CSE 431/531: Algorithm Analysis and Design (Spring 2024)
Divide-and-Conquer

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Outline

Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

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- trivial algorithm runs in exponential time
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- main focus of analysis: correctness of algorithm

Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

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Running time analysis

- recursive programs: recurrence

merge-sort(A, n)

```
if n = 1 then
    return A
else
    B ← merge-sort(A[1..⌊n/2⌋], ⌊n/2⌋)
    C ← merge-sort(A[⌊n/2⌋ + 1..n], ⌈n/2⌉)
    return merge(B, C, ⌊n/2⌋, ⌈n/2⌉)
```

merge-sort(A, n)

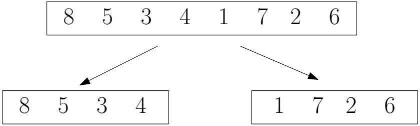
1: **if** $n = 1$ **then**
2:     **return** A
3: **else**
4:     $B \leftarrow$ merge-sort$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
5:     $C \leftarrow$ merge-sort$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], \lceil n/2 \rceil\Big)$
6:     **return** merge$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$

- Divide: trivial
- Conquer: 4, 5
- Combine: 6

merge-sort()

8	5	3	4	1	7	2	6

merge-sort()



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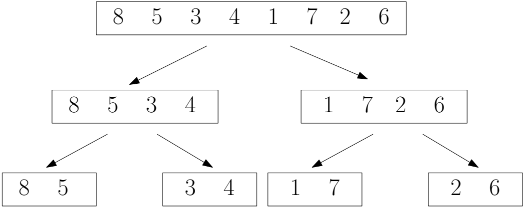

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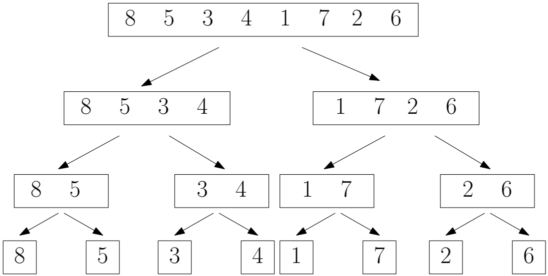

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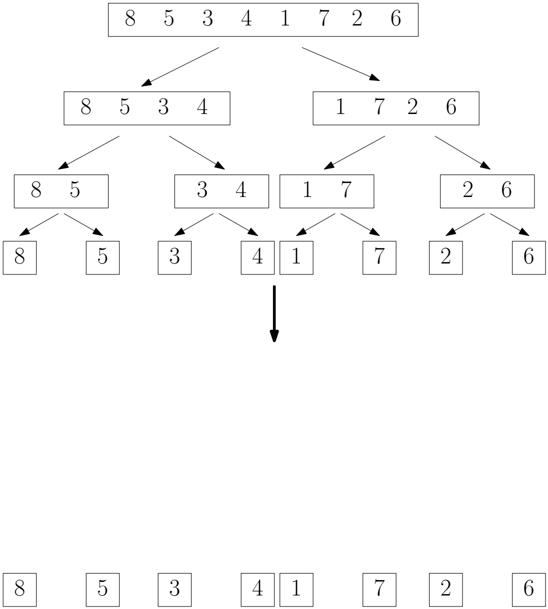

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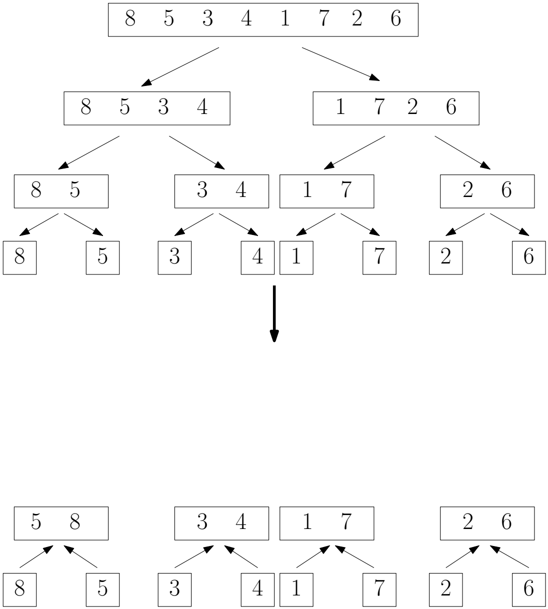

