- Example Input: ($a$: 18, $b$: 3, $c$: 4, $d$: 6, $e$: 10)

**Q:** What types of decisions should we make?

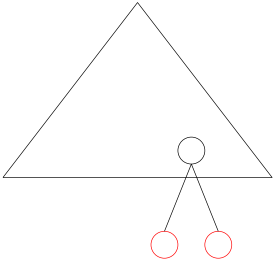
- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.

- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

**A:** We can choose two letters and make them brothers in the tree.

# Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree



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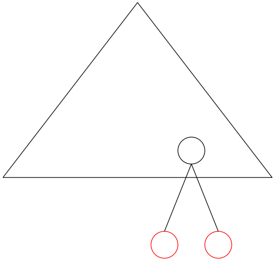

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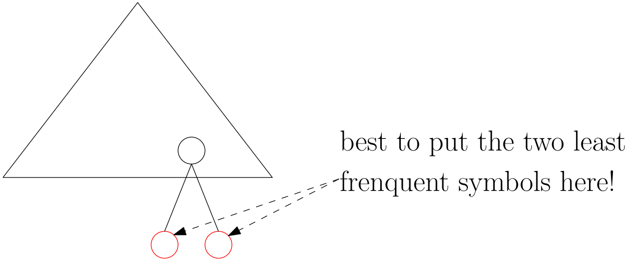

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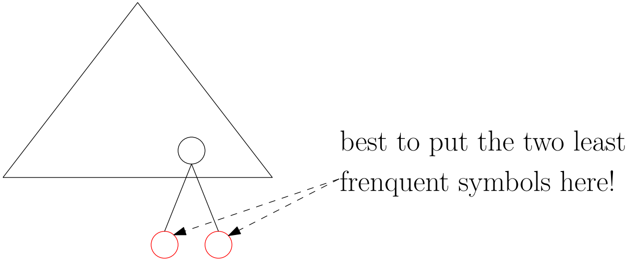


**Lemma** It is safe to make the two least frequent letters brothers.

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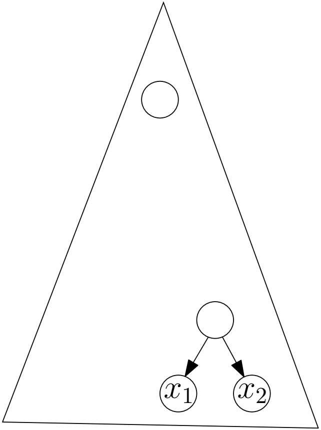
**Q:** Is the residual problem another instance of the best prefix codes problem?

**A:** Yes, though it is not immediate to see why.

- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$ the depth of letter $x$ in our output encoding tree.




$$
\begin{aligned}
& \sum_{x \in S} f_x d_x \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
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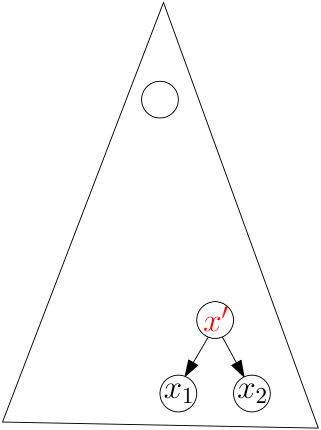


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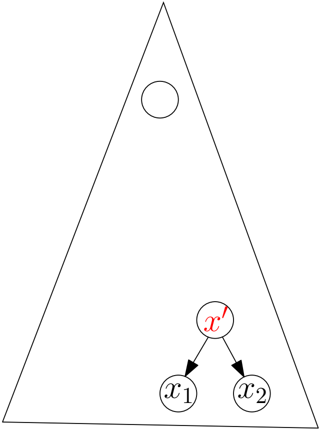


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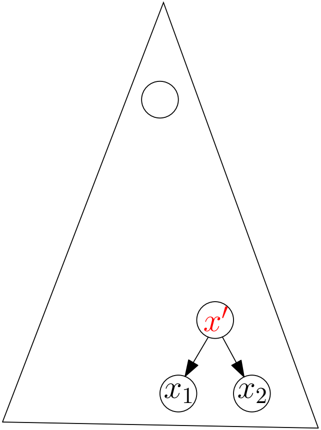


