Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x+2)x^2 + (-5x+4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x-3)x^2 + (6x-5)$$

- $p(x)$: degree of $n-1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right) x^{n/2} + p_L q_L \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

$$
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$$

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$$\begin{aligned} \text{multiply}(p, q) = {} & \text{multiply}(p_H, q_H) \times x^n \\ & + \big(\text{multiply}(p_H, q_L) + \text{multiply}(p_L, q_H)\big) \times x^{n/2} \\ & + \text{multiply}(p_L, q_L) \end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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&+ \text{multiply}(p_L, q_L)
\end{aligned}
$$

- Recurrence: $T(n) = 4T(n/2) + O(n)$

Divide-and-Conquer for Polynomial Multiplication

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- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

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pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\
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Reduce Number from 4 to 3

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&= p_H q_H x^n + \big(p_H q_L + p_L q_H\big) x^{n/2} + p_L q_L
\end{aligned}
$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \textsf{multiply}(p_H, q_H)$$
$$r_L = \textsf{multiply}(p_L, q_L)$$

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \mathsf{multiply}(p_H, q_H)$$
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$$\begin{aligned}\mathsf{multiply}(p, q) = {} & r_H \times x^n \\ & + \big(\mathsf{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L\big) \times x^{n/2} \\ & + r_L\end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$

Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \,..\, n/2-1], A_H \leftarrow A[n/2 \,..\, n-1]$
$B_L \leftarrow B[0 \,..\, n/2-1], B_H \leftarrow B[n/2 \,..\, n-1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n-1)$ 0's
for $i \leftarrow 0$ to $n-2$ **do**
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i+n] \leftarrow C[i+n] + C_H[i]$
    $C[i+n/2] \leftarrow C[i+n/2] + C_M[i] - C_L[i] - C_H[i]$
return C

Outline

Methods for Solving Recurrences

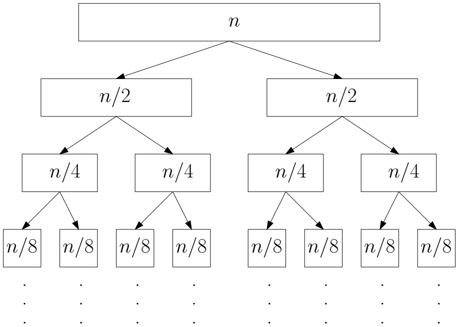
- The recursion-tree method
- The master theorem

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$

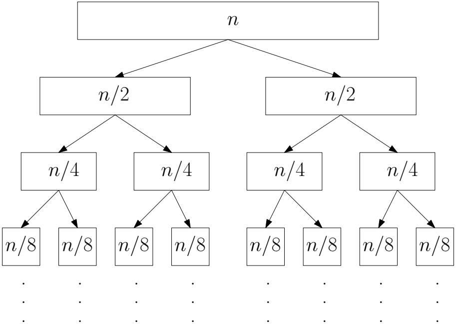
Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$



Recursion-Tree Method

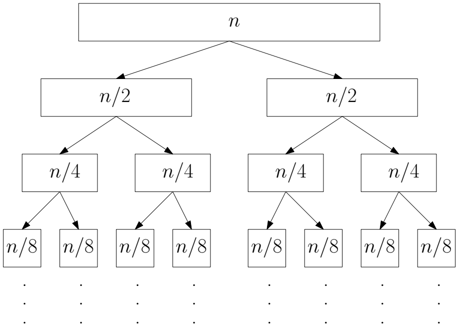
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$

Recursion-Tree Method

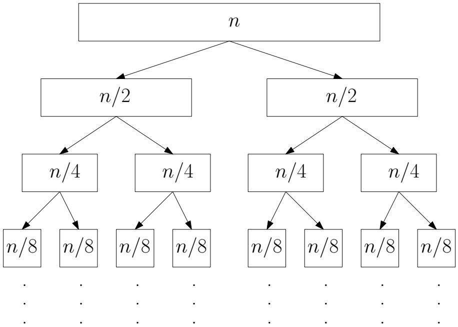
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

n

Recursion-Tree Method

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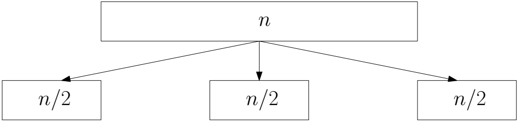