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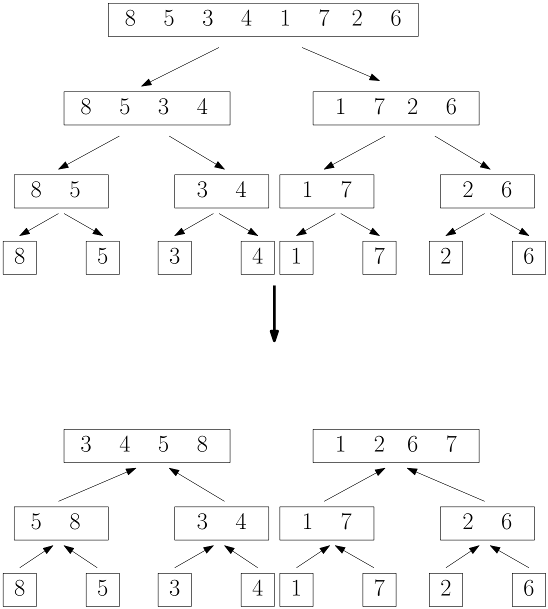

8 5 3 4 1 7 2 6
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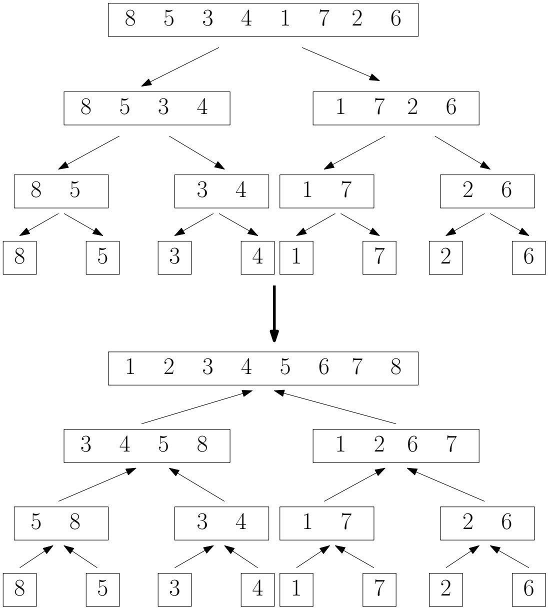
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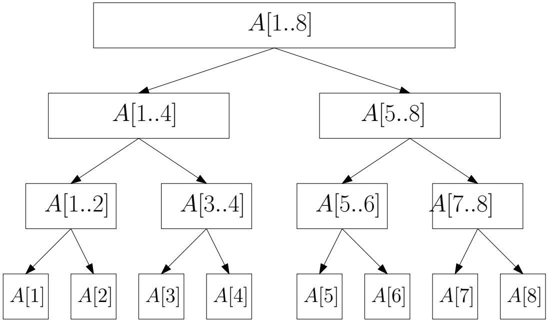
6

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$
- Better than insertion sort

Running Time for Merge-Sort

Implementation

- Divide $A[a, b]$ by $q = \lfloor (a+b)/2 \rfloor$: $A[a, q]$ and $A[q+1, b]$; or $A[a, q-1]$ and $A[q, b]$?

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Stable sorting algorithm

- Stable sorting algorithm has the property that equal items will appear in the final sorted list in the same relative order that they appeared in the initial input.

Running Time for Merge-Sort Using Recurrence

- $T(n) =$ running time for sorting n numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

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Example:

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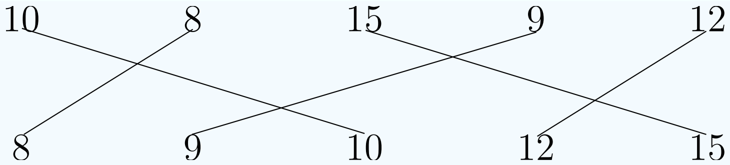

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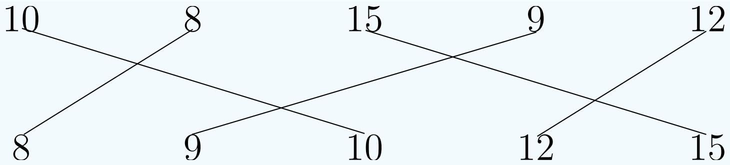


