Find Common Subsequence

$i \leftarrow n, j \leftarrow m, S \leftarrow ()$
while $i > 0$ and $j > 0$ **do**
    if $\pi[i, j] =$ "$\nwarrow$" **then**
        add $A[i]$ to beginning of S, $i \leftarrow i - 1, j \leftarrow j - 1$
    else if $\pi[i, j] =$ "$\uparrow$" **then**
        $i \leftarrow i - 1$
    else
        $j \leftarrow j - 1$
return S

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A and a string B

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

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Obs. $\#\text{OPs} = \text{length}(A) + \text{length}(B) - 2 \cdot \text{length}(\text{LCS}(A, B))$

Variants of Problem

Edit Distance with Insertions, Deletions and Replacing

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each time we can delete a letter from A, insert a letter to A or change a letter

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Example:

- $A =$ ocurrance, $B =$ occurrence.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance with Replacing: Reduction to a Variant of LCS

- Need to match letters in A and B, every letter is matched at most once and there should be no crosses.
- However, we can match two different letters: Matching a same letter gives score 2, matching two different letters gives score 1.
- Need to maximize the score.
- DP recursion for the case $i > 0$ and $j > 0$:

$$opt[i,j] = \begin{cases} opt[i-1,j-1] + 2 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i-1,j] \\ opt[i,j-1] \\ opt[i-1,j-1]+1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

- Relation : #OPs $=$ length(A) $+$ length(B) - max_score

Edit Distance (with Replacing): using DP directly

- $opt[i, j], 0 \le i \le n, 0 \le j \le m$: edit distance between $A[1 \,..\, i]$ and $B[1 \,..\, j]$.

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Outline

Computing the Length of LCS

for $j \leftarrow 0$ to m **do**
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m **do**
        if $A[i] = B[j]$ **then**
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$
        else if $opt[i, j-1] \geq opt[i-1, j]$ **then**
            $opt[i, j] \leftarrow opt[i, j-1]$
        else
            $opt[i, j] \leftarrow opt[i-1, j]$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n+m)$

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Q: How to use this observation to reduce space?

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Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

for $j \leftarrow 0$ to m **do**
  $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
  $opt[i \text{ mod } 2, 0] \leftarrow 0$
  for $j \leftarrow 1$ to m **do**
    if $A[i] = B[j]$ **then**
      $opt[i \text{ mod } 2, j] \leftarrow opt[i - 1 \text{ mod } 2, j - 1] + 1$
    else if $opt[i \text{ mod } 2, j - 1] \geq opt[i - 1 \text{ mod } 2, j]$ **then**
      $opt[i \text{ mod } 2, j] \leftarrow opt[i \text{ mod } 2, j - 1]$
    else
      $opt[i \text{ mod } 2, j] \leftarrow opt[i - 1 \text{ mod } 2, j]$
return $opt[n \text{ mod } 2, m]$

How to Recover LCS Using Linear Space?

- Only keep the last two rows: only know how to match $A[n]$

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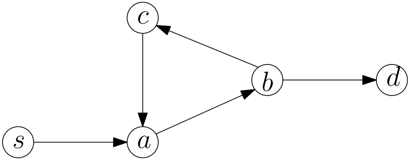
  - Space: $O(m+n)$
  - Time: $O(nm)$

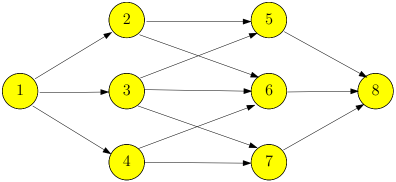
Outline

Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



not a DAG



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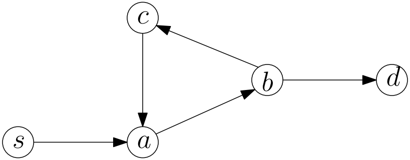

