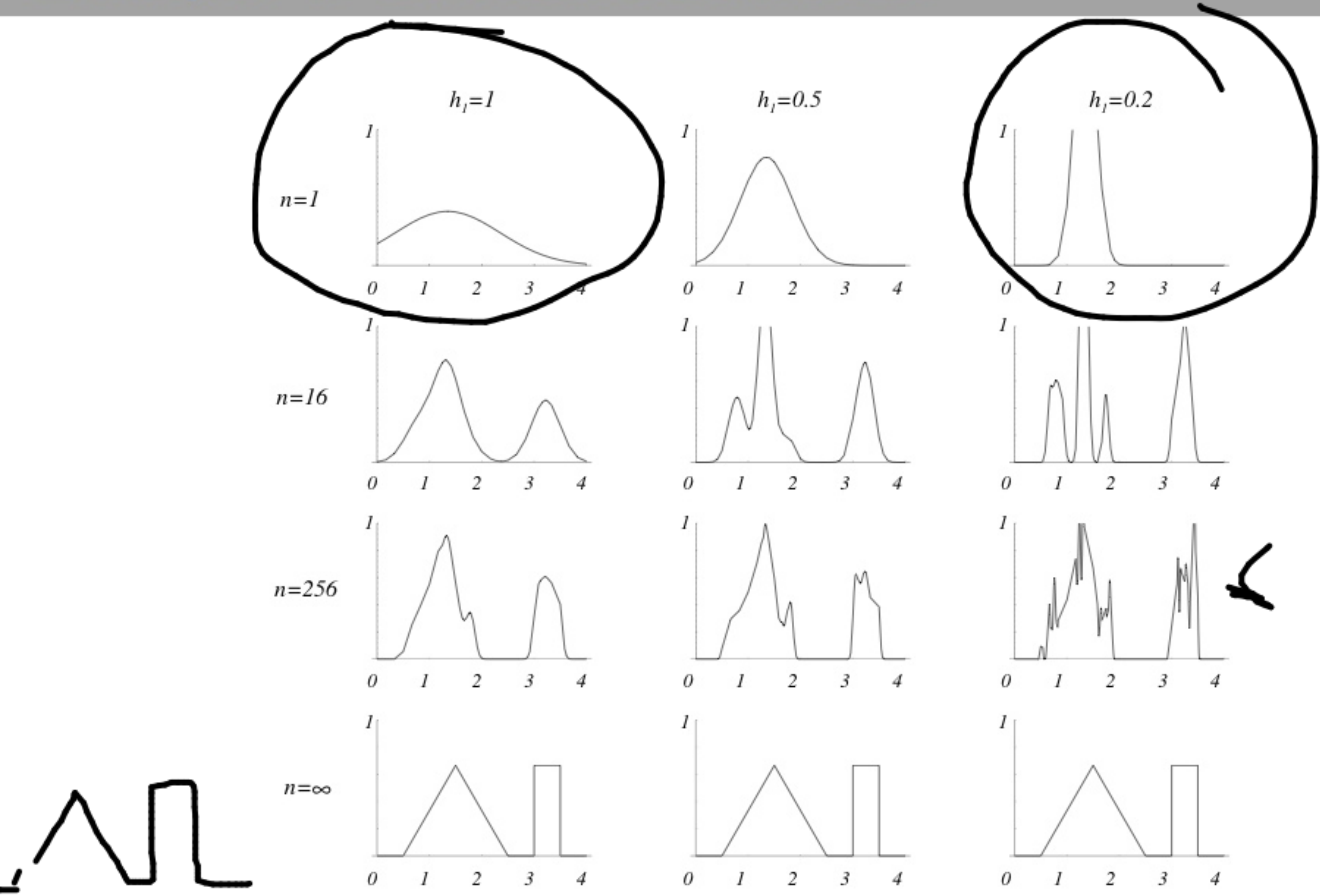
- If $V_n$ is too large, the estimate will suffer from too little resolution.
- If $V_n$ is too small, the estimate will suffer from too much variability.
- In theory (with an unlimited number of samples), we can let $V_n$ slowly approach zero as $n$ increases and then $p_n(\mathbf{x})$ will converge to the unknown $p(\mathbf{x})$. But, in practice, we can, at best, seek some compromise.

