Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$

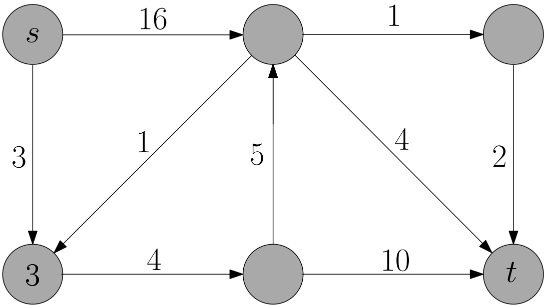
Output: a shortest path from s to t in G

Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$

Output: a shortest path from s to t in G

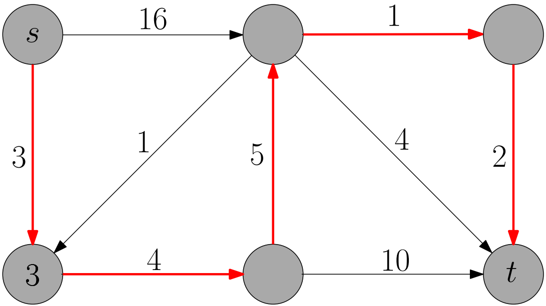


Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$

Output: a shortest path from s to t in G

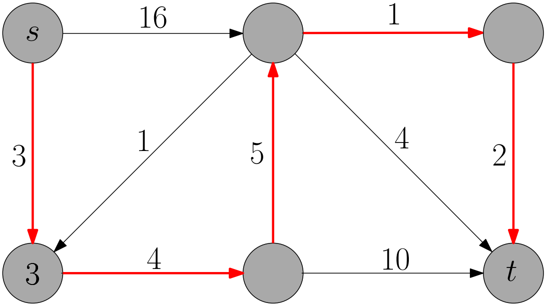


Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$

Output: a shortest path from s to t in G



- Algorithm: Dijkstra's algorithm ...

Examples

Shortest Path

Input: directed graph $G = (V, E)$ (may have negative edges), $s, t \in V$

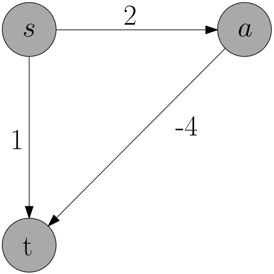
Output: a shortest path from s to t in G

Examples

Shortest Path

Input: directed graph $G = (V, E)$ (may have negative edges), $s, t \in V$

Output: a shortest path from s to t in G

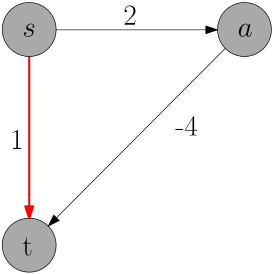


Examples

Shortest Path

Input: directed graph $G = (V, E)$ (may have negative edges), $s, t \in V$

Output: a shortest path from s to t in G

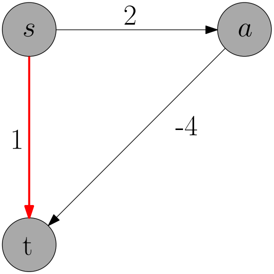


Examples

Shortest Path

Input: directed graph $G = (V, E)$ (may have negative edges), $s, t \in V$

Output: a shortest path from s to t in G



- Algorithm: Bellman-Ford algorithm, Floyd-Warshall ...

Algorithm = Computer Program?

- Algorithm: "abstract", can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: "concrete", implementation of algorithm, using a particular programming language

Pseudo-Code

Python program:

- def euclidean(a: int, b: int):
-         c = 0
-         while b > 0:
-                 c = b
-                 b = a % b
-                 a = c
-         return a

Pseudo-Code:

Euclidean(a, b)

1: **while** $b > 0$ **do**
2: $\quad (a, b) \leftarrow (b, a \bmod b)$
3: **return** a

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible
2. efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g, python)

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible
2. efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g, python)
3. fundamental

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible
2. efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g, python)
3. fundamental
4. it is fun!

Outline

Sorting Problem

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

Insertion-Sort

- At the end of j-th iteration, the first j numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

1: **for** $j \leftarrow 2$ to n **do**
2: $\quad key \leftarrow A[j]$
3: $\quad i \leftarrow j - 1$
4: $\quad$ **while** $i > 0$ and $A[i] > key$ **do**
5: $\quad\quad A[i+1] \leftarrow A[i]$
6: $\quad\quad i \leftarrow i - 1$
7: $\quad A[i+1] \leftarrow key$

- $j = 6$
- $key = 15$

12	21	35	53	59	15
				$\uparrow$	
				i	

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 53 & 59 & \color{red}{59} \\ & & & & \uparrow & \\ & & & & i & \end{array}$$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

for $j \leftarrow 2$ to n **do**
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ **do**
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$