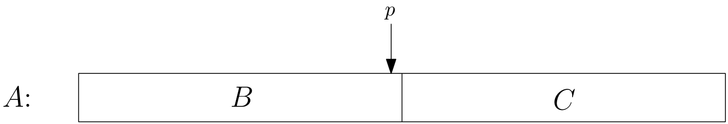
- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

Naive Algorithm for Counting Inversions

count-inversions(A, n)

$c \leftarrow 0$
for every $i \leftarrow 1$ to $n-1$ **do**
    for every $j \leftarrow i+1$ to n **do**
        if $A[i] > A[j]$ **then** $c \leftarrow c+1$
return c

Divide-and-Conquer



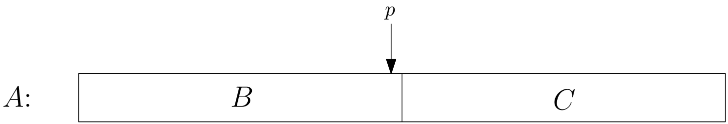
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$
  $$m = \big|\{(i,j) : B[i] > C[j]\}\big|$$

Q: How fast can we compute m, via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer



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Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:	3	8	12	20	32	48

total$=$ 0

C:	5	7	9	25	29

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$+0$

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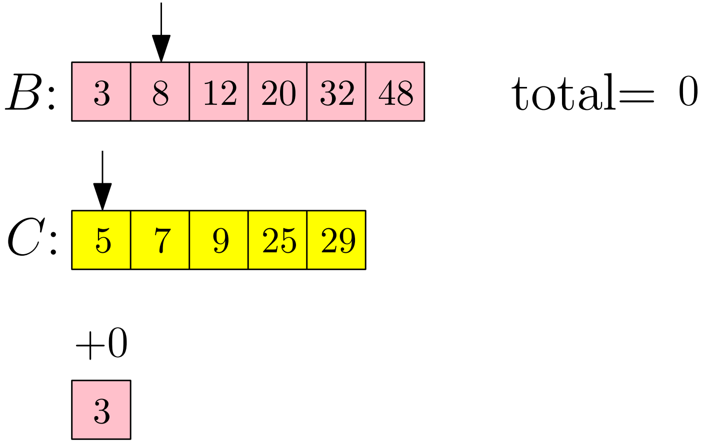

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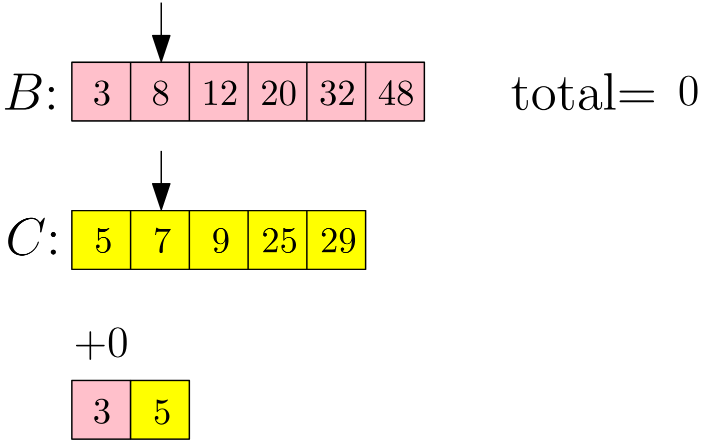

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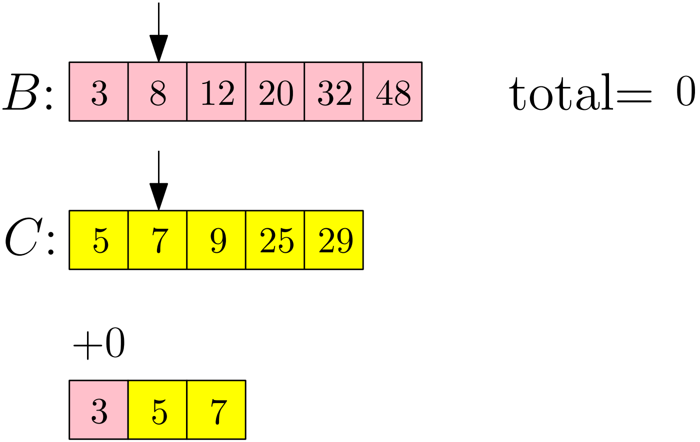