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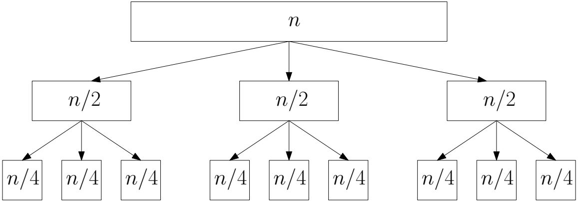

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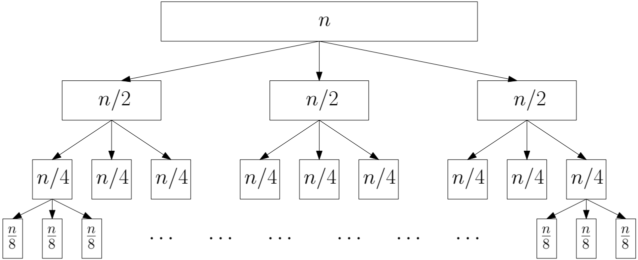

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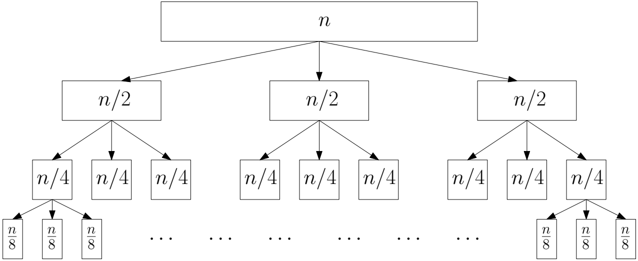
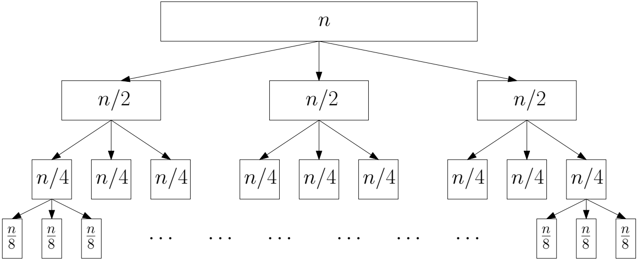


- Total running time at level i?

Recursion-Tree Method

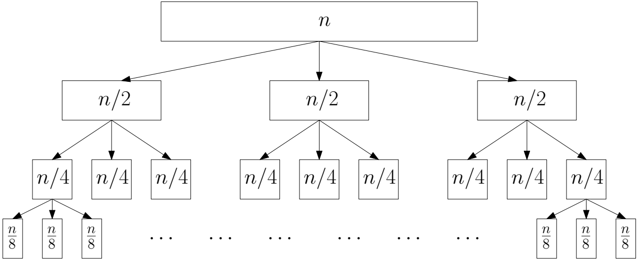
- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$



- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
- Index of last level?

Recursion-Tree Method

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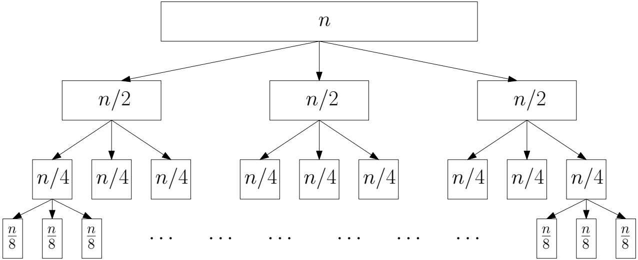

- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
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Recursion-Tree Method

