Algorithm Analysis

- It means: estimating the resourses required.
- The resources of algorithms: time and space.
- We mainly consider time: harder to estimate; often more critical.
- The efficiency of an algorithm is measured by a runtime function $T(n)$.
- n is the size of the input.
- Strictly speaking, n is the # of bits needed to represent input.
- Commonly, n is the # of items in the input, if each item is of fixed size.
- This makes no difference in asymptotic analysis in most cases.

Example 1

An array of k int. Strictly speaking $n = 32k$ bits. However, since int has fixed size of 32 bits, we can use $n = k$ as input size.

Example 2

The input is one integer of k digits long. Since its size is not fixed (k can be arbitrarily large). The input size is **not** $n = 1$. It is $n = 4k$ bits long.

What's $T(n)$?

- Defining $T(n)$ as the real run time is meaningless, because the real run time depends on many factors, such as the machine speed, the programming language used, the quality of compilers etc. These **are not** the properties of the algorithm.
- $T(n) \stackrel{\text{def}}{=}$ the number of basic instructions performed by the algorithm.
- Basic instructions: $+, -, *, /$, read from/write into a memory location, comparison, branching to another instruction ...
- These are **not** basic instructions: input/output statement, $\sin(x), \exp(x) \ldots$. These actions are done by function calls, not by a single machine instruction.
- Knowing $T(n)$ and the machine speed, we can estimate the real runtime.
- Example 3: The machine speed is 10^8 ins/sec. $T(n) = 10^6$. The real runtime would be about $10^{-8} \times 10^6 = 0.01$ sec.

Example 4: Consider this simple program:

```
s = 0
for i = 1 to n do
    for j = 1 to n do
        s = s + i + j
    end for
end for
```

- $T(n) = ?$ It's hard to get the exact expression of $T(n)$ even for this very simple program.
- Also, the exact value of $T(n)$ depends on factors such as prog language, compiler. These are not the properties of the loop. They should **not** be our concern.
- We can see: the loop iterates n^2 times, and loop body takes constant number of instructions.
- So $T(n) = an^2 + bn + c$ for some constants a, b, c.
- We say the growth rate of $T(n)$ is n^2. This is the sole property of the algorithm and is our main concern.

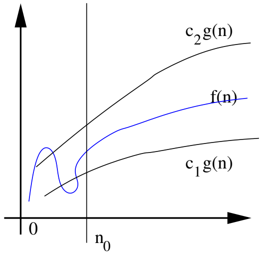
Growth rate functions

We want to define the precise meaning of growth rate.

Definition 1:

$$\Theta(g(n)) = \{f(n) \quad | \quad \exists c_1 > 0, c_2 > 0, n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

If $f(n) \in \Theta(g(n))$, we also write $f(n) = \Theta(g(n))$ and say: the growth rate of $f(n)$ is the same as the growth rate of $g(n)$.



Example 5

$f(n) = \frac{1}{12}n^2 + 60n - 4 \in \Theta(n^2)$ (or write $f(n) = \Theta(n^2)$.)

Proof: We need to find c_1 and n_0 so that $\forall n \geq n_0$,

$$c_1 n^2 \leq \frac{1}{12}n^2 + 60n - 4$$

Pick $c_1 = 1/12$, the above becomes: $0 \leq 60n - 4$. This is true for all $n \geq n_0 = 1$. We also need to find c_2 and n_0 so that $\forall n \geq n_0$,

$$\frac{1}{12}n^2 + 60n - 4 \leq c_2 n^2$$

For any $n \geq 1$, we have:

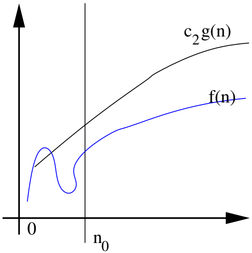
$$\frac{1}{12}n^2 + 60n - 4 < n^2 + 60n \leq n^2 + 60n^2 = 61n^2$$

So if $c_1 = 1/12, c_2 = 61$ and $n_0 = 1$, all the required conditions hold.

Definition 2:

$$O(g(n)) = \{f(n) \quad | \quad \exists c_2 > 0, n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq f(n) \leq c_2 g(n)\}$$

If $f(n) \in O(g(n))$, we also write $f(n) = O(g(n))$ and say: the growth rate of $f(n)$ is at most the growth rate of $g(n)$.



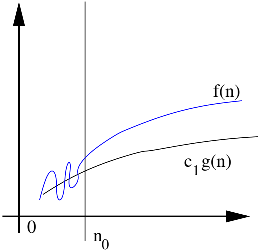
Example 6

$f(n) = 10n - 4 \in O(0.01n^2)$ (or write $f(n) = O(0.01n^2)$.)

