“Evidence” for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.

Outline

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest path from s to t

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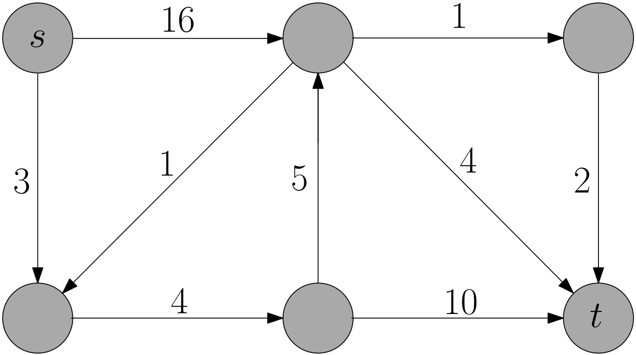

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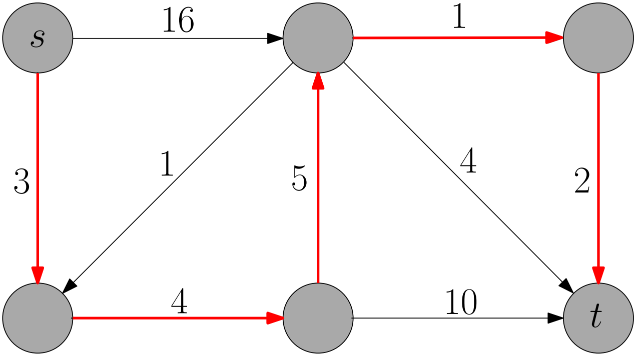

Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$

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Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

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Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi[v], v \in V \setminus s$: the parent of v in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from s to v

Q: How to compute shortest paths from s when all edges have weight 1?

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A: Breadth first search (BFS) from source s

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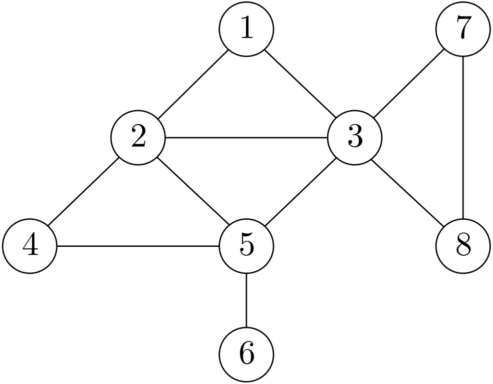

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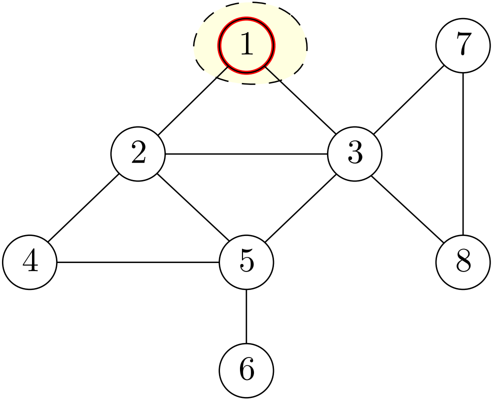

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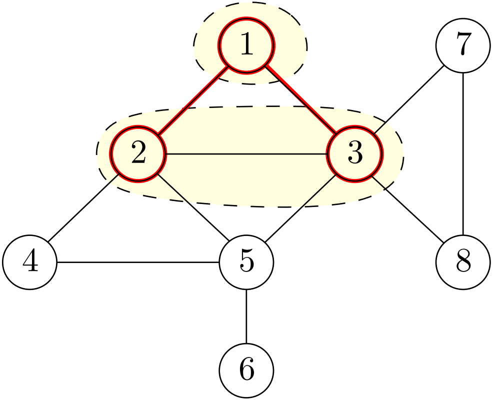

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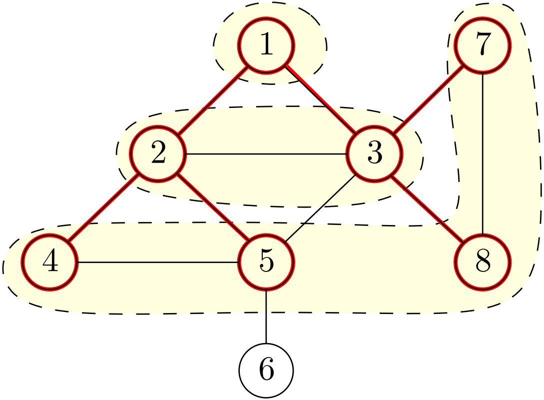

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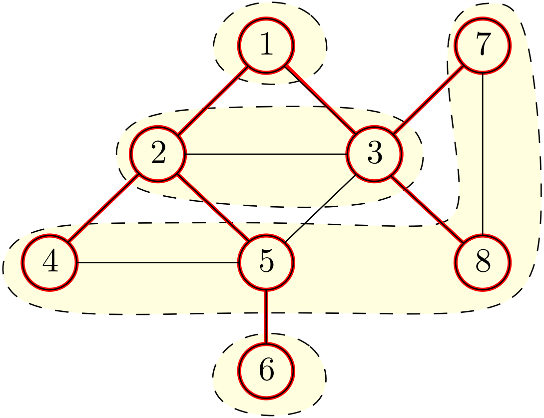

Assumption Weights $w(u, v)$ are integers (w.l.o.g).

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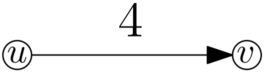

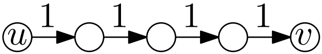

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Shortest Path Algorithm by Running BFS

1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
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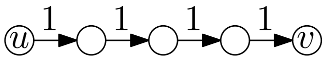

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Shortest Path Algorithm by Running BFS

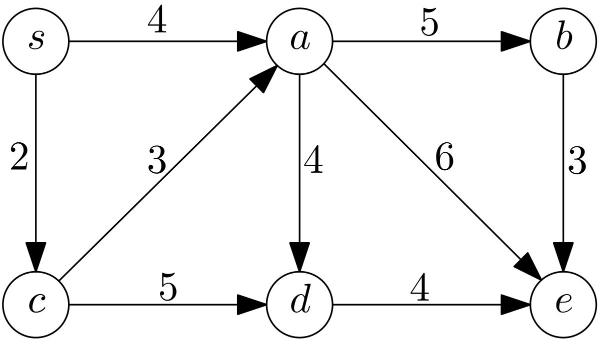
1: replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2: run BFS virtually
3: $\pi[v] \leftarrow$ vertex from which v is visited
4: $d[v] \leftarrow$ index of the level containing v

- Problem: $w(u,v)$ may be too large!