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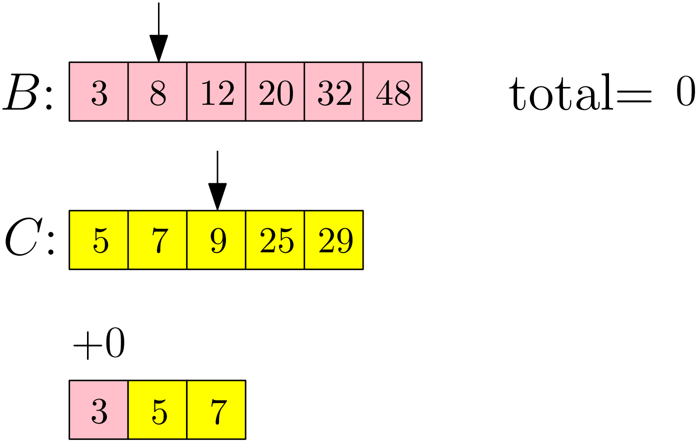

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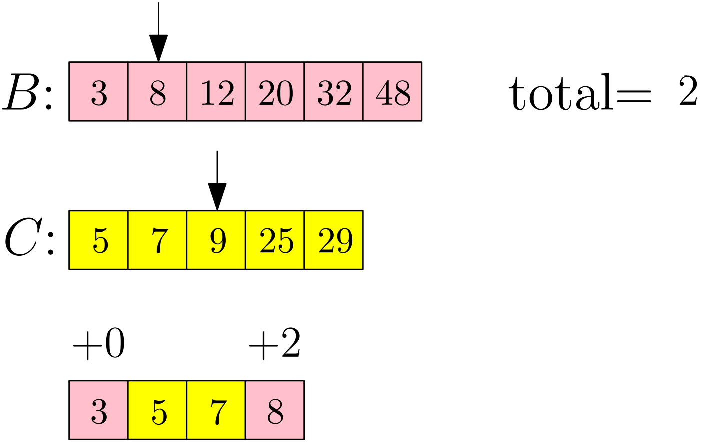

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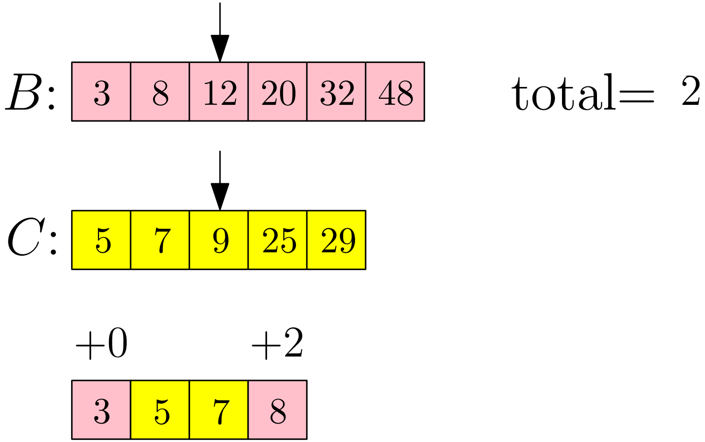

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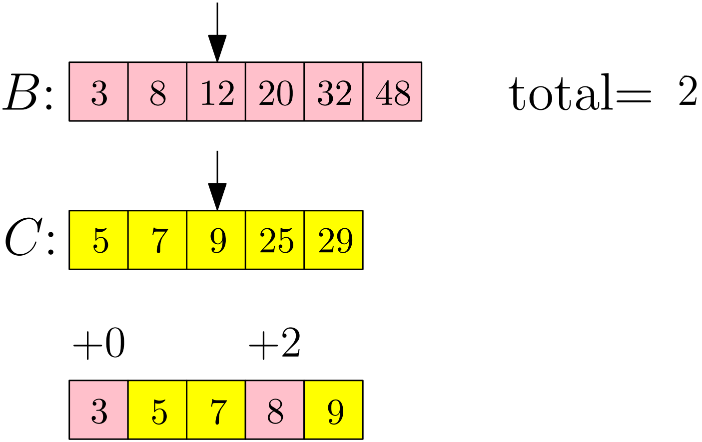