Definition 3:

$$\Omega(g(n)) = \{f(n) \mid \exists c_1 > 0, n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n)\}$$

If $f(n) \in \Omega(g(n))$, we also write $f(n) = \Omega(g(n))$ and say: the growth rate of $f(n)$ is at least the growth rate of $g(n)$.



Definition 4:

$$o(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq f(n) \leq cg(n)\}$$

If $f(n) \in o(g(n))$, we also write $f(n) = o(g(n))$ and say: the growth rate of $f(n)$ is strictly less than the growth rate of $g(n)$.

Example:

$f(n) = 2n$ and $g(n) = n^2$. Then:
$f(n) = O(g(n)), f(n) = o(g(n))$, but $f(n) \neq \Theta(g(n))$,

Definition 5:

$$\omega(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq cg(n) \leq f(n)\}$$

If $f(n) \in \omega(g(n))$, we also write $f(n) = \omega(g(n))$ and say: the growth rate of $f(n)$ is strictly bigger than the growth rate of $g(n)$.

The properties of growth rate functions:

The meaning of these notations (roughly speaking):

if	the growth-rate is
$f(n) = \Theta(g(n))$	$=$
$f(n) = O(g(n))$	$\leq$
$f(n) = \Omega(g(n))$	$\geq$
$f(n) = o(g(n))$	$<$
$f(n) = \omega(g(n))$	$>$

Some properties of growth rate functions:

- $f(n) = \Theta(g(n)) \Longleftrightarrow f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
- $f(n) = O(g(n)) \Longleftrightarrow g(n) = \Omega(f(n))$
- $f(n) = o(g(n)) \Longleftrightarrow g(n) = \omega(f(n))$
- $f(n) = O(g(n))$ and $g(n) = O(h(n)) \Longrightarrow f(n) = O(h(n))$ if we replace O by $\Theta, \Omega, o, \omega$, it holds true.
- Read Ch. 3 for more relations and properties.

Importance of the growth rate

The growth rate of the runtime function is the most important property of an algorithm. Assuming 10^9 instruction/sec, The real runtime:

$f(n)$	$n = 10$	30	50	1000
$\log_2 n$	3.3 ns	4.9 ns	5.6 ns	9.9 ns
n	10 ns	30 ns	50 ns	1 μs
n^2	0.1 μs	0.9μs	2.5 μs	1 ms
n^3	1 μs	27 μs	125 μs	1 sec
n^5	0.1 ms	24.3 ms	0.3 sec	277 h
2^n	1 μs	1 sec	312 h	3.4 $\cdot 10^{281}$ Cent

- If $T(n) = n^k$ for some constant $k > 0$, the runtime is polynomial.
- If $T(n) = a^n$ for some constant $a > 1$, the runtime is exponential.

- $T(n) = 2^n$, $n = 360$ and assuming 10^9 instructions/sec.
- $T(360) = 2^{360} = (2^{10})^{36} \approx (10^3)^{36} = 10^{108}$ instructions.
- This translates into: 10^{99} CPU sec, about $3 \cdot 10^{91}$ years.
- For comparison: the age of the universe: about $1.5 \cdot 10^{10}$ years.
- The number of atoms in the known universe: $\leq 10^{80}$.
- If every atom in the known universe is a supercomputer and starts at the beginning of the big bang, we have only done $\frac{1.5 \cdot 10^{10} \times 10^{80}}{3 \cdot 10^{91}} = 5\%$ of the needed computations!
- Moore's law: CPU speed doubles every 18 months. Then, instead of solving the problem of size $n =$ say 100, we can solve the problem of size 101.
- An exponential time algorithm **cannot** be used to solve problems of realistic input size, no matter how powerful the computers are!

An example

Some simple looking problems indeed require exp runtime. Here is a very important application that depends on this fact.

P1: Factoring Problem

Input: an integer X.
Output: Find its prime factorization.

If $X = 117$, the output: $X = 3 \cdot 3 \cdot 13$.

P2: Primality Testing

Input: an integer X.
Output: "yes" if X is a prime number; "no" if not.

- If $X = 117$, output "no".
- If $X = 456731$, output = ?

- P1 and P2 are related.
- If we can solve P1, we can solve P2 immediately.
- The reverse is not true: even if we know X is not a prime, how to find its prime factors?
- P1 is harder than P2.
- How to solve P1?

