Matrix Chain Multiplication: Design DP

- Assume the last step is $(A_1 A_2 \cdots A_i)(A_{i+1} A_{i+2} \cdots A_n)$
- Cost of last step: $r_1 \times c_i \times c_n$
- Optimality for sub-instances: we need to compute $A_1 A_2 \cdots A_i$ and $A_{i+1} A_{i+2} \cdots A_n$ optimally
- $opt[i, j]$: the minimum cost of computing $A_i A_{i+1} \cdots A_j$

$$opt[i,j] = \begin{cases} 0 & i = j \\ \min_{k:i \le k < j} \left(opt[i,k] + opt[k+1,j] + r_i c_k c_j \right) & i < j \end{cases}$$

Matrix Chain Multiplication: Design DP

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

```
1: let opt[i,i] ← 0 for every i = 1, 2, ··· , n
2: for ℓ ← 2 to n do
3:     for i ← 1 to n − ℓ + 1 do
4:         j ← i + ℓ − 1
5:         opt[i,j] ← ∞
6:         for k ← i to j − 1 do
7:             if opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j] then
8:                 opt[i,j] ← opt[i,k] + opt[k+1,j] + r_i c_k c_j
9: return opt[1,n]
```

Recover the Optimum Way of Multiplication

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

```
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3:      for i ← 1 to n − ℓ + 1 do
4:          j ← i + ℓ − 1
5:          opt[i,j] ← ∞
6:          for k ← i to j − 1 do
7:              if opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j] then
8:                  opt[i,j] ← opt[i,k] + opt[k+1,j] + r_i c_k c_j
9:                  π[i,j] ← k
10: return opt[1,n]
```

Constructing Optimal Solution

Print-Optimal-Order(i, j)

```
1: if i = j then
2:     print("A"_i)
3: else
4:     print("(")
5:     Print-Optimal-Order(i, π[i, j])
6:     Print-Optimal-Order(π[i, j] + 1, j)
7:     print(")")
```

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[1,2] &= opt[1,1] + opt[2,2] + 3 \times 5 \times 2 = 30, && \pi[1,2] = 1 \\
opt[2,3] &= opt[2,2] + opt[3,3] + 5 \times 2 \times 6 = 60, && \pi[2,3] = 2 \\
opt[3,4] &= opt[3,3] + opt[4,4] + 2 \times 6 \times 9 = 108, && \pi[3,4] = 3 \\
opt[4,5] &= opt[4,4] + opt[5,5] + 6 \times 9 \times 4 = 216, && \pi[4,5] = 4 \\
opt[1,3] &= \min\{opt[1,1] + opt[2,3] + 3 \times 5 \times 6, \\
&\qquad\qquad opt[1,2] + opt[3,3] + 3 \times 2 \times 6\} \\
&= \min\{0 + 60 + 90, 30 + 0 + 36\} = 66, && \pi[1,3] = 2 \\
opt[2,4] &= \min\{opt[2,2] + opt[3,4] + 5 \times 2 \times 9, \\
&\qquad\qquad opt[2,3] + opt[4,4] + 5 \times 6 \times 9\} \\
&= \min\{0 + 108 + 90, 60 + 0 + 270\} = 198, && \pi[2,4] = 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[3,5] &= \min\{opt[3,3] + opt[4,5] + 2 \times 6 \times 4, \\
&\qquad\qquad opt[3,4] + opt[5,5] + 2 \times 9 \times 4\} \\
&= \min\{0 + 216 + 48, 108 + 0 + 72\} = 180, \\
\pi[3,5] &= 4, \\
opt[1,4] &= \min\{opt[1,1] + opt[2,4] + 3 \times 5 \times 9, \\
&\qquad\qquad opt[1,2] + opt[3,4] + 3 \times 2 \times 9, \\
&\qquad\qquad opt[1,3] + opt[4,4] + 3 \times 6 \times 9\} \\
&= \min\{0 + 198 + 135, 30 + 108 + 54, 66 + 0 + 162\} = 192, \\
\pi[1,4] &= 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[2,5] &= \min\{opt[2,2] + opt[3,5] + 5 \times 2 \times 4, \\
&\qquad opt[2,3] + opt[4,5] + 5 \times 6 \times 4, \\
&\qquad opt[2,4] + opt[5,5] + 5 \times 9 \times 4\} \\
&= \min\{0 + 180 + 40, 60 + 216 + 120, 198 + 0 + 180\} = 220, \\
opt[1,5] &= \min\{opt[1,1] + opt[2,5] + 3 \times 5 \times 4, \\
&\qquad opt[1,2] + opt[3,5] + 3 \times 2 \times 4, \\
&\qquad opt[1,3] + opt[4,5] + 3 \times 6 \times 4, \\
&\qquad opt[1,4] + opt[5,5] + 3 \times 9 \times 4\} \\
&= \min\{0 + 220 + 60, 30 + 180 + 24, \\
&\qquad 66 + 216 + 72, 192 + 0 + 108\} \\
&= 234, \\
\pi[1,5] &= 2.
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

opt, π	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	0, /	30, 1	66, 2	192, 2	234, 2
$i=2$		0, /	60, 2	198, 2	220, 2
$i=3$			0, /	108, 3	180, 4
$i=4$				0, /	216, 4
$i=5$					0, /