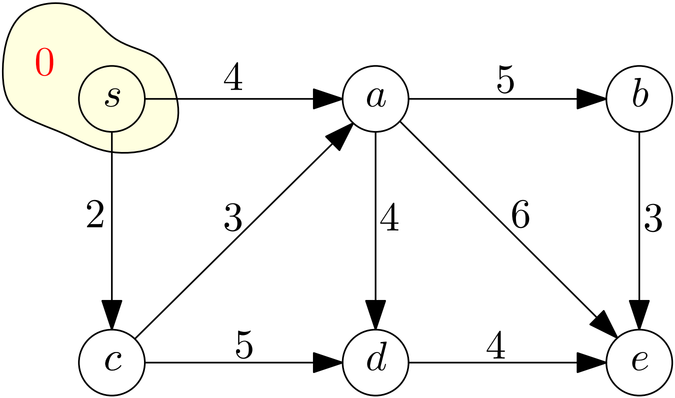
Shortest Path Algorithm by Running BFS Virtually

$S \leftarrow \{s\}, d(s) \leftarrow 0$
while $|S| \leq n$ **do**
    find a $v \notin S$ that minimizes $\min\limits_{u \in S:(u,v) \in E} \{d[u] + w(u,v)\}$
    $S \leftarrow S \cup \{v\}$
    $d[v] \leftarrow \min_{u \in S:(u,v) \in E} \{d[u] + w(u,v)\}$