# Example: Revisiting the Univariate Guassian Kernel



# Example: A Bimodal Distribution



# Parzen Window-Based Classifiers

- Estimate the densities for each category.
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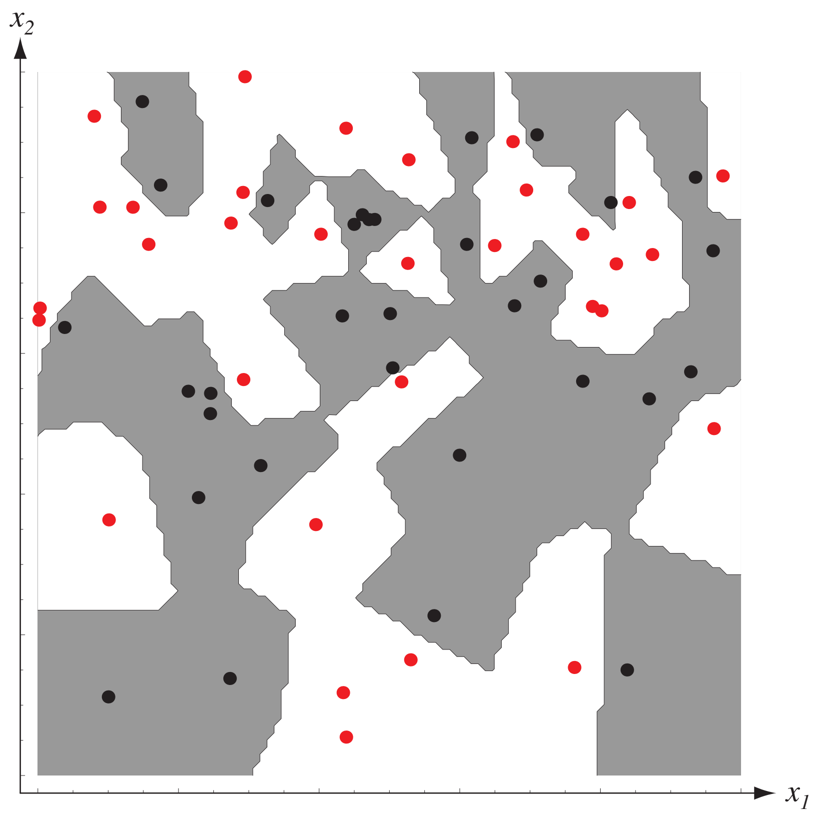
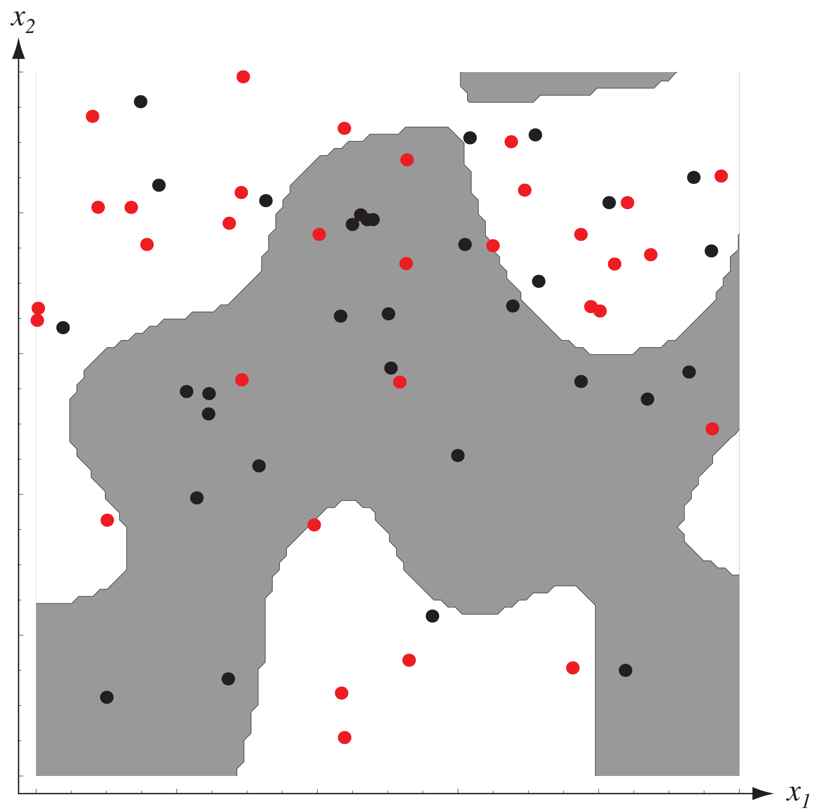
- As you guessed it, the **decision regions for a Parzen window-based classifier depend upon the kernel function**.



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- One possibility is to use cross-validation. Break up the data into a training set and a validation set. Then, perform training on the training set with varying bandwidths. Select the bandwidth that minimizes the error on the validation set.
- There is little theoretical justification for choosing one window width over another.

# $k_n$ Nearest Neighbor Methods

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- The basic idea here is to center our window around $\mathbf{x}$ and let it grow until it captures $k_n$ samples, where $k_n$ is a function of $n$.
  - These samples are the $k_n$ nearest neighbors of $\mathbf{x}$.
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  - In either case, we estimate $p_n(\mathbf{x})$ according to