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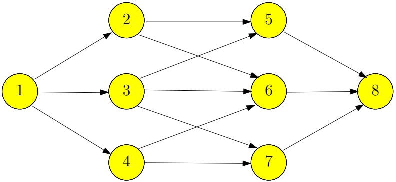


a DAG

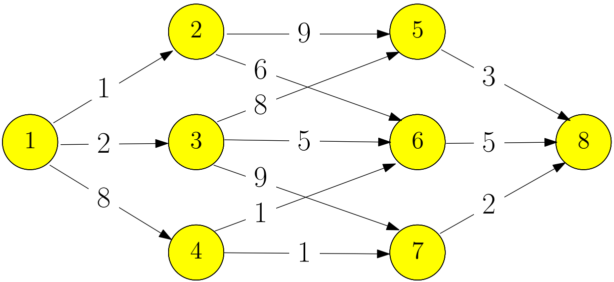
Lemma A directed graph is a DAG if and only its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots , n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i, for every $i \in V$



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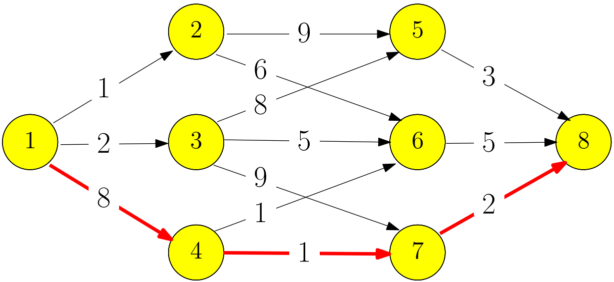

Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

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$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

Shortest Paths in DAG

```
f[1] ← 0
for i ← 2 to n do
    f[i] ← ∞
    for each incoming edge (j, i) of i do
        if f[j] + w(j, i) < f[i] then
            f[i] ← f[j] + w(j, i)
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            $\pi(i) \leftarrow j$

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            π(i) ← j
```
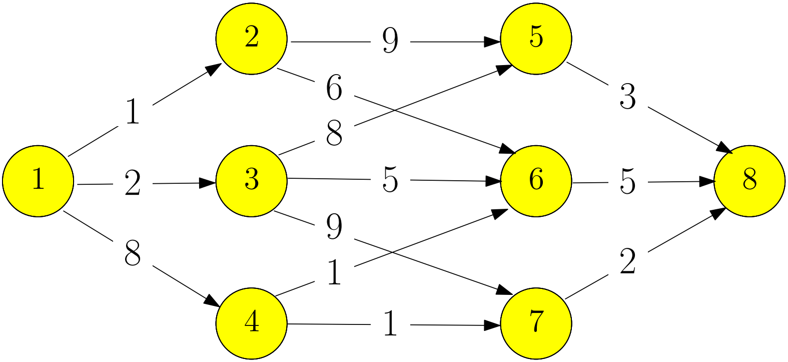

print-path(t)

```
if t = 1 then
    print(1)
    return
print-path(π(t))
print(",", t)
```

Example



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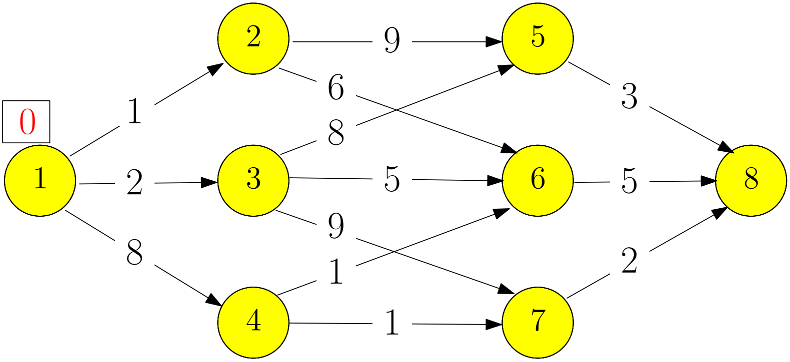

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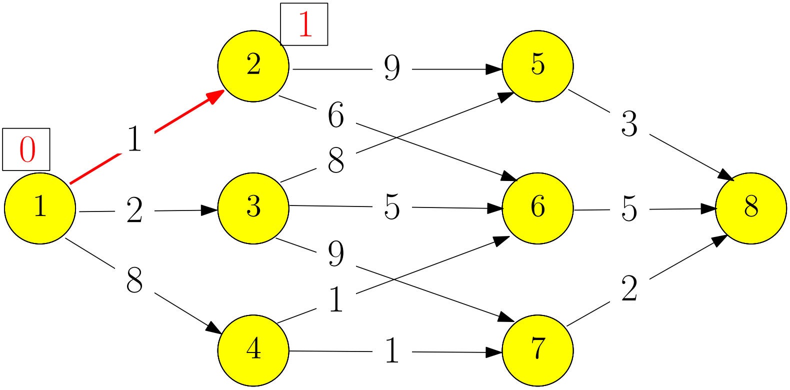