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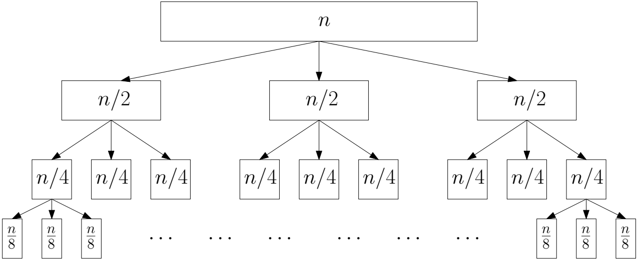

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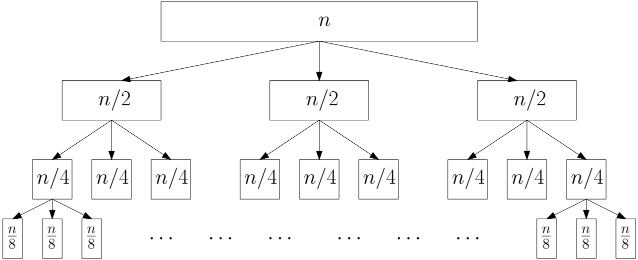

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- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n\left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$

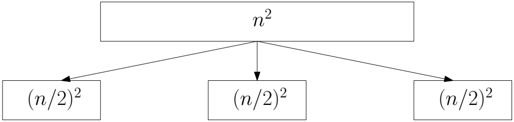
Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$

n^2

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



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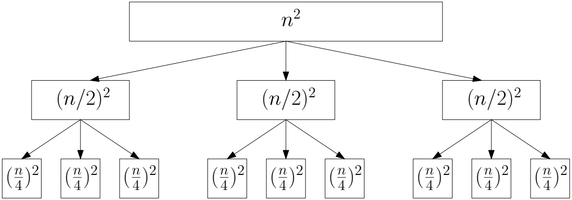

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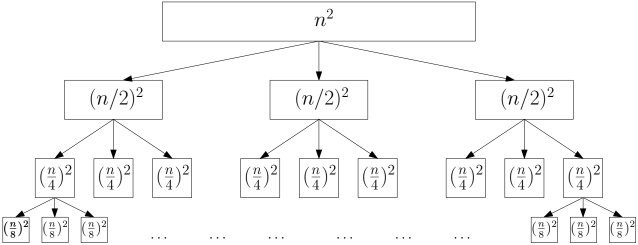

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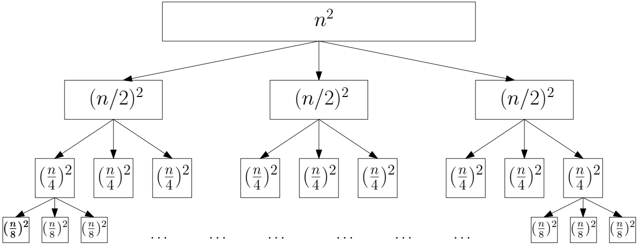

- Total running time at level i?

Recursion-Tree Method

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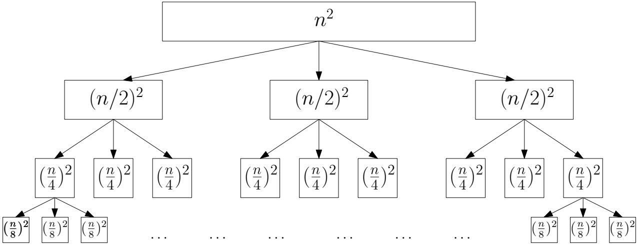

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$

Recursion-Tree Method

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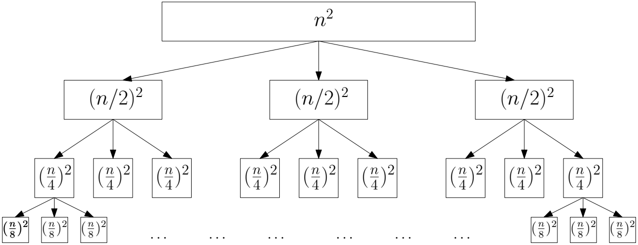

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level?