$$p_n(\mathbf{x}) = \frac{k_n}{nV_n} \tag{22}$$

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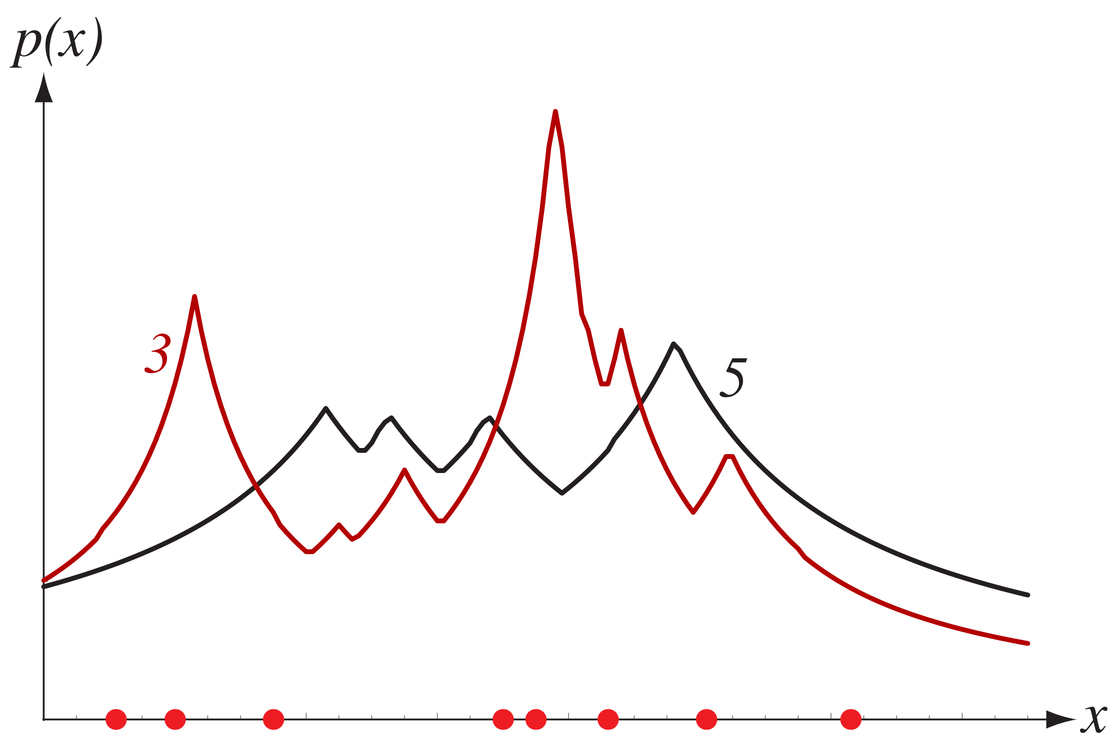
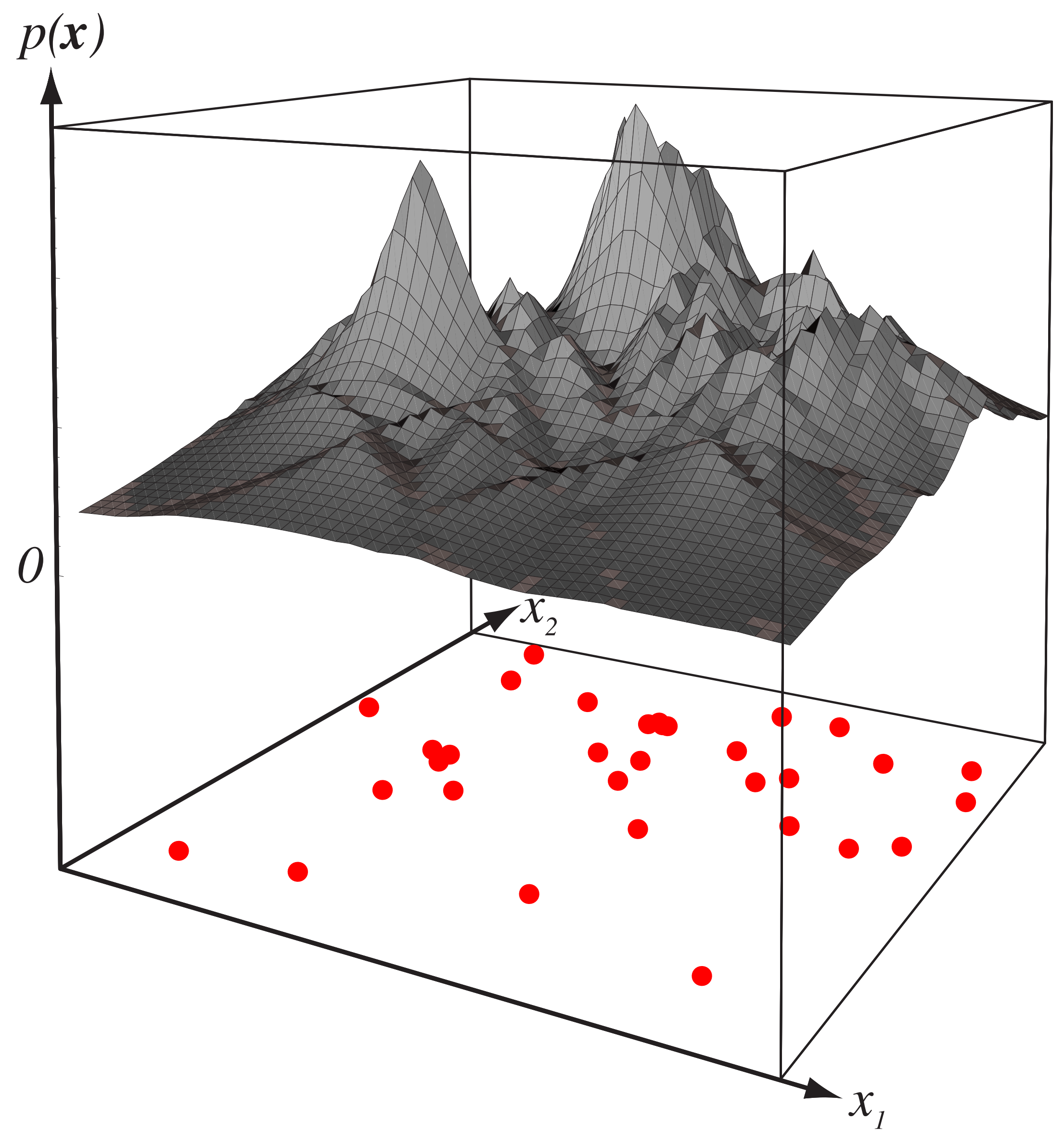
- Recall these conditions from the earlier discussion; these will ensure that $p_n(\mathbf{x})$ converges to $p(\mathbf{x})$ as $n$ approaches infinity.