Print-Optimal-Order(1,5)
- Print-Optimal-Order(1, 2)
  - Print-Optimal-Order(1, 1)
  - Print-Optimal-Order(2, 2)
- Print-Optimal-Order(3, 5)
  - Print-Optimal-Order(3, 4)
    - Print-Optimal-Order(3, 3)
    - Print-Optimal-Order(4, 4)
  - Print-Optimal-Order(5, 5)

Optimum way for multiplication: $((A_1A_2)((A_3A_4)A_5))$

Outline

Optimum Binary Search Tree

Def. Binary search tree (BST), also called an ordered or sorted binary tree, is a rooted binary tree data structure with the key of each internal node being greater than all the keys in the respective node's left subtree and less than the ones in its right subtree.

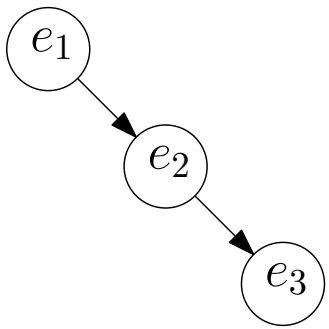
Optimum Binary Search Tree

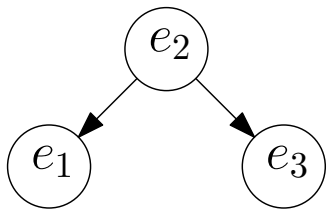
- n elements $e_1 < e_2 < e_3 < \cdots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \cdots, e_n\}$ with the minimum accessing cost:

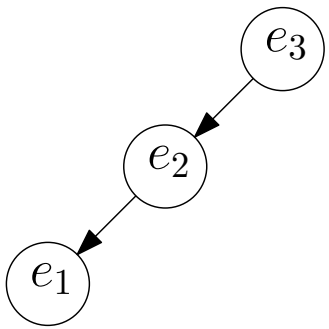
$$\sum_{i=1}^{n} f_i \times (\text{depth of } e_i \text{ in the tree})$$

Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$







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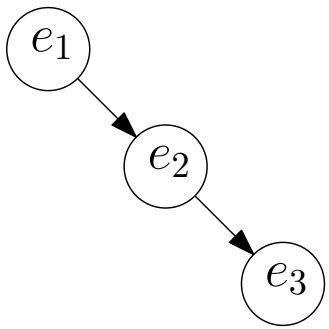
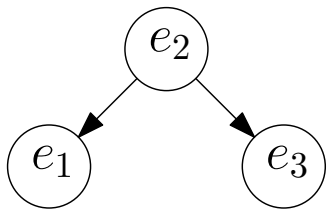
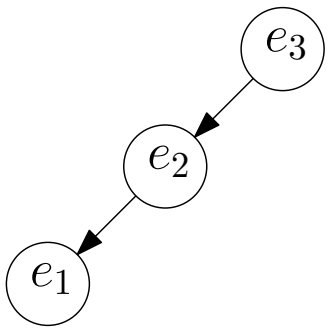


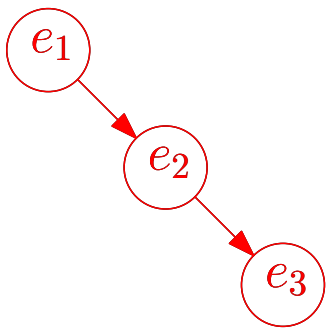


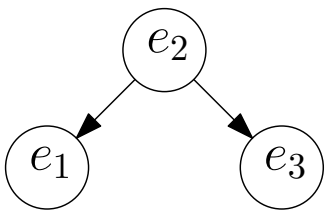


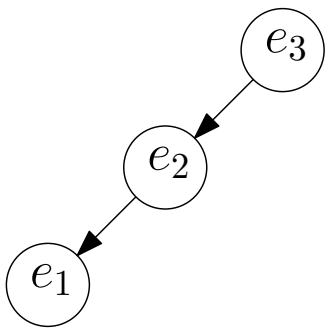
- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$