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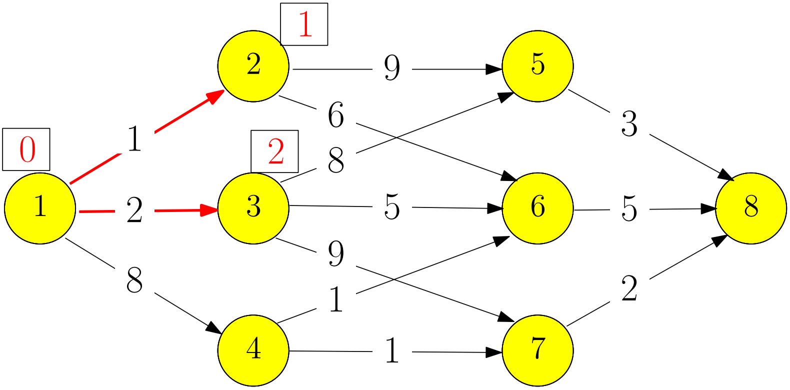

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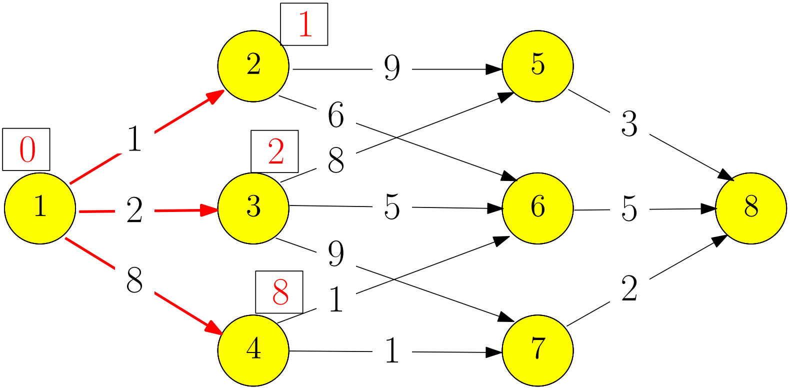

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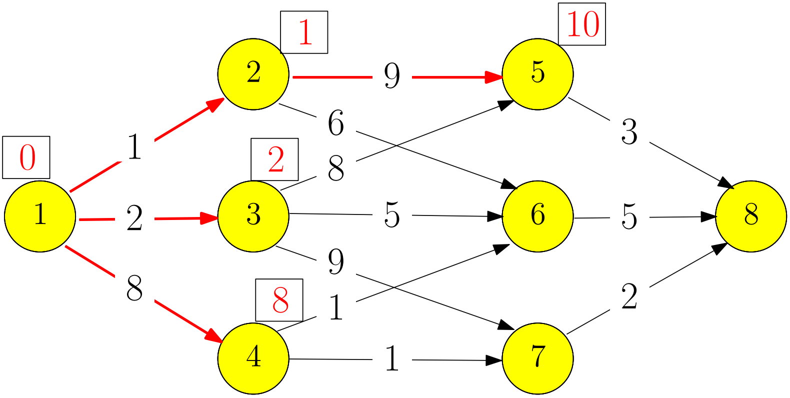

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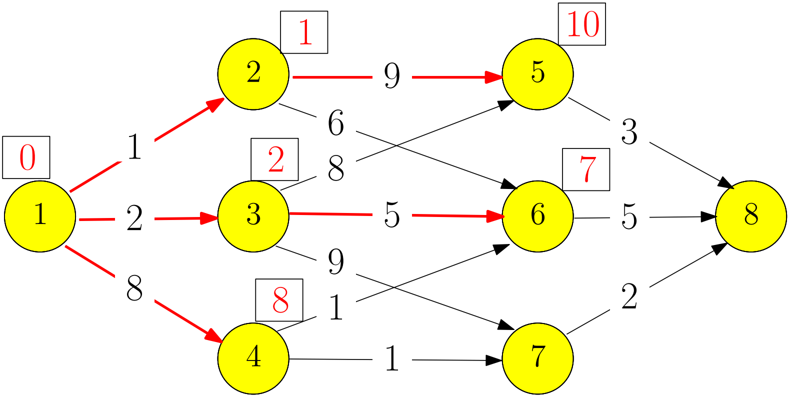