# Examples of $k_n$-NN Estimation

- Notice the discontinuities in the slopes of the estimate.



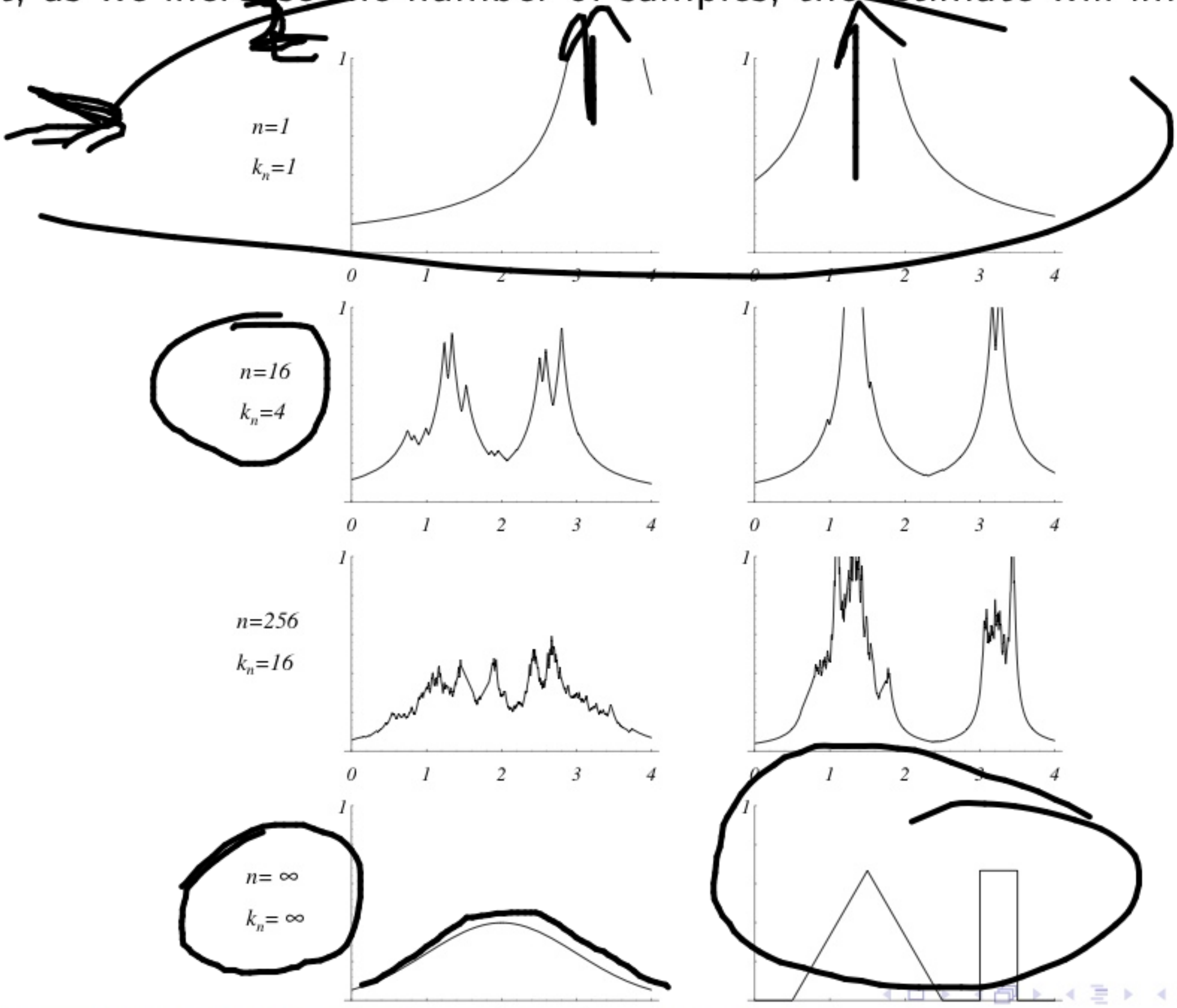



# $k$-NN Estimation From 1 Sample

- We don't expect the density estimate from 1 sample to be very good, but in the case of $k$-NN it will diverge!
- With $n = 1$ and $k_n = \sqrt{n} = 1$, the estimate for $p_n(x)$ is

$$p_n(x) = \frac{1}{2|x - x_1|} \tag{23}$$

But, as we increase the number of samples, the estimate will improve.