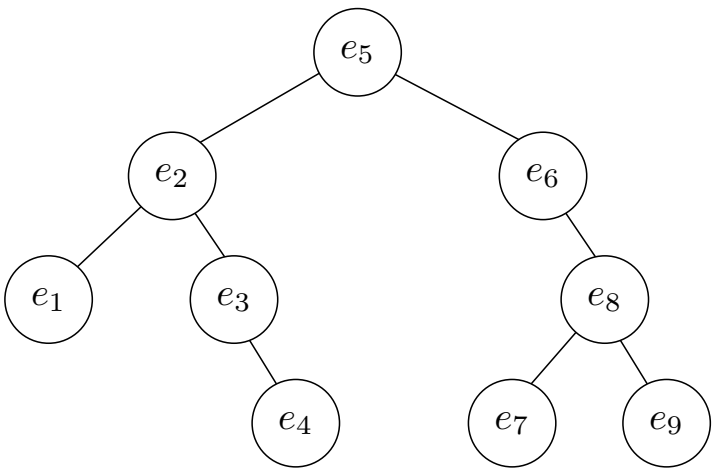




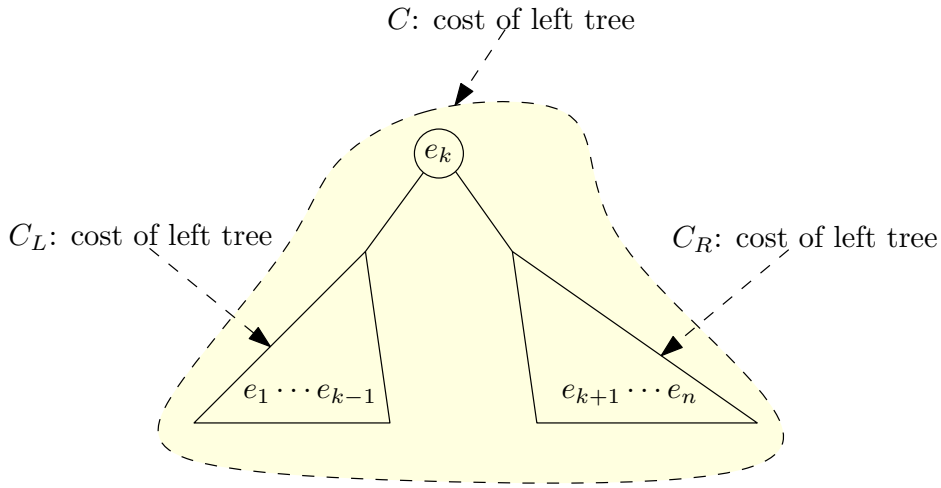


- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

- suppose we decided to let e_k be the root
- $e_1, e_2, \cdots, e_{k-1}$ are on left sub-tree
- $e_{k+1}, e_{k+2}, \cdots, e_n$ are on right sub-tree
- d_j: depth of e_j in our tree
- C, C_L, C_R: cost of tree, left sub-tree and right sub-tree



- $d_1 = 3, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 1,$
- $d_6 = 2, d_7 = 4, d_8 = 3, d_9 = 4,$
- $C = 3f_1 + 2f_2 + 3f_3 + 4f_4 + f_5 + 2f_6 + 4f_7 + 3f_8 + 4f_9$
- $C_L = 2f_1 + f_2 + 2f_3 + 3f_4$
- $C_R = f_6 + 3f_7 + 2f_8 + 3f_9$
- $C = C_L + C_R + \sum_{j=1}^{9} f_j$



$$
\begin{aligned}
C &= \sum_{\ell=1}^{n} f_\ell d_\ell = \sum_{\ell=1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= \sum_{\ell=1}^{k-1} f_\ell (d_\ell - 1) + \sum_{\ell=k+1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= C_L + C_R + \sum_{\ell=1}^{n} f_\ell
\end{aligned}
$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively

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- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

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- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$opt[1, n] = \qquad (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
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$$opt[1, n] = \min_{k:1 \le k \le n} (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

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- In order to minimize C, need to minimize C_L and C_R respectively
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$$opt[1, n] = \min_{k:1 \le k \le n} \left(opt[1, k-1] + opt[k+1, n]\right) + \sum_{\ell=1}^{n} f_\ell$$

- In general, $opt[i, j] =$

$$\begin{cases} 0 & \text{if } i = j + 1 \\ \min_{k:i \le k \le j} \left(opt[i, k-1] + opt[k+1, j]\right) + \sum_{\ell=i}^{j} f_\ell & \text{if } i \le j \end{cases}$$

Optimum Binary Search Tree

1: $fsum[0] \leftarrow 0$
2: **for** $i \leftarrow 1$ to n **do** $fsum[i] \leftarrow fsum[i-1] + f_i$
$\triangleright\ fsum[i] = \sum_{j=1}^{i} f_j$
3: **for** $i \leftarrow 0$ to n **do** $opt[i+1, i] \leftarrow 0$
4: **for** $\ell \leftarrow 1$ to n **do**
5: **for** $i \leftarrow 1$ to $n - \ell + 1$ **do**
6: $j \leftarrow i + \ell - 1$, $opt[i, j] \leftarrow \infty$
7: **for** $k \leftarrow i$ to j **do**
8: **if** $opt[i, k-1] + opt[k+1, j] < opt[i, j]$ **then**
9: $opt[i, j] \leftarrow opt[i, k-1] + opt[k+1, j]$
10: $\pi[i, j] \leftarrow k$
11: $opt[i, j] \leftarrow opt[i, j] + fsum[j] - fsum[i-1]$

Printing the Tree

Print-Tree(i, j)

```
1: if i > j then
2:     return
3: else
4:     print('(')
5:     Print-Tree(i, π[i, j] − 1)
6:     print(π[i, j])
7:     Print-Tree(π[i, j] + 1, j)
8:     print(')')
```

Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree: $opt[i, j]$ = value of instances defined by matrices i to j