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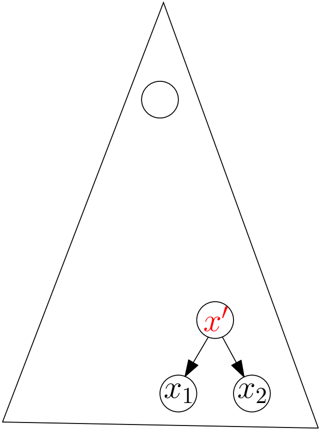


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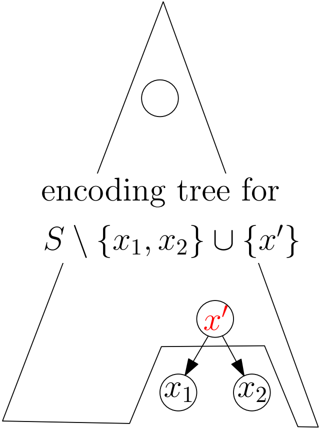


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In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that $d$ is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector $f$!

# Example

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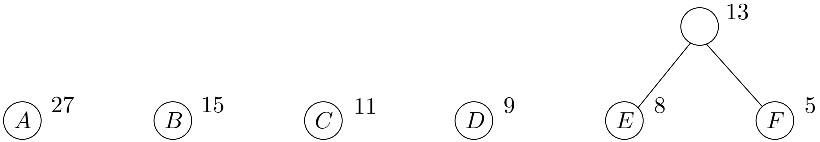

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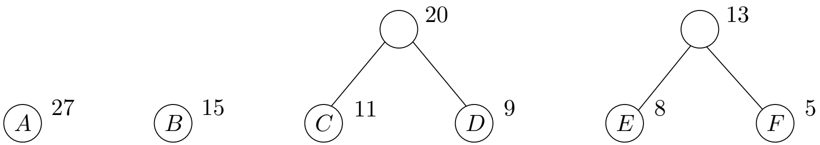

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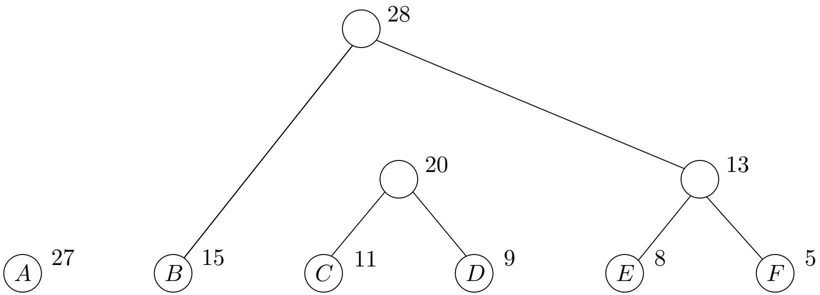

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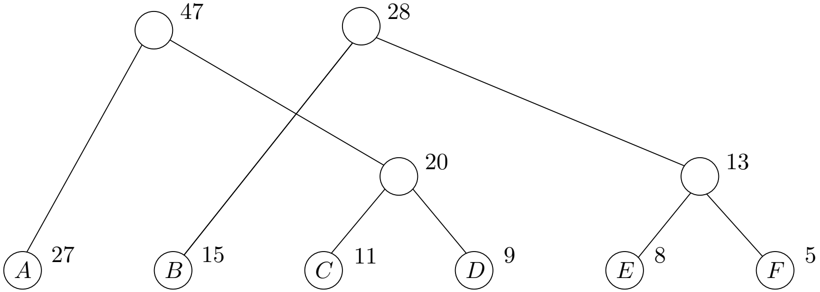