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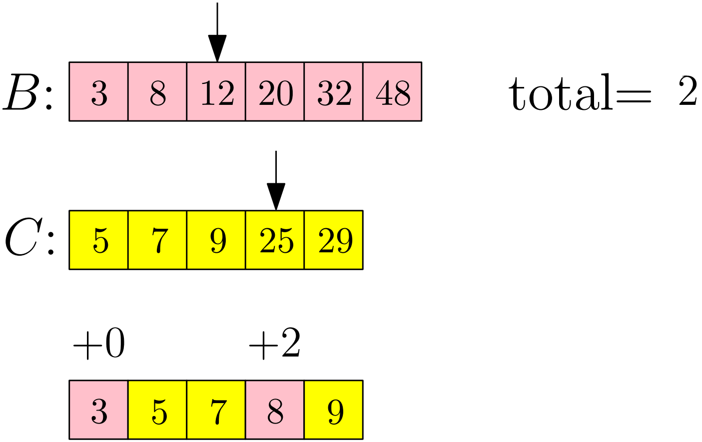

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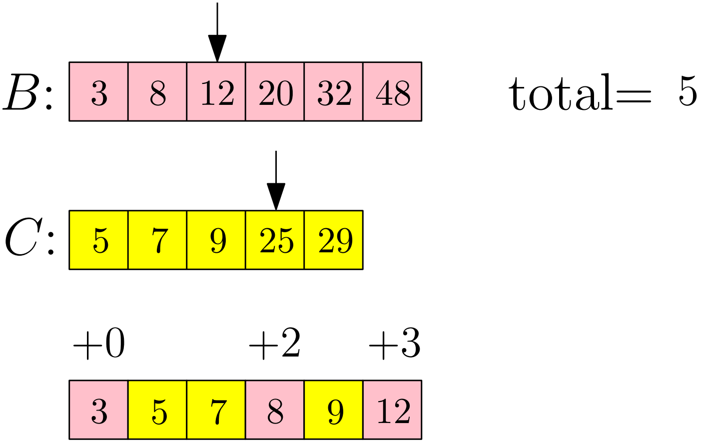

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:

3	8	12	20	32	48

total$=$ 5

C:

5	7	9	25	29

+0			+2		+3
3	5	7	8	9	12

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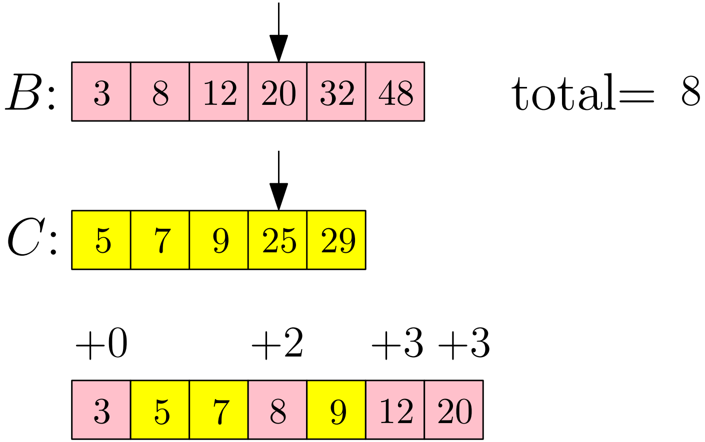

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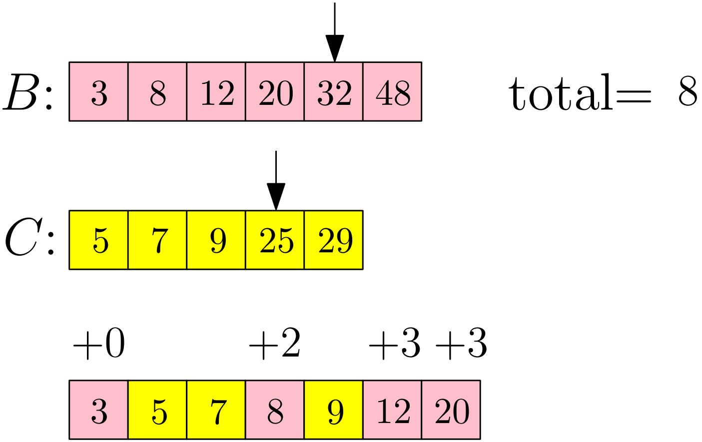