Virtual BFS: Example

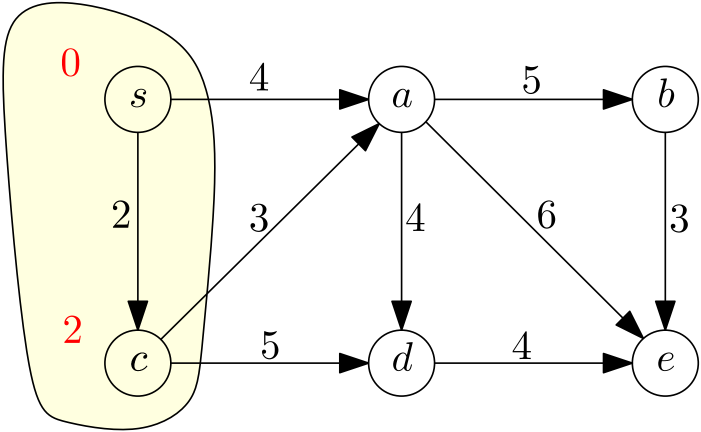


Virtual BFS: Example



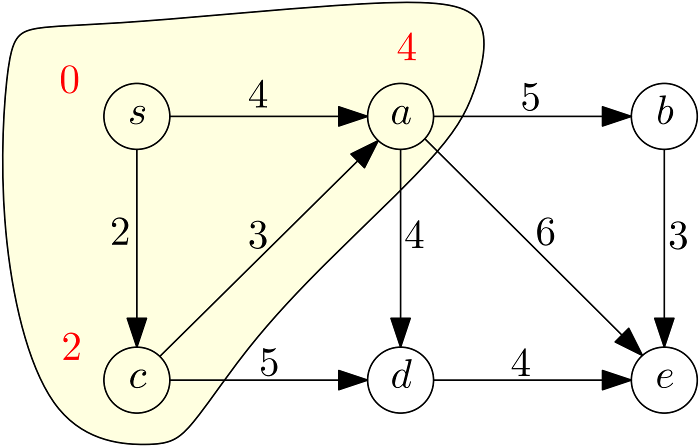
Time 0

Virtual BFS: Example



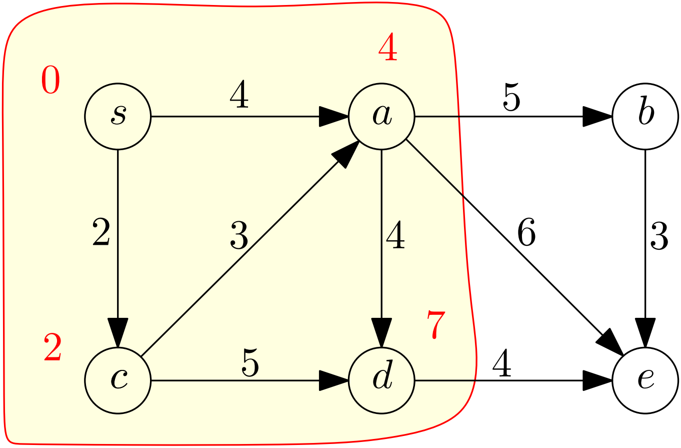
Time 2

Virtual BFS: Example



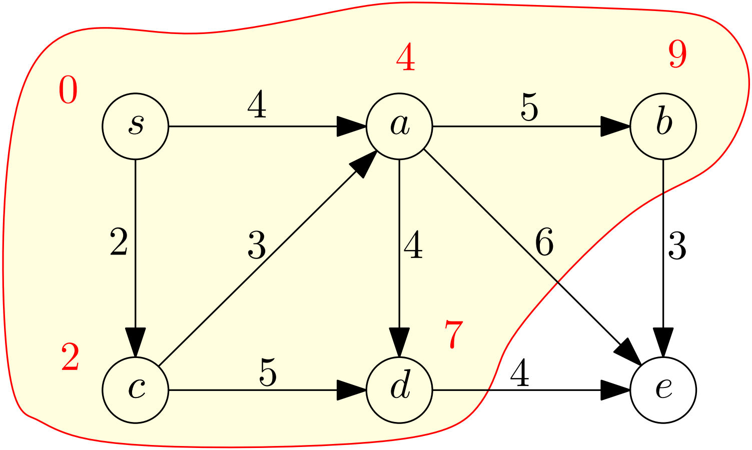
Time 4

Virtual BFS: Example



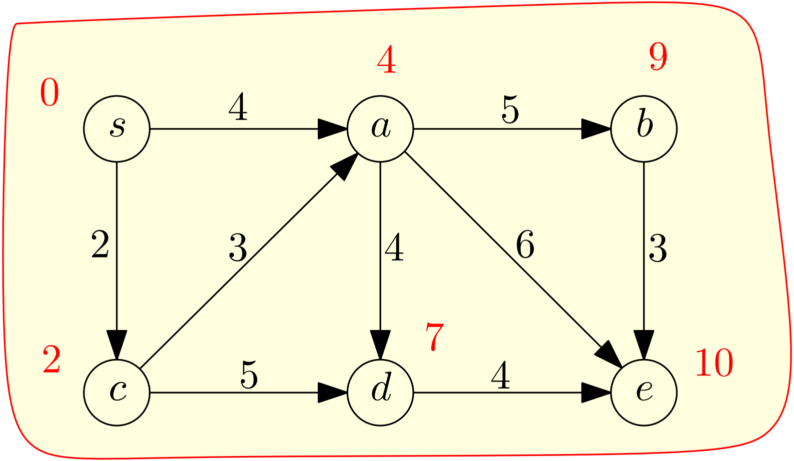
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

Dijkstra's Algorithm

Dijkstra(G, w, s)

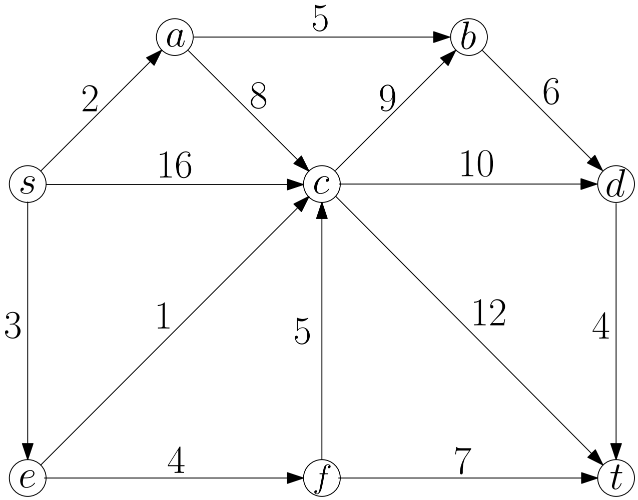
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    add u to S
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
return (d, π)

Dijkstra's Algorithm

Dijkstra(G, w, s)

```
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
while S ≠ V do
    u ← vertex in V \ S with the minimum d[u]
    add u to S
    for each v ∈ V \ S such that (u, v) ∈ E do
        if d[u] + w(u, v) < d[v] then
            d[v] ← d[u] + w(u, v)
            π[v] ← u
return (d, π)
```

- Running time $= O(n^2)$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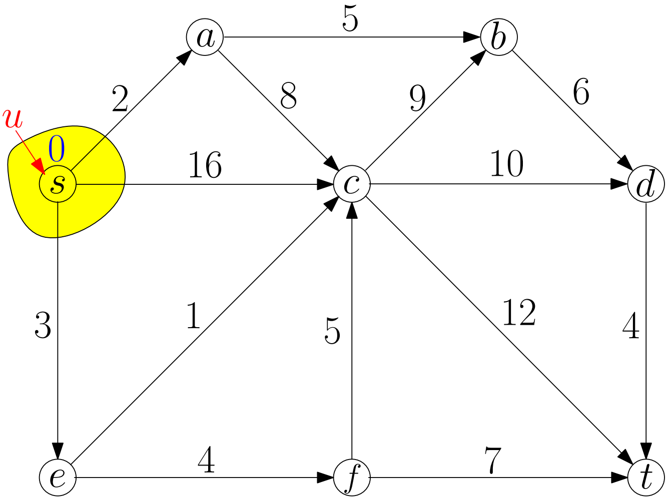