Recursion-Tree Method

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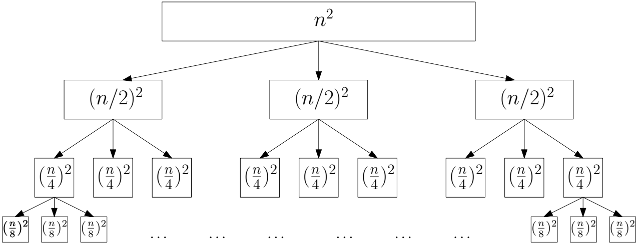

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Recursion-Tree Method

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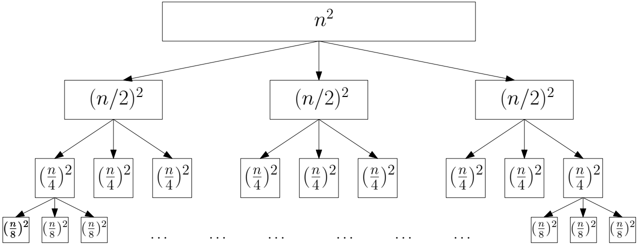

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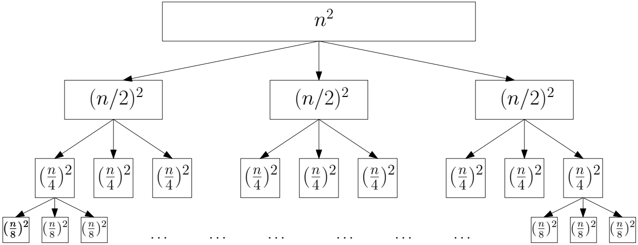

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- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 =$$

Recursion-Tree Method

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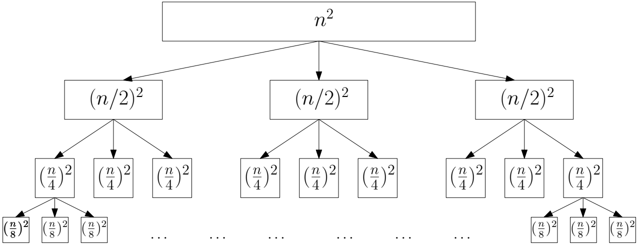

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- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$				$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

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Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

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Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} & \text{if } c < \lg_b a \\ & \text{if } c = \lg_b a \\ & \text{if } c > \lg_b a \end{cases}$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
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Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} ?? & \text{if } c < \lg_b a \\ & \text{if } c = \lg_b a \\ & \text{if } c > \lg_b a \end{cases}$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
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Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
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Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
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Master Theorem

Recurrences	a	b	c	time
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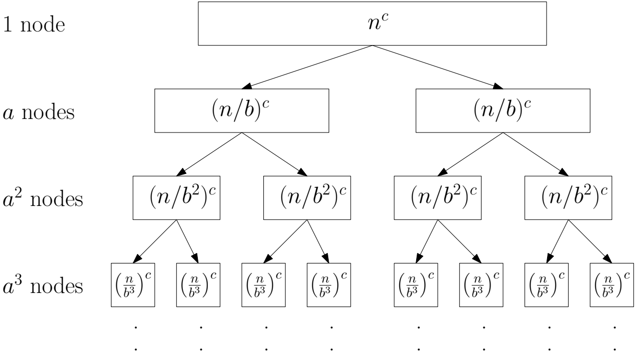
Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



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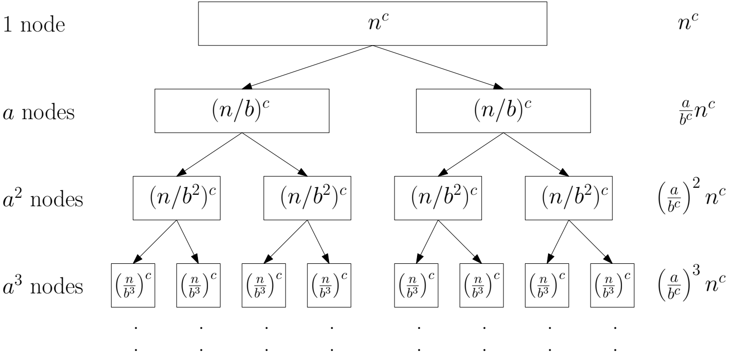