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- But, like the Parzen window size, one choice is as good as another absent any additional information.
- Similarly, in classification scenarios, we can base our judgement on classification error.

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- The estimate for the joint probability is thus

$$p_n(\mathbf{x}, \omega_i) = \frac{k_i}{nV} \tag{24}$$

$$p_n(x) = \sum_{\omega_i} p(x, \omega_i)$$

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- A reasonable estimate for the posterior is thus

$$P_n(\omega_i|\mathbf{x}) = \frac{p_n(\mathbf{x}, \omega_i)}{\sum_c p_n(\mathbf{x}, \omega_c)} = \frac{k_i}{k} \tag{25}$$

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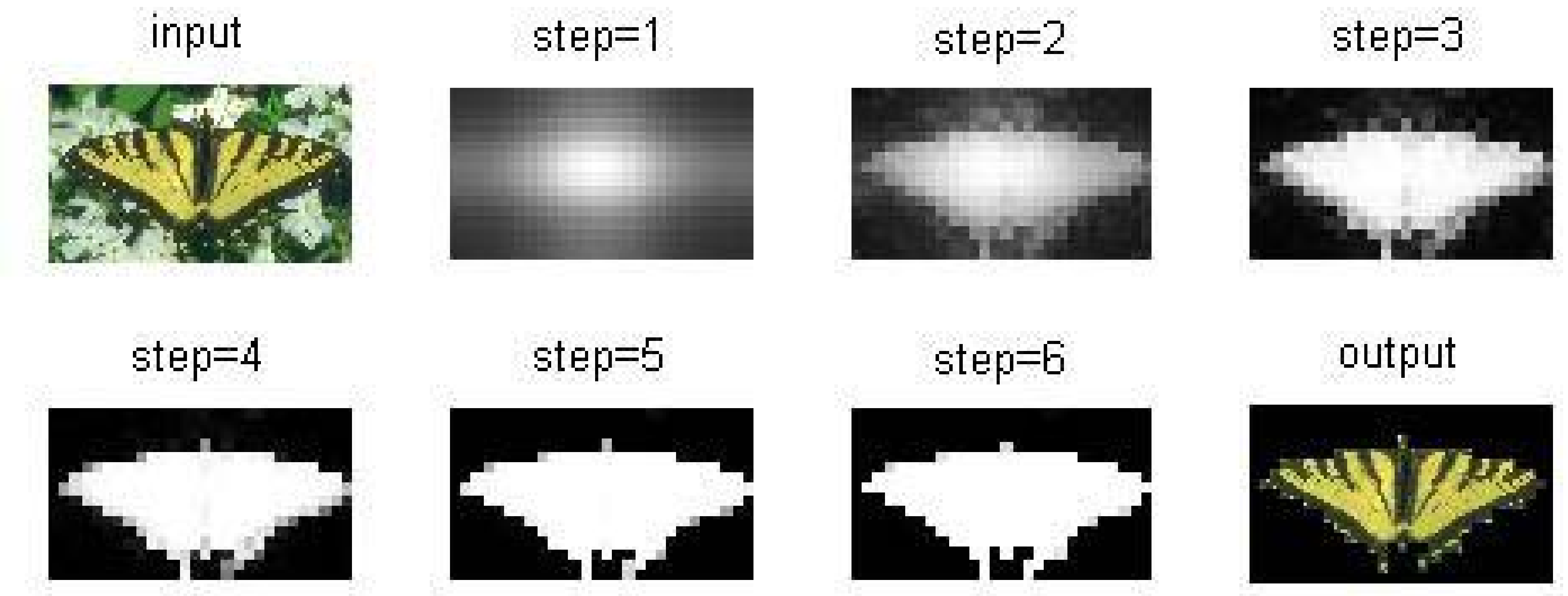
- Hence, the posterior probability for $\omega_i$ is simply the fraction of samples within the window that are labeled $\omega_i$. This is a simple and intuitive result.

# Example: Figure-Ground Discrimination

**Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.**

- Figure-ground discrimination is an important low-level vision task.
- Want to separate the pixels that contain some foreground object (specified in some meaningful way) from the background.



# Example: Figure-Ground Discrimination

**Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.**

- This paper presents a method for figure-ground discrimination based on non-parametric densities for the foreground and background.
- They use a subset of the pixels from each of the two regions.
- They propose an algorithm called **iterative sampling-expectation** for performing the actual segmentation.
- The required input is simply a region of interest (mostly) containing the object.

# Example: Figure-Ground Discrimination

**Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.**

- Given a set of $n$ samples $S = \{\mathbf{x}_i\}$ where each $\mathbf{x}_i$ is a $d$-dimensional vector.
- We know the kernel density estimate is defined as

$$\hat{p}(\mathbf{y}) = \frac{1}{n\sigma_1 \ldots \sigma_d} \sum_{i=1}^{n} \prod_{j=1}^{d} \varphi\left(\frac{\mathbf{y}_j - \mathbf{x}_{ij}}{\sigma_j}\right) \tag{26}$$

where the same kernel $\varphi$ with different bandwidth $\sigma_j$ is used in each dimension.