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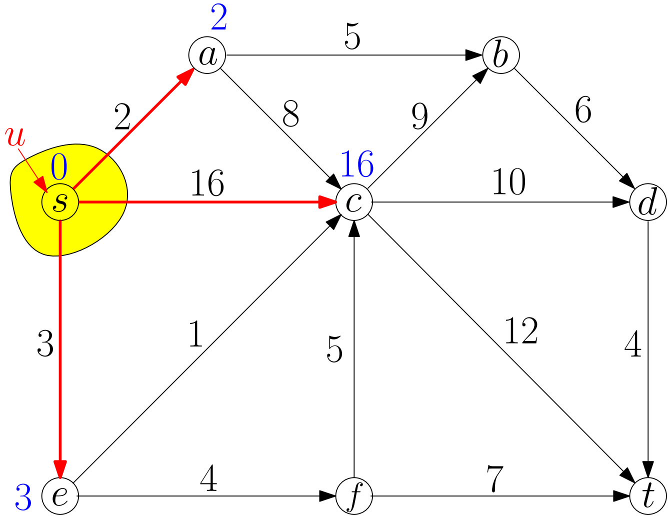
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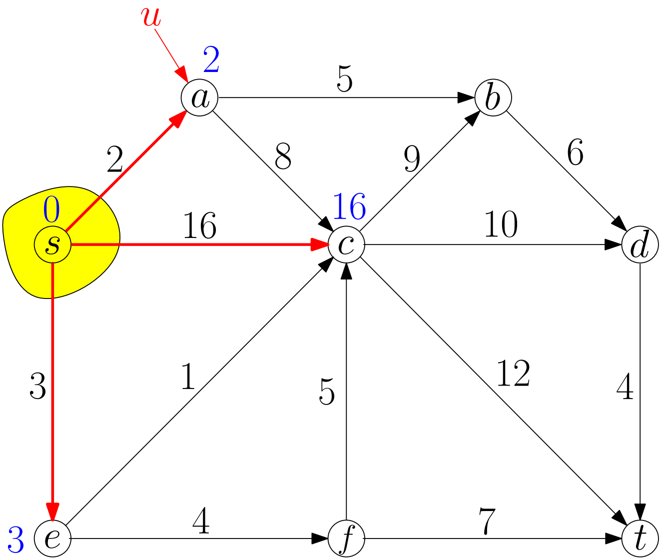
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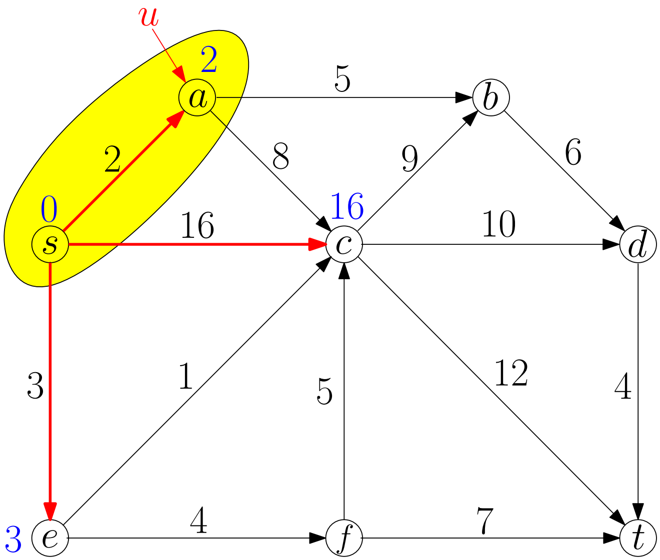
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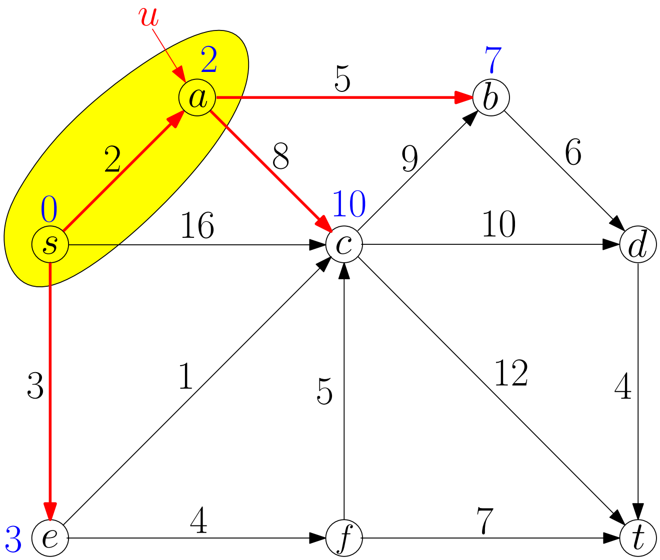
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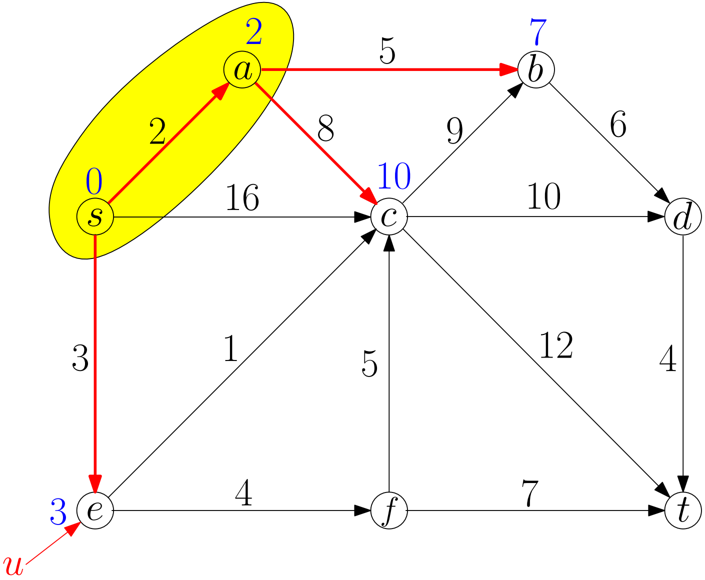