Find-Factor(X)

1: **if** X is even **then**
2: return "2 is a factor"
3: **end if**
4: **for** $i = 3$ **to** $\sqrt{X}$ **by** $+2$ **do**
5: test **if** $X\%i = 0$, if yes, output "i is a factor"
6: **end for**
7: return "X is a prime."

- To solve P1, we call **Find-Factor(X)** to find the smallest prime factor i of X. Then call **Find-Factor(X/i)** ...
- The runtime of **Find-Factor**: X is not a fixed-size object. So the input size n is the $\#$ of bits needed to represent X.
- X is n bits long, the value of X is $\leq 2^n$.
- In the worst case, we need to perform $\frac{1}{2}\sqrt{2^n} = \frac{1}{2}(1.414)^n$ divisions. So this is an exp time algorithm.
- Minor improvements can be (and had been) made. But basically, we have to perform most of these tests. No poly-time algorithm for Factoring is known.
- It is strongly believed, (but not proven), no poly-time algorithm for solving the Factoring problem exists.

Encryption

- A customer (Alice) wants to send a message M to her bank (Bob).
- If an intruder (Evil) intercepts M, we must make sure Evil cannot understand it.
- So M must be encrypted:
  - Alice computes an encrypted message $C = P_A(M)$ ($P_A()$ is the encryption function), and send C to Bob.
  - Bob receives C, and computes $M = S_A(C)$ ($S_A()$ is the decryption function), to retrieve the original M
  - Even if Evil sees C, he doesn't know $S_A()$, so cannot recover M.

- 1-1 Encryption:
  - Alice and Bob agree a particular method (secret key) for encryption.
  - Only Alice and Bob know this particular secret key, and keep it secret.
  - For another customer (Dave), Bob and Dave must use a different key.
- There are many different ways for 1-1 Encryption. It is not hard.
- However, Bob is dealing with many customers, and Alice is dealing with many banks, on-line accounts ...
- It would be a nightmare if we have to arrange a different key for each (Alice, Bob) pair.

RSA Public-Key Cryptosystem

- Invented by Rivest, Shamir and Aldeman in 1977. Most of current computer security systems are based on this.
- Everyone uses the same public key for encryption.
- Bob: chose a pair of large prime numbers x and y, say 128 digits each.
- Bob: compute $X = x \cdot y$.
- Bob: computes two numbers d and e, such that $d \cdot e = 1 \pmod{[(x-1) \cdot (y-1)]}$. (This is easy to do, see Sec. 31.7)
- The pair (X, e) is the public key. Bob makes it public.
- (x, y, d) is the secret key. Only Bob knows it.

Example

$x = 7$, $y = 29$. Then $X = 7 \cdot 29 = 203$, and $(x-1) \cdot (y-1) = 168$.
Pick $e = 11$ and $d = 107$, then $11 \cdot 107 = 1177 = 1 \pmod{168}$.
Thus $(203, 11)$ is the public key. $(7, 29, 107)$ is secret key.

- Alice (and Dave and everyone else): Get public key (X, e) $(= (203, 11)$ in our example).
- Treat her message M as an integer. (It can be just the value of the binary string representing M. For example $M = 100$.)
- Compute the encrypted message $C = P_A(M) \stackrel{\text{def}}{=} M^e \pmod X$. (In our example $C = 100^{11} \pmod{203} = 4$).
- Send $C(= 4)$ to Bob.
- Bob: Receiving $C(= 4)$. Recover the original message $M = S_A(C) \stackrel{\text{def}}{=} C^d \pmod X$. (In our example $4^{107} \pmod{203} = 100$).
- Because of the the choice of e, d, the number theory ensures the result M is the same as the original message M. (Namely $(M^e)^d = M \pmod X$ for all M.)

- If Evil intercepts C, he doesn't know the secret key d, so he cannot recover $M = C^d \pmod{X}$.
- But Evil knows X (since this is public).
- If Evil can factor $X = x \cdot y$, he can calculate d. Then he knows every thing that Bob knows.
- But he must factor a 256 digit number X. This requires about $\sqrt{10^{256}} = 10^{128} \approx 2^{426}$ divisions. This will need much much much .... longer time than the previous 2^{360} example!

- RSA received 2002 Turing Award (the Nobel prize equivalent in CS) for this (and related) work.
- This system works because the strong (but not proven) belief: The Factoring (P1) problem cannot be solved in poly-time.
- For long time, it is not known if the problem P2 (Primality Testing) can be solved in poly-time.
- In 2001, Agrawal, Kayal and Saxena found a poly-time algorithm for solving P2.
- Had they found a poly-time algorithm for solving P1 (Factoring), RSA system (and the entire computer security industry) would have collapsed overnight!