# The Representation

## Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.

- The representation used here is a function of RGB:

$$r = R/(R+G+B) \tag{27}$$
$$g = G/(R+G+B) \tag{28}$$
$$s = (R+G+B)/3 \tag{29}$$

- Separating the chromaticity from the brightness allows them to us a wider bandwidth in the brightness dimension to account for variability due to shading effects.
- And, much narrower kernels can be used on the $r$ and $g$ chromaticity channels to enable better discrimination.

# The Color Density

**Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.**

- Given a sample of pixels $S = \{\mathbf{x}_i = (r_i, g_i, s_i)\}$, the color density estimate is given by

$$\hat{P}(\mathbf{x} = (r, g, s)) = \frac{1}{n} \sum_{i=1}^{n} K_{\sigma_r}(r - r_i) K_{\sigma_g}(g - g_i) K_{\sigma_s}(s - s_i) \tag{30}$$

where we have simplified the kernel definition:

$$K_\sigma(t) = \frac{1}{\sigma} \varphi\left(\frac{t}{\sigma}\right) \tag{31}$$

- They use Gaussian kernels

$$K_\sigma(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2\right] \tag{32}$$

with a different bandwidth in each dimension.

# Data-Driven Bandwidth

## Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.

- The bandwidth for each channel is calculated directly from the image based on sample statistics.

$$\sigma \approx 1.06\hat{\sigma}n^{-1/5} \tag{33}$$

where $\hat{\sigma}^2$ is the sample variance.

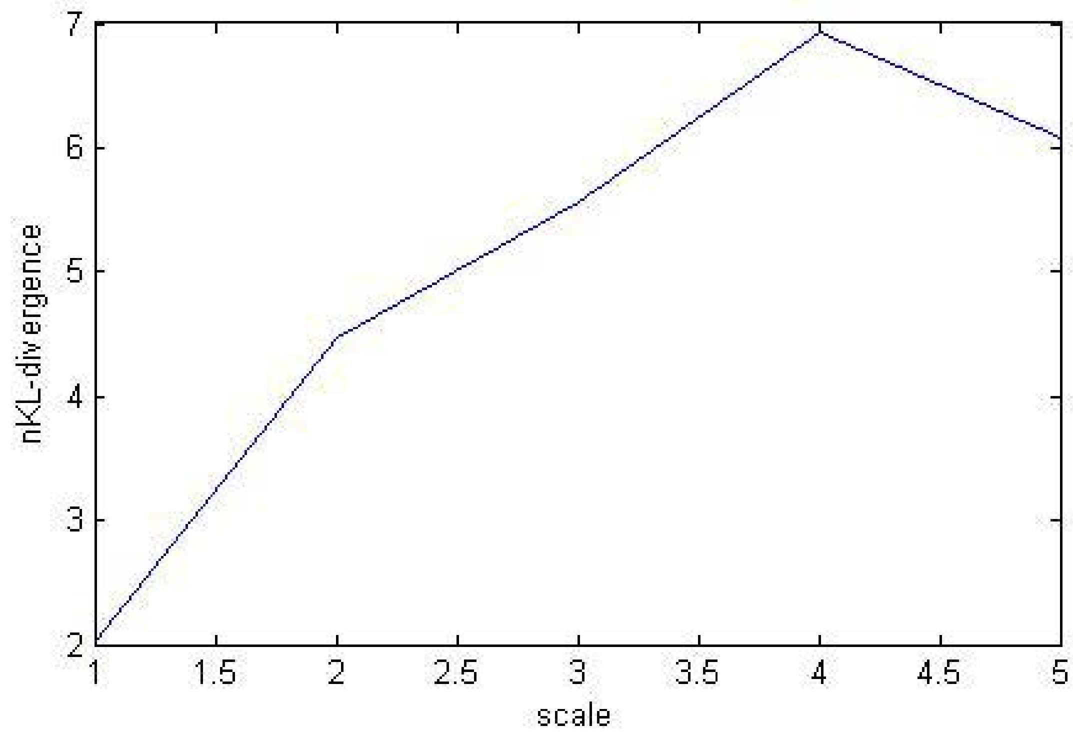
# Initialization: Choosing the Initial Scale