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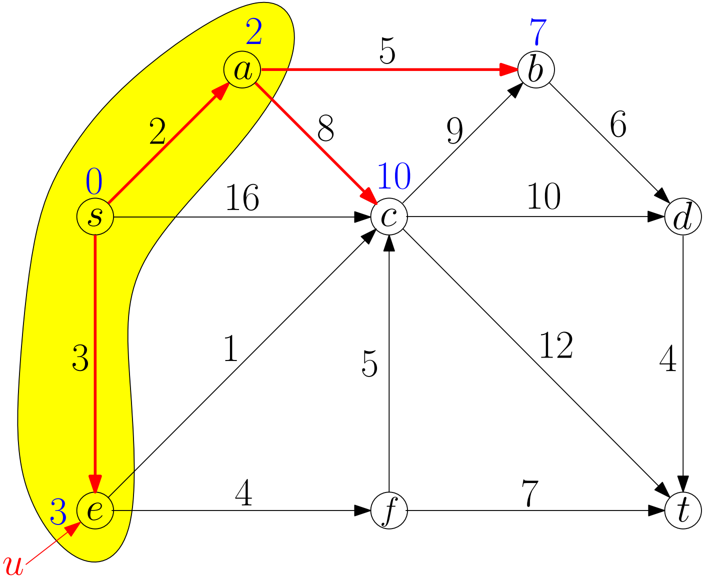
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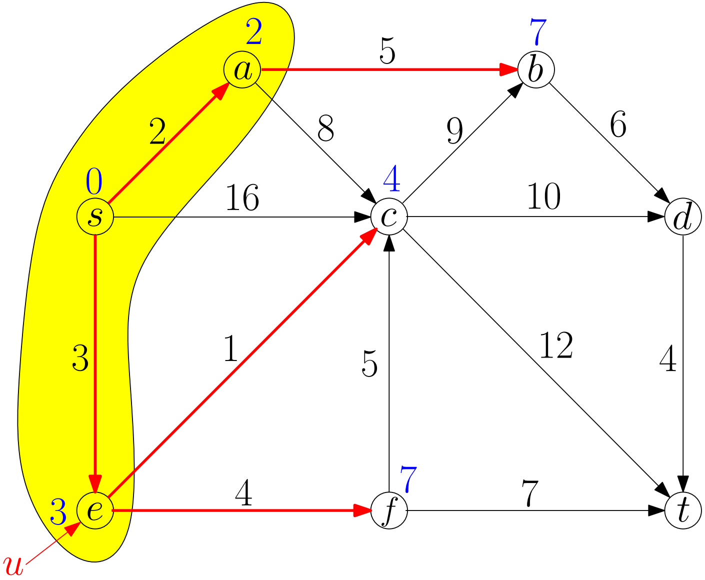
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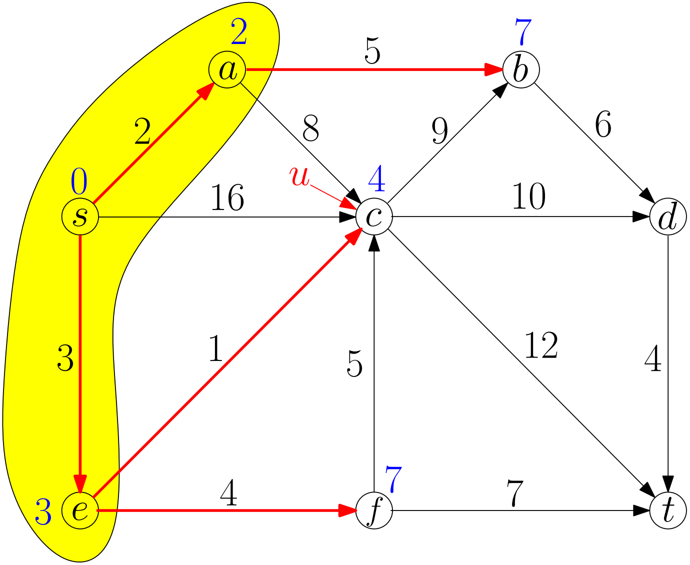
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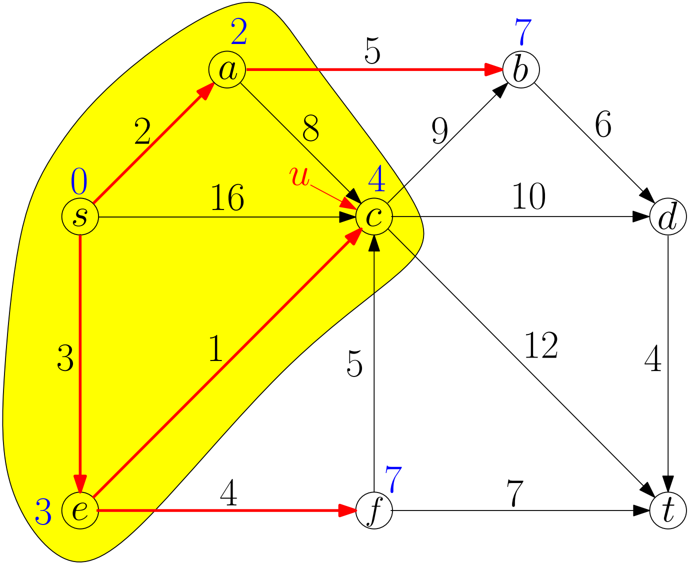