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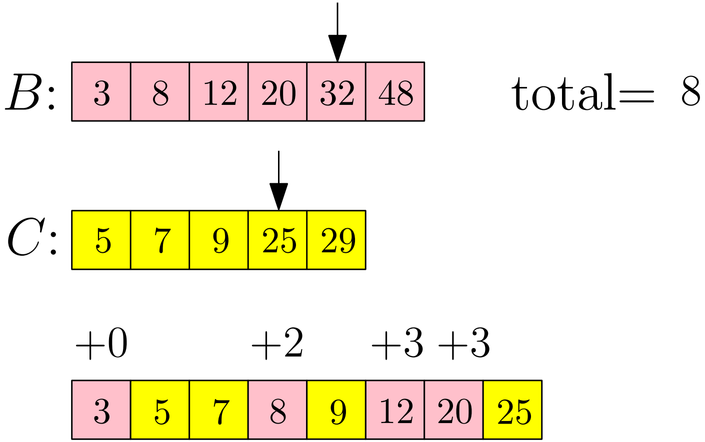

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:

3	8	12	20	32	48

total$=$ 8

C:

5	7	9	25	29

$+0$ $+2$ $+3$ $+3$

3	5	7	8	9	12	20	25

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Count pairs i, j such that $B[i] > C[j]$:

B:

3	8	12	20	32	48

total= 8

C:

5	7	9	25	29

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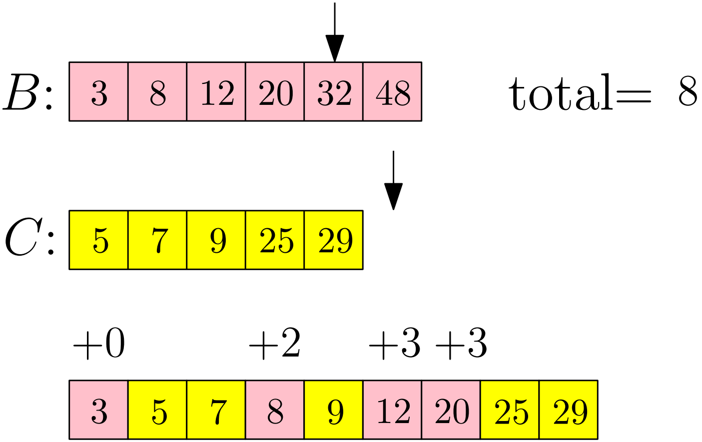

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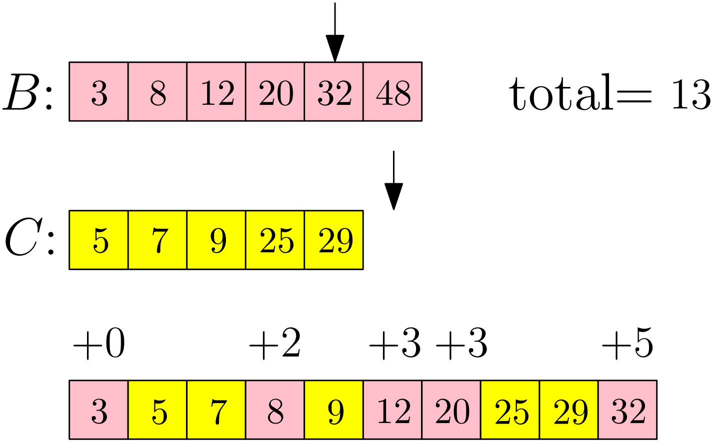

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:

3	8	12	20	32	48

total $= 13$

C:

5	7	9	25	29

+0			+2		+3	+3			+5
3	5	7	8	9	12	20	25	29	32

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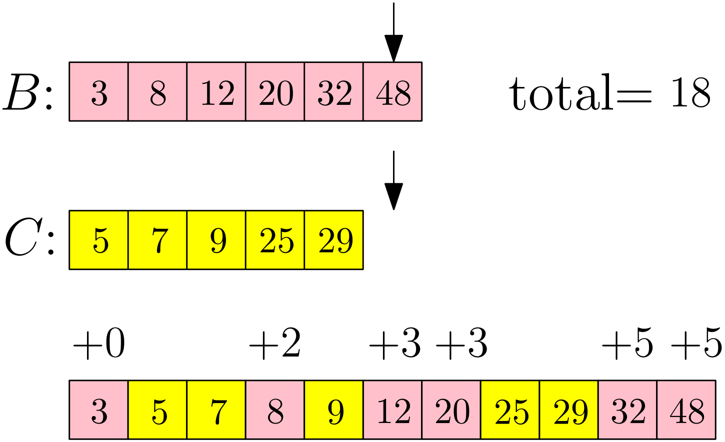

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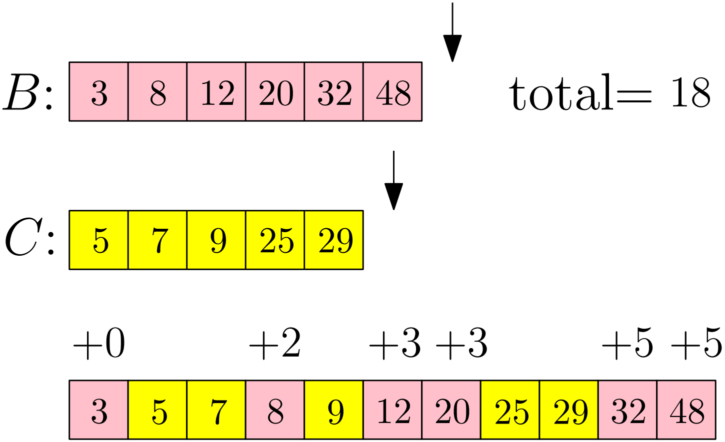

Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

$count \leftarrow 0$;
$A \leftarrow$ array of size $n_1 + n_2$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ or $j \leq n_2$ **do**
    if $j > n_2$ or $(i \leq n_1$ and $B[i] \leq C[j])$ **then**
        $A[i+j-1] \leftarrow B[i]$; $i \leftarrow i+1$
        $count \leftarrow count + (j-1)$
    else
        $A[i+j-1] \leftarrow C[j]$; $j \leftarrow j+1$
return $(A, count)$

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A:

sort-and-count(A, n)

if $n = 1$ **then**
  return $(A, 0)$
else
  $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
  $(C, m_2) \leftarrow$ sort-and-count$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], \lceil n/2 \rceil\Big)$
  $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
  return $(A, m_1 + m_2 + m_3)$

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- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$

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else
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    (C, m_2) ← sort-and-count(A[⌊n/2⌋ + 1..n], ⌈n/2⌉)
    (A, m_3) ← merge-and-count(B, C, ⌊n/2⌋, ⌈n/2⌉)
    return (A, m_1 + m_2 + m_3)
```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$
- Running time $= O(n \lg n)$

Outline

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Quicksort vs Merge-Sort

	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

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Quicksort

quicksort(A, n)

if $n \leq 1$ **then return** A
$x \leftarrow$ lower median of A
$A_L \leftarrow$ array of elements in A that are less than x \\\\ Divide
$A_R \leftarrow$ array of elements in A that are greater than x \\\\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$ \\\\ Conquer
$B_R \leftarrow$ quicksort$(A_R,$ length of $A_R)$ \\\\ Conquer
$t \leftarrow$ number of times x appear A
return concatenation of B_L, t copies of x, and B_R

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1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)

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Q: How to remove this assumption?

A:

1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
2. Choose a pivot randomly and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

```
if n ≤ 1 then return A
x ← a random element of A (x is called a pivot)
A_L ← array of elements in A that are less than x        \\ Divide
A_R ← array of elements in A that are greater than x     \\ Divide
B_L ← quicksort(A_L, length of A_L)                      \\ Conquer
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t ← number of times x appear A
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Randomized Algorithm Model

Assumption There is a procedure to produce a random real number in $[0, 1]$.

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A: No! The execution of a computer programs is deterministic!

- In practice: use pseudo-random-generator, a deterministic algorithm returning numbers that "look like" random
- In theory: assume they can.

Quicksort Using A Random Pivot

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Lemma The expected running time of the algorithm is $O(n \lg n)$.

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- To partition the array into two parts, we only need $O(1)$ extra space.