extract_min()

```
ret ← A[1]
A[1] ← A[s]
p[A[1]] ← 1
s ← s − 1
if s ≥ 1 then
    heapify_down(1)
return ret
```

decrease_key(v, key_val

```
key[v] ← key_value
heapify-up(p[v])
```

heapify-down(i)

```
while 2i ≤ s do
    if 2i = s or key[A[2i]] ≤ key[A[2i + 1]] then
        j ← 2i
    else
        j ← 2i + 1
    if key[A[j]] < key[A[i]] then
        swap A[i] and A[j]
        p[A[i]] ← i, p[A[j]] ← j
        i ← j
    else break
```

- Running time of heapify_up and heapify_down: $O(\lg n)$

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data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

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Outline

Encoding Letters Using Bits

- 8 letters a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per letter

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \rightarrow 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?

Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.

Q: What is the issue with the following encoding scheme?

- a: 0 b: 1 c: 00

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A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

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- a: 0 b: 1 c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

Solution

Use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of letters is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

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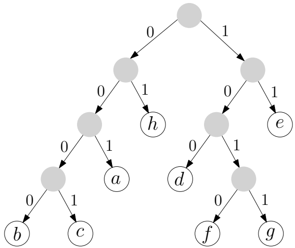
001	0000	0001	100
e	f	g	h
11	1010	1011	01



Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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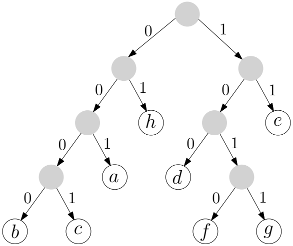

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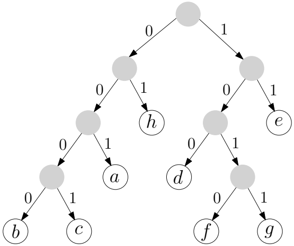

- 0001001100000001011110100001001

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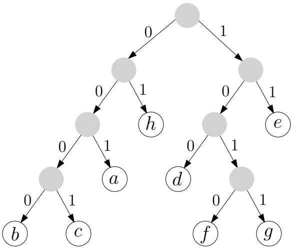

- 0001/001100000001011110100001001
- c

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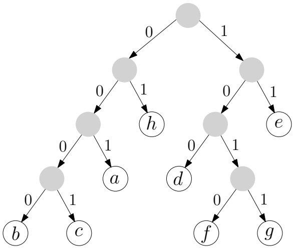

- 0001/001/100000001011110100001001
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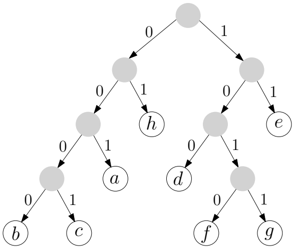

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- cad

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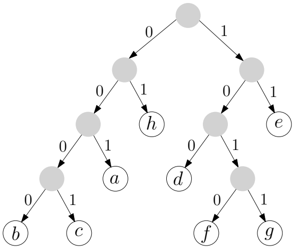

- 0001/001/100/0000/01011110100001001
- cadb

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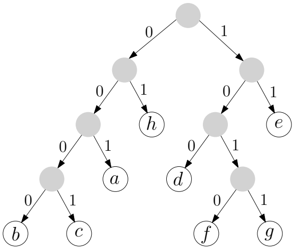

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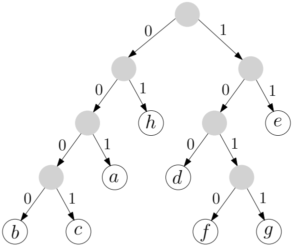

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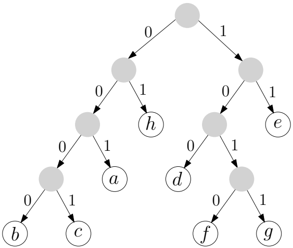

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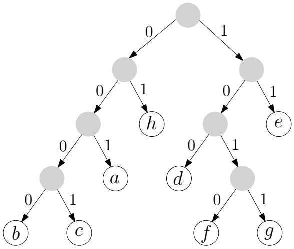

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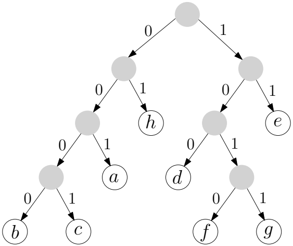

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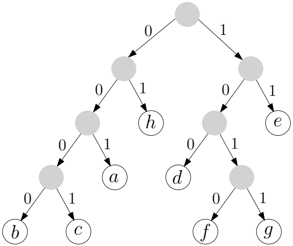