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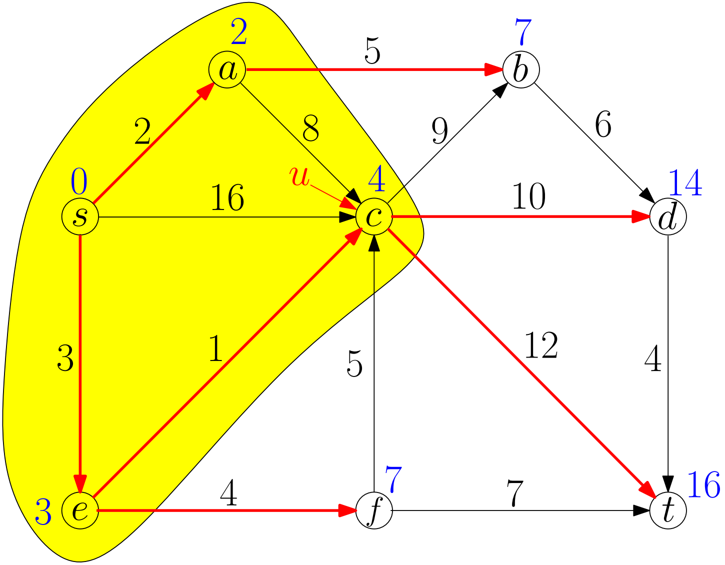
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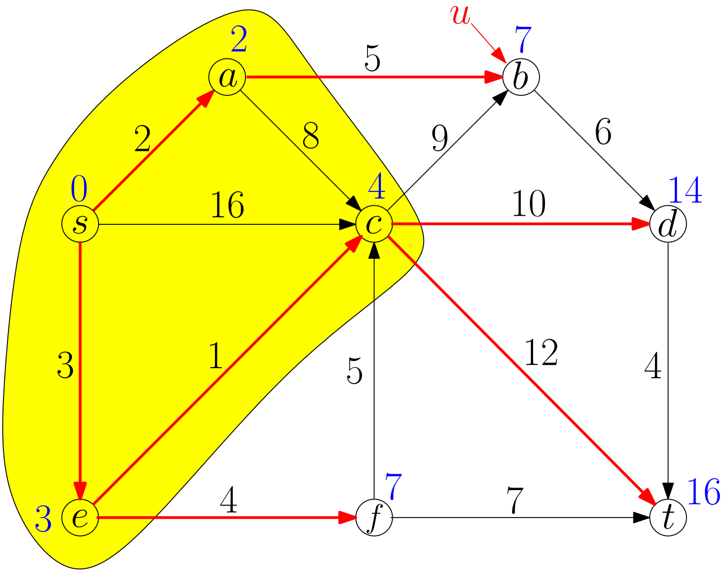
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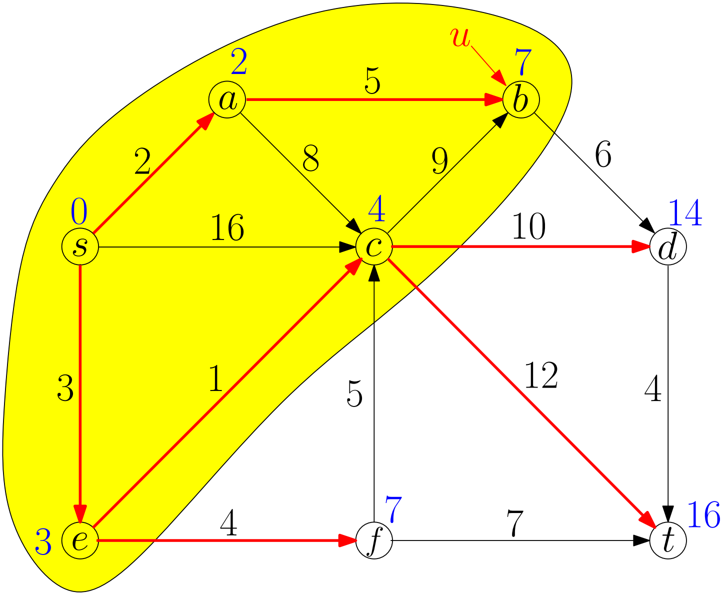
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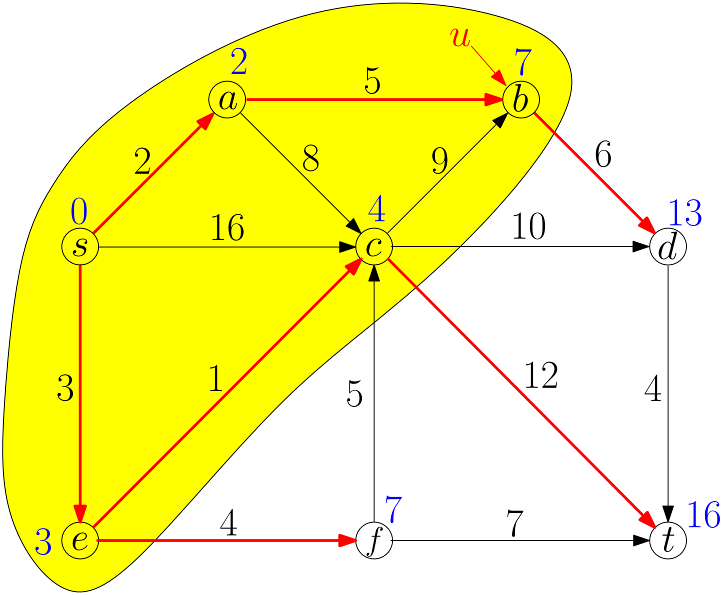
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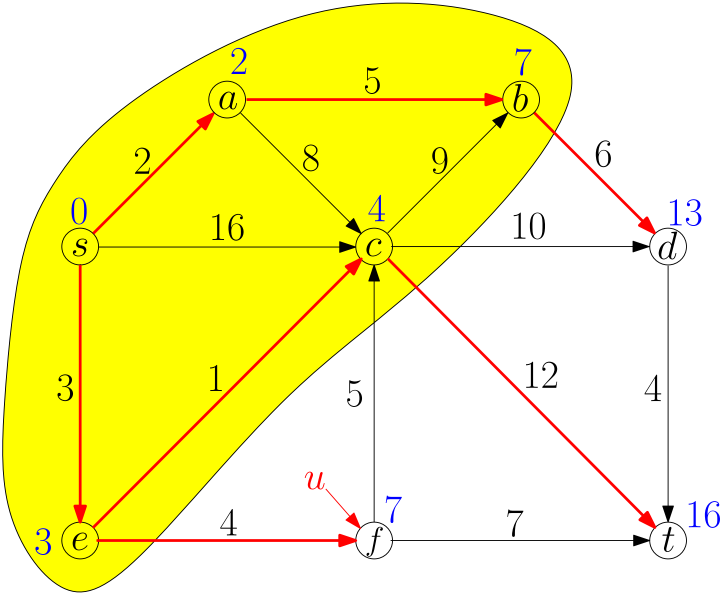

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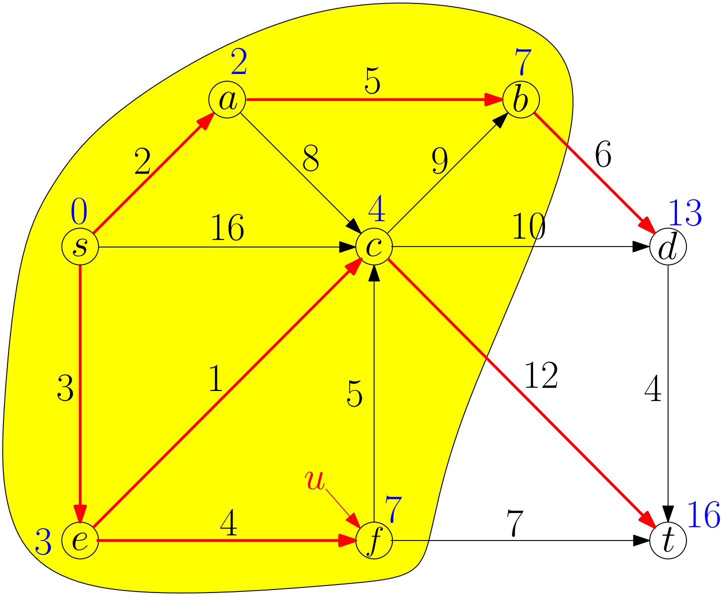
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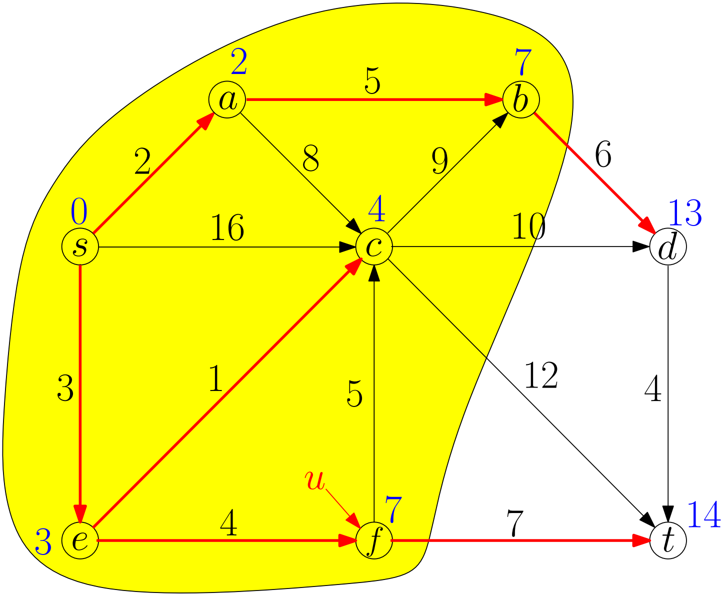