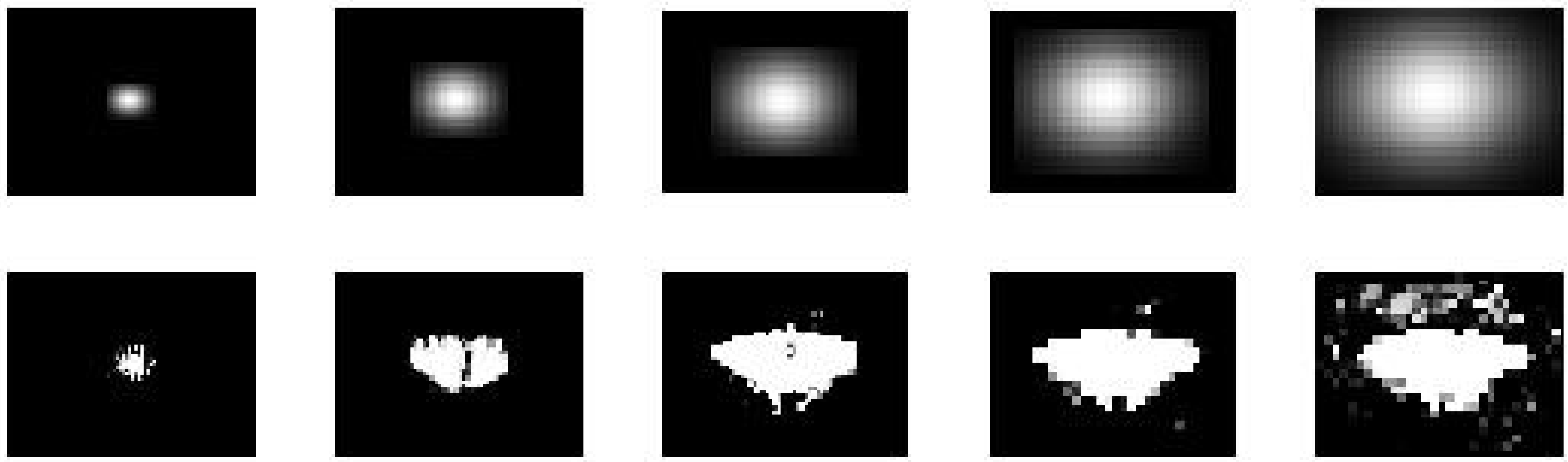
**Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.**

- For initialization, they compute a distance between the foreground and background distribution by varying the scale of a single Gaussian kernel (on the foreground).
- To evaluate the "significance" of a particular scale, they compute the normalized KL-divergence:

$$\text{nKL}(\hat{P}_{fg}||\hat{P}_{bg}) = \frac{-\sum_{i=1}^{n} \hat{P}_{fg}(\mathbf{x}_i) \log \frac{\hat{P}_{fg}(\mathbf{x}_i)}{\hat{P}_{bg}(\mathbf{x}_i)}}{\sum_{i=1}^{n} \hat{P}_{fg}(\mathbf{x}_i)} \tag{34}$$

where $\hat{P}_{fg}$ and $\hat{P}_{bg}$ are the density estimates for the foreground and background regions respectively. To compute each, they use about 6% of the pixels (using all of the pixels would lead to quite slow performance).


nKL-divergence
scale
7
6
5
4
3
2
1
1.5
2
2.5
3
3.5
4
4.5
5



# Iterative Sampling-Expectation Algorithm

**Source: Zhao and Davis. Iterative Figure-Ground Discrimination. ICPR 2004.**

- Given the initial segmentation, they need to refine the models **and** labels to adapt better to the image.
- However, this is a chicken-and-egg problem. If we know the labels, we could compute the models, and if we knew the models, we could compute the best labels.

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- However, this is a chicken-and-egg problem. If we know the labels, we could compute the models, and if we knew the models, we could compute the best labels.
- They propose an EM algorithm for this. The basic idea is to alternate between estimating the probability that each pixel is of the two classes, and then given this probability to refine the underlying models.
- EM is guaranteed to converge (but only to a local minimum).

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- The overall probability of a pixel belonging to the foreground class

$$\hat{P}_{fg}(\mathbf{y}) = \frac{1}{Z} \sum_{i=1}^{n} \hat{P}_{fg}(\mathbf{x}_i) \prod_{j=1}^{d} K\left(\frac{y_j - x_{ij}}{\sigma_j}\right) \tag{35}$$

# Results: Stability

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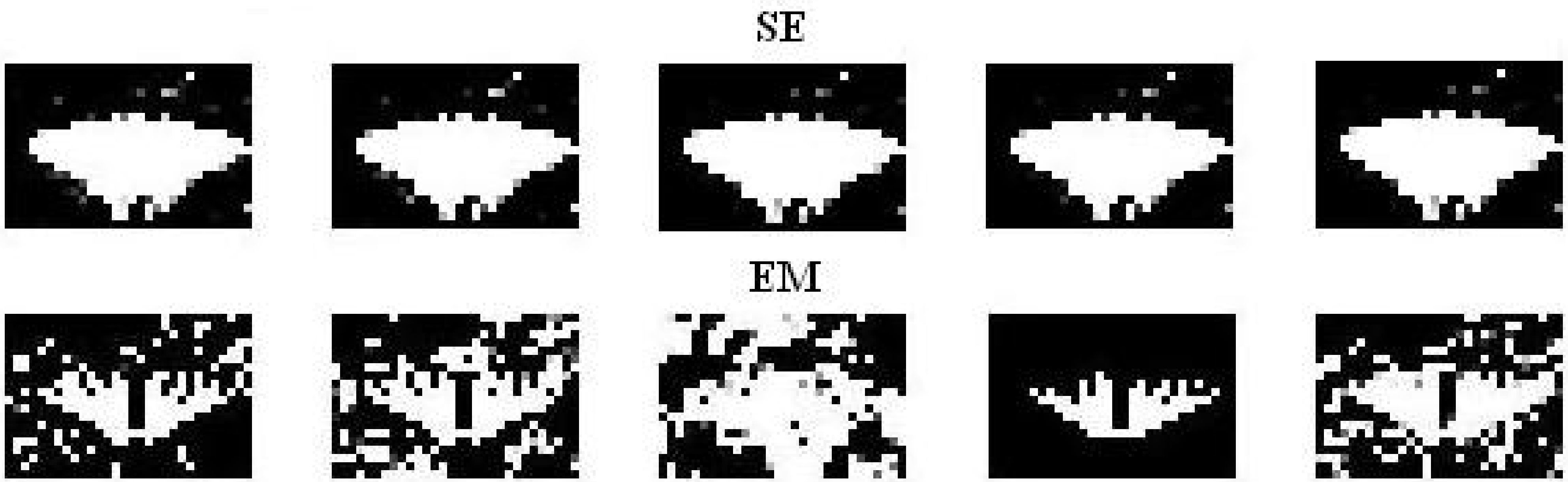