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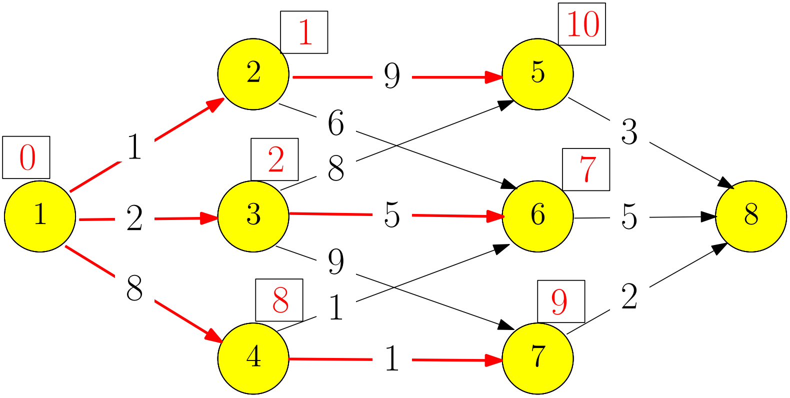

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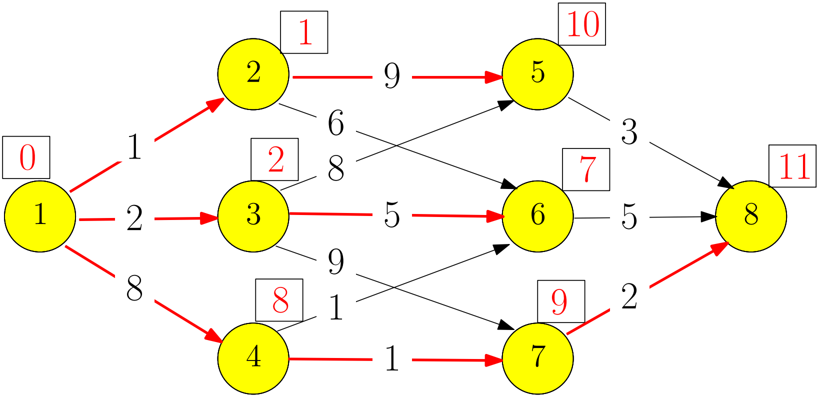

Variant: Heavest Path in a Directed Acyclic Graph

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Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the path with the largest weight (the heaviest path) from 1 to n.

- $f[i]$: weight of the heaviest path from 1 to i

$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

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Outline

Matrix Chain Multiplication

Matrix Chain Multiplication

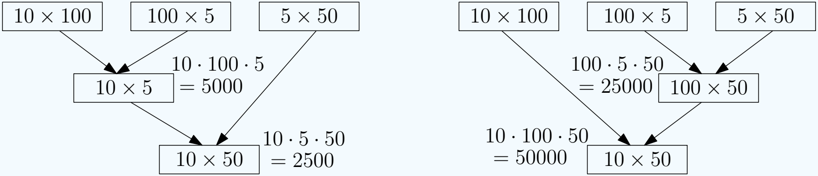
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$



cost $= 5000 + 2500 = 7500$ cost $= 25000 + 50000 = 75000$

- $(A_1 A_2) A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1 (A_2 A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Example:

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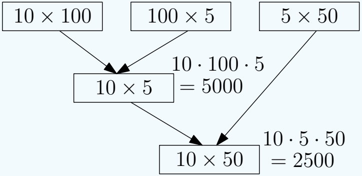

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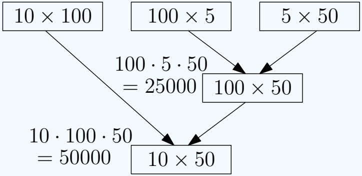

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Matrix Chain Multiplication: Design DP