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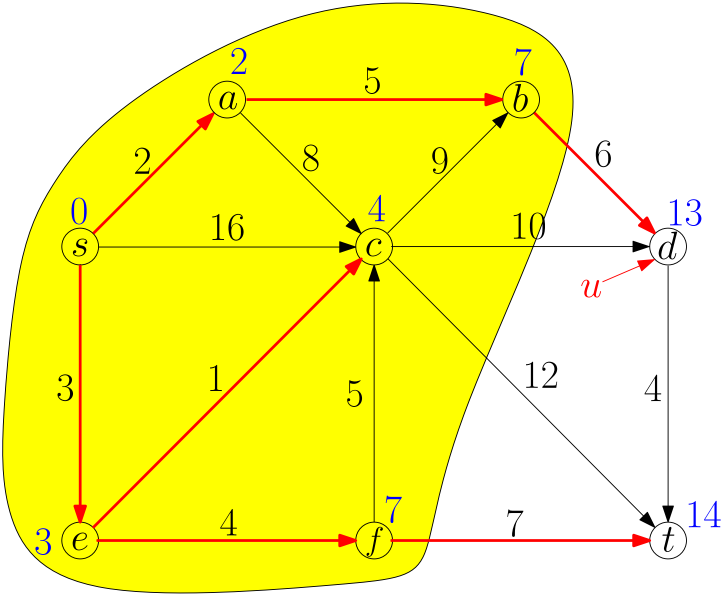

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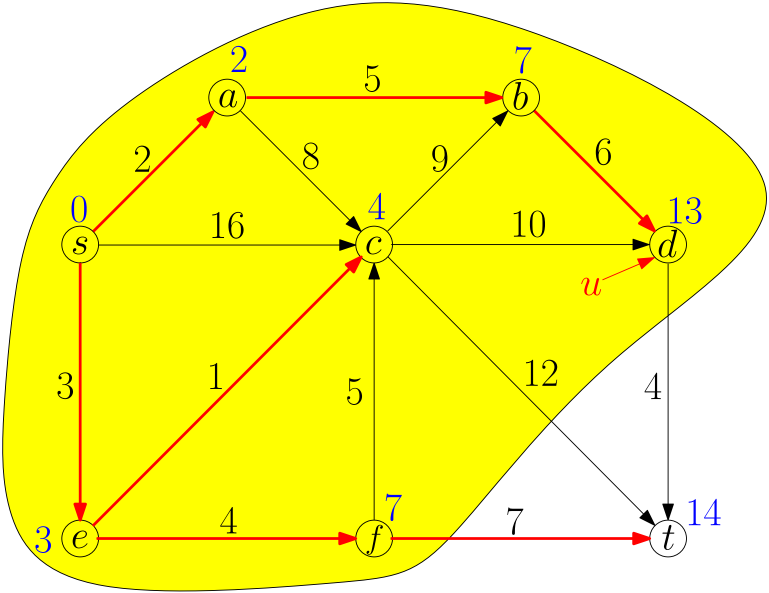
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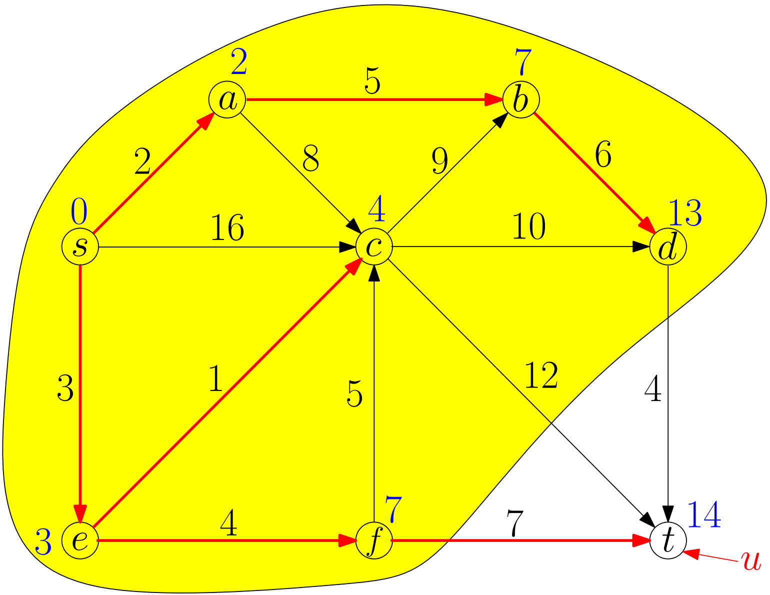

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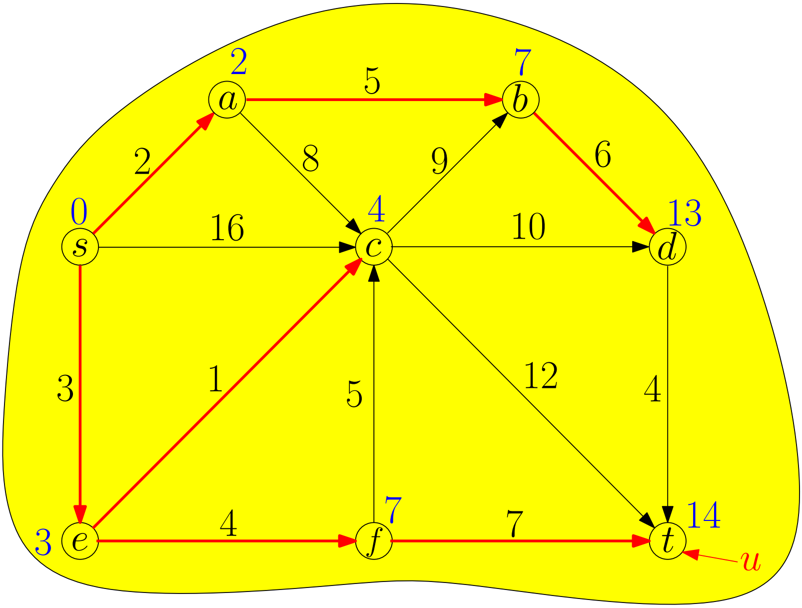