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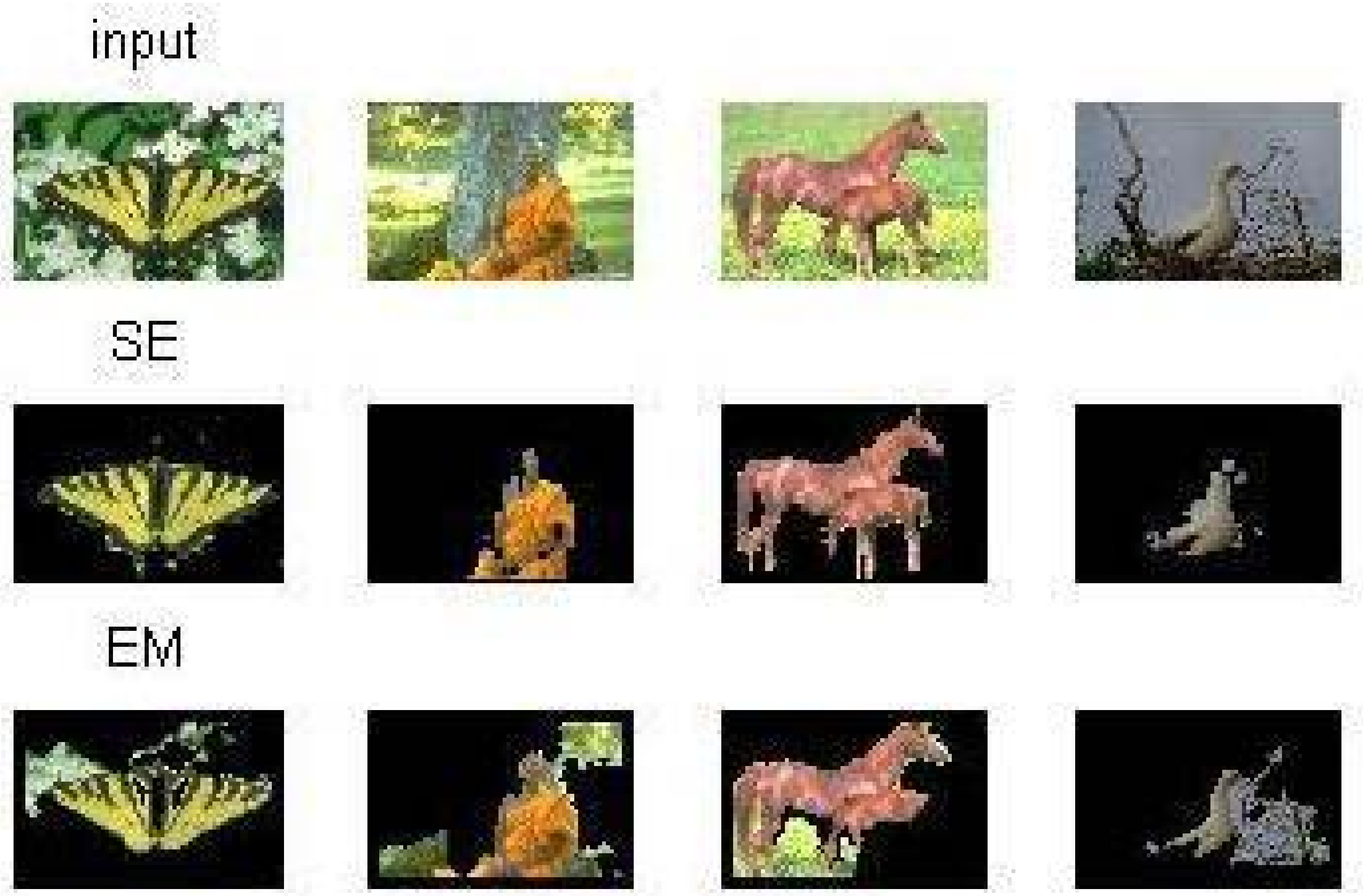

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