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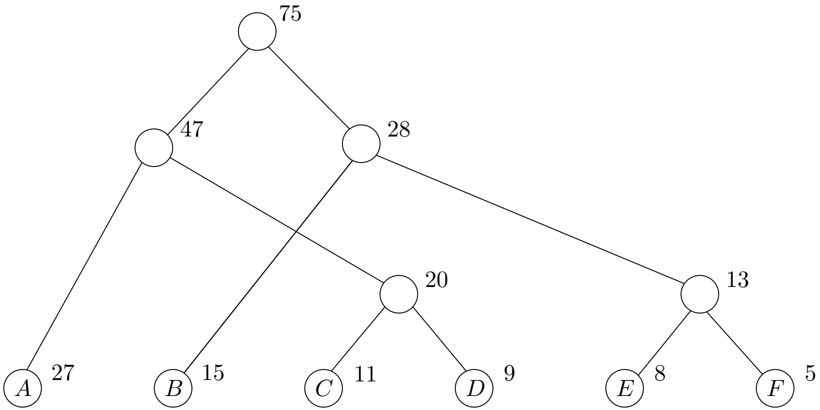

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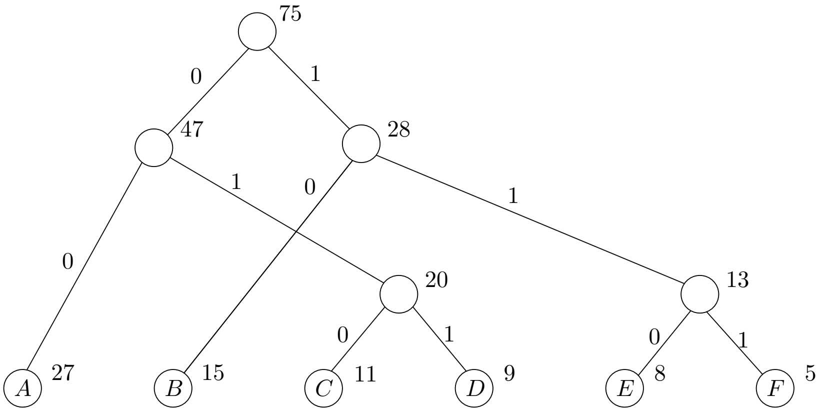

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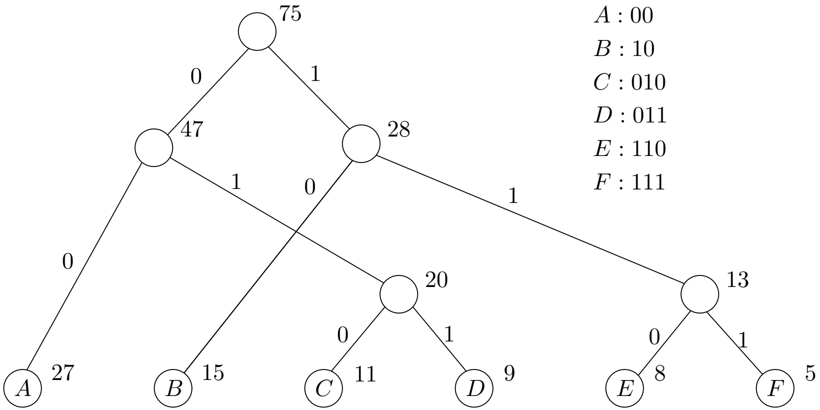


$A$ : 00

$B$ : 10

$C$ : 010

$D$ : 011

$E$ : 110

$F$ : 111

**Def.** The codes given the greedy algorithm is called the Huffman codes.

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### Huffman$(S, f)$

**while** $|S| > 1$ **do**
    let $x_1, x_2$ be the two letters with the smallest $f$ values
    introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
    let $x_1$ and $x_2$ be the two children of $x'$
    $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
**return** the tree constructed

# Algorithm using Priority Queue

**Huffman($S, f$)**

$Q \leftarrow$ build-priority-queue($S$)
**while** $Q.\text{size} > 1$ **do**
    $x_1 \leftarrow Q.\text{extract-min}()$
    $x_2 \leftarrow Q.\text{extract-min}()$
    introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
    let $x_1$ and $x_2$ be the two children of $x'$
    $Q.\text{insert}(x', f_{x'})$
**return** the tree constructed

# Outline

# Summary for Greedy Algorithms

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**Def.** A strategy is “safe” if there is always an optimum solution that “agrees with” the decision made according to the strategy.

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- Offline caching: trivial
- Huffman codes: merge two letters into one