- $j = 6$
- $key = 15$

12 21 35 53 59 59

$\uparrow$
i

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 53 & \color{red}{53} & 59 \\ & & & \uparrow & & \\ & & & i & & \end{array}$$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

for $j \leftarrow 2$ to n **do**
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ **do**
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 53 & 53 & 59 \\ & & \uparrow & & & \\ & & i & & & \end{array}$$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 35 & 53 & 59 \\ & & \uparrow & & & \\ & & i & & & \end{array}$$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

$$\begin{array}{cccccc} 12 & 21 & 35 & 35 & 53 & 59 \\ & \uparrow & & & & \\ & i & & & & \end{array}$$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

1: **for** $j \leftarrow 2$ to n **do**
2: $\quad key \leftarrow A[j]$
3: $\quad i \leftarrow j - 1$
4: $\quad$ **while** $i > 0$ and $A[i] > key$ **do**
5: $\quad\quad A[i+1] \leftarrow A[i]$
6: $\quad\quad i \leftarrow i - 1$
7: $\quad A[i+1] \leftarrow key$

- $j = 6$
- $key = 15$

12	21	21	35	53	59
	$\uparrow$				
	i				

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

12	21	21	35	53	59
$\uparrow$					
i					

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

12 15 21 35 53 59

$\uparrow$

i

Outline

Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 5 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph
- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.

Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
- possible definition of size :
  - Sorting problem: # integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: # edges in graph
- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
- A2: Worst-case analysis:
  - Running time for size n = worst running time over all possible arrays of length n

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?
- A: They do not matter!

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?
- A: They do not matter!

Important idea: asymptotic analysis

- Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$
- $n^2/100 - 3n + 10 = O(n^2)$

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n + 10 = O(n^2)$

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or $\#$ instructions?

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or # instructions?
- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or 10^{-8} seconds
  - program 2 requires 2 instructions, or 10^{-9} seconds

Asymptotic Analysis: O-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n + 10 = O(n^2)$

O-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or $\#$ instructions?
- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or 10^{-8} seconds
  - program 2 requires 2 instructions, or 10^{-9} seconds
  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$
- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.
- For Algorithm 1: if we increase n by a factor of 2, running time increases by a factor of 4

Asymptotic Analysis: O-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$

- Does not tell which algorithm is faster for a specific n!
- Algorithm 2 will eventually beat algorithm 1 as n increases.

- For Algorithm 1: if we increase n by a factor of 2, running time increases by a factor of 4
- For Algorithm 2: if we increase n by a factor of 2, running time increases by a factor of 2

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

for $j \leftarrow 2$ to n **do**
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ **do**
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- Worst-case running time for iteration j of the outer loop?

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

```
for j ← 2 to n do
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- Worst-case running time for iteration j of the outer loop?
  Answer: $O(j)$

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

for $j \leftarrow 2$ to n **do**
  $key \leftarrow A[j]$
  $i \leftarrow j - 1$
  while $i > 0$ and $A[i] > key$ **do**
    $A[i+1] \leftarrow A[i]$
    $i \leftarrow i - 1$
  $A[i+1] \leftarrow key$

- Worst-case running time for iteration j of the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)$
  $= O(\frac{n(n+1)}{2} - 1) = O(n^2)$

Computation Model

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?

Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer n and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

Outline

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

Asymptotically Positive Functions

Def. $f : \mathbb{N} \rightarrow \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$?

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$? No

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$? No
- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\big\}.$$

O-Notation: Asymptotic Upper Bound

O-Notation For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

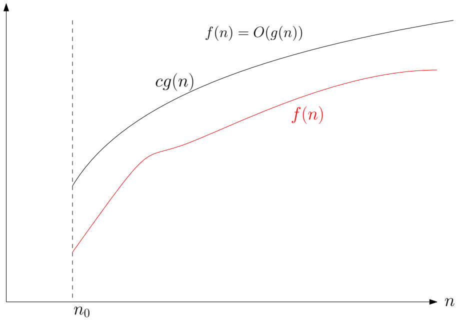
- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some $c > 0$ and every large enough n.

O-Notation: Asymptotic Upper Bound

O-Notation For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some $c > 0$ and every large enough n.



O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some $c > 0$ and every large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some $c > 0$ and every large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$

Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$

$$= -n^2 + 42n \le 0.$$

$$3n^2 + 2n \le c(n^2 - 10n)$$

□

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- ? $n^{100} \in O(2^n)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- ? $n^{100} \in O(2^n)$
- ? $\sin n \in O(1/2)$

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \le cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- ? $n^{100} \in O(2^n)$
- ? $\sin n \in O(1/2)$
- ? $n^3 \notin O(10n^2)$

O-Notation For a function $g(n)$,

$$O(g(n)) = \Big\{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\Big\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- ? $n^{100} \in O(2^n)$
- ? $\sin n \in O(1/2)$
- ? $n^3 \notin O(10n^2)$

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$		

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$

"$=$" is asymmetric! Following equalities are wrong:

- $O(n^3 - 10n) = 3n^2 + 2n$
- $O(n^2 + 5n) = 3n^2 + 2n$
- $O(n^2) = 3n^2 + 2n$

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$

"$=$" is asymmetric! Following equalities are wrong:

- $O(n^3 - 10n) = 3n^2 + 2n$
- $O(n^2 + 5n) = 3n^2 + 2n$
- $O(n^2) = 3n^2 + 2n$

- Analogy: Mike is a student. ~~A student is Mike.~~