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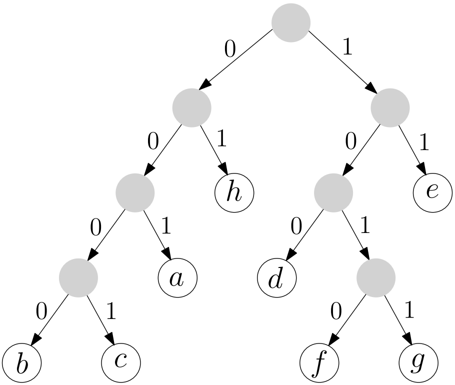
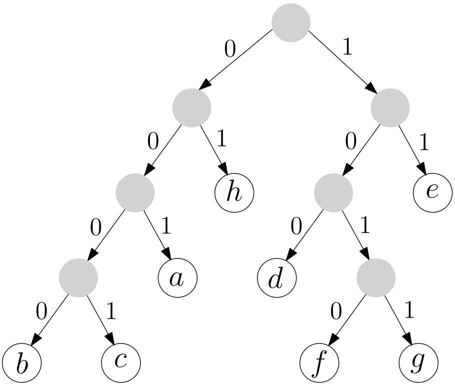
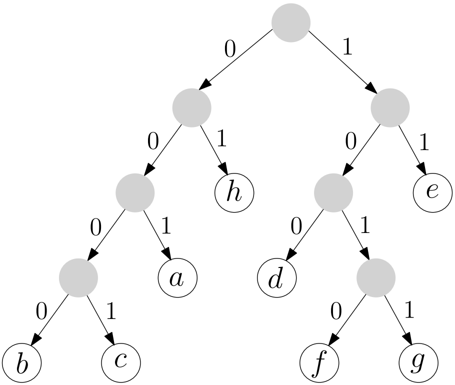


Properties of Encoding Tree



Properties of Encoding Tree

- Rooted binary tree



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- Rooted binary tree
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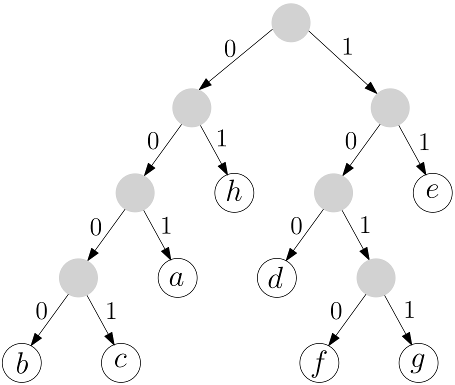

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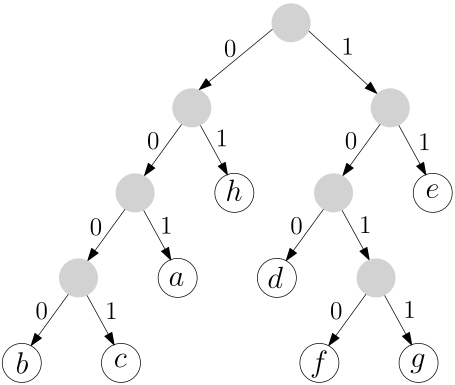

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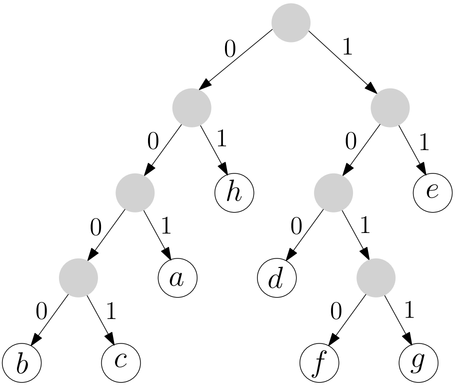

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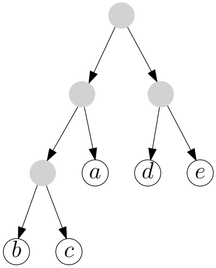
Best Prefix Codes

Input: frequencies of letters in a message

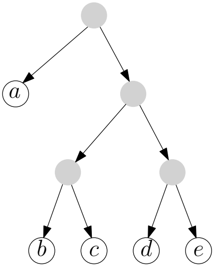
Output: prefix coding scheme with the shortest encoding for the message

example

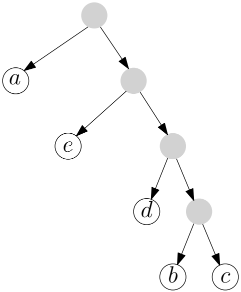
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



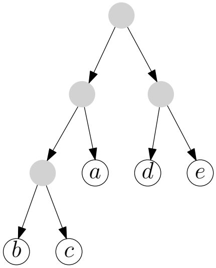
scheme 2



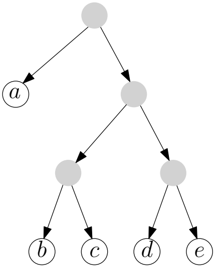
scheme 3

example

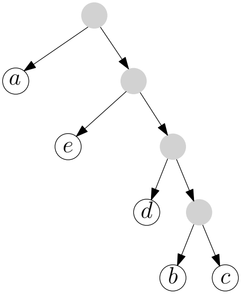
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total = 89
scheme 2 length	1	3	3	3	3	total = 87
scheme 3 length	1	4	4	3	2	total = 84



scheme 1



scheme 2



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A: We can choose two letters and make them brothers in the tree.