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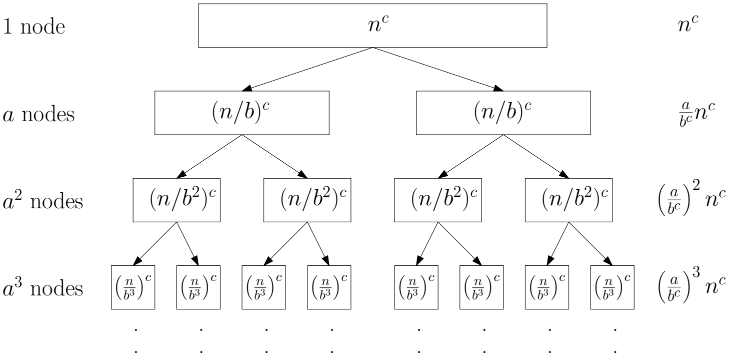

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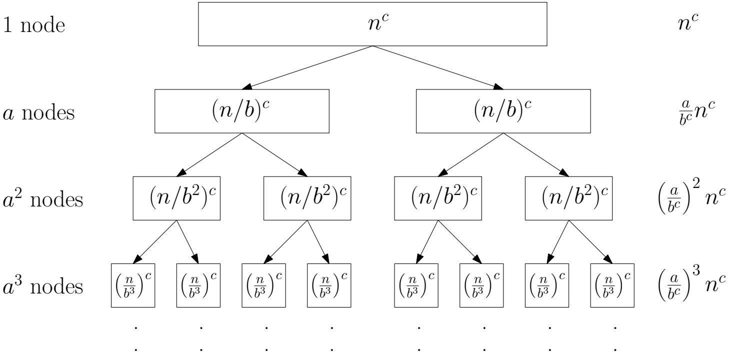

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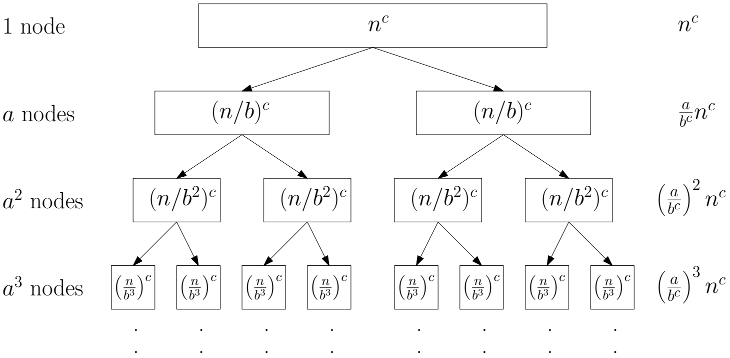

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- $c = \lg_b a$: all levels have same time: $n^c \lg_b n = O(n^c \lg n)$
- $c > \lg_b a$: top-level dominates: $O(n^c)$

Outline

Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

n-th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

1: if $n = 0$ return 0
2: if $n = 1$ return 1
3: return Fib$(n-1)$ + Fib$(n-2)$

Q: Is the running time of the algorithm polynomial or exponential in n?

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- Running time is at least $\Omega(F_n)$

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A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n: Reasonable Algorithm

Fib(n)

```
F[0] ← 0
F[1] ← 1
for i ← 2 to n do
    F[i] ← F[i − 1] + F[i − 2]
return F[n]
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- Dynamic Programming

Computing F_n: Reasonable Algorithm

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- Dynamic Programming
- Running time = ?

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- Dynamic Programming
- Running time $= O(n)$

Computing F_n: Even Better Algorithm

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\cdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

power(n)

if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$R \leftarrow$ power($\lfloor n/2 \rfloor$)

$R \leftarrow R \times R$

if n is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

return R

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Running time $= O(\lg n)$: We Cheated!

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- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
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Fixing the Problem

To compute F_n, we need $O(\lg n)$ basic arithmetic operations on integers

Summary: Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

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- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions:
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- To improve running time, design better algorithm for "combine" step, or reduce number of recursions, ...