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Improved Running Time using Priority Queue

Dijkstra(G, w, s)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return (π, d)

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

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$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

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- In transition graphs, negative weights make sense

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- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)

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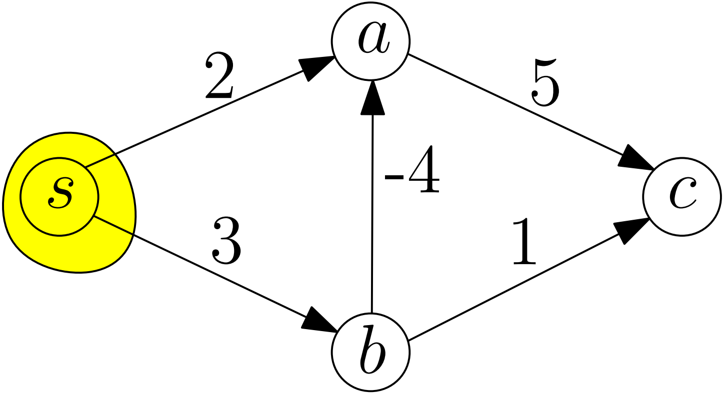
- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights



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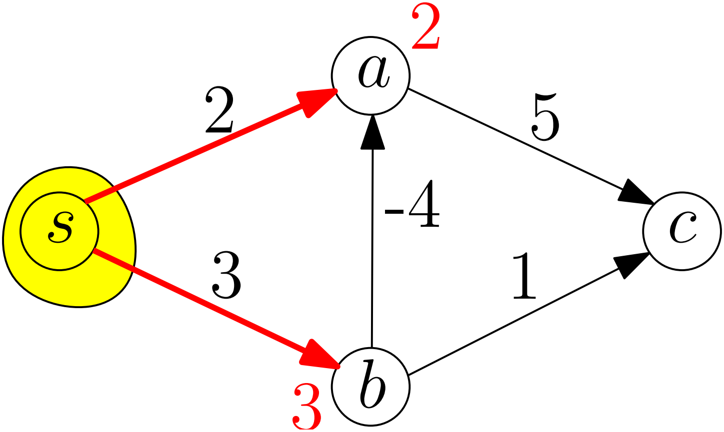

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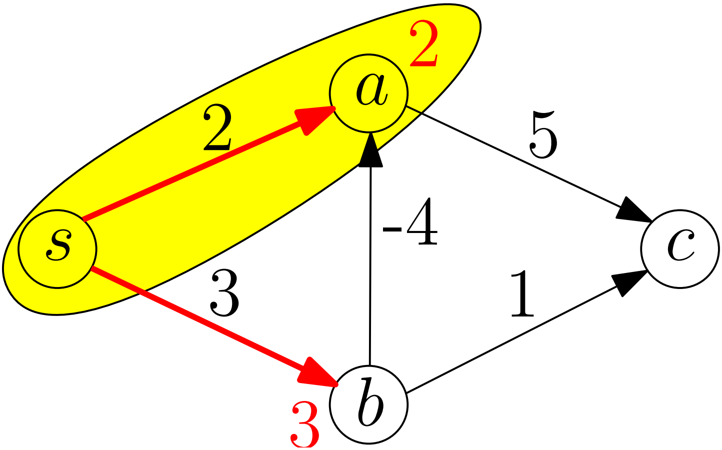

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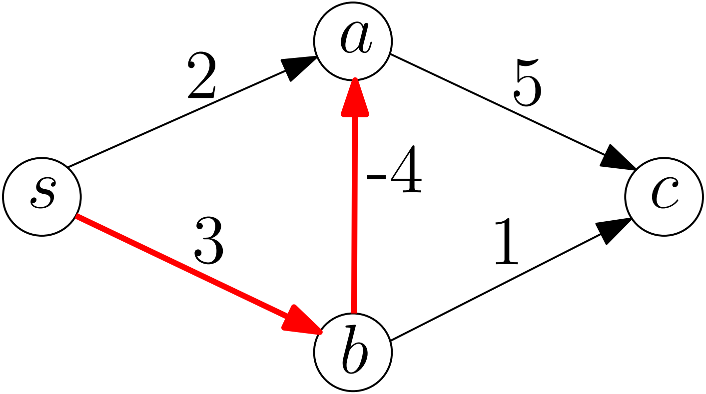

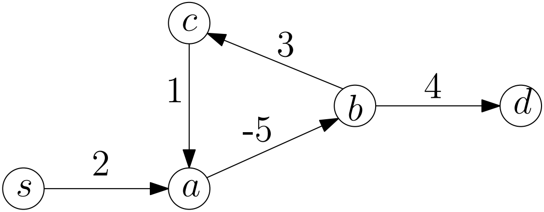

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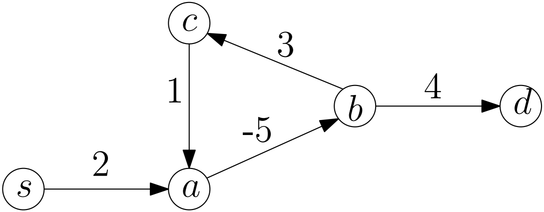
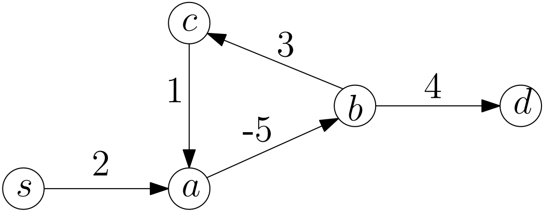
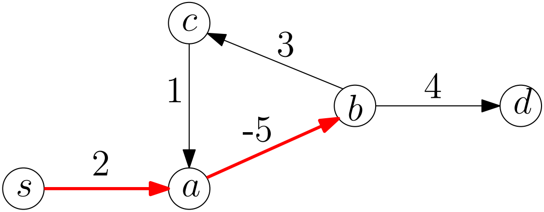


Q: What is the length of the shortest path from s to d?



Q: What is the length of the shortest path from s to d?

A: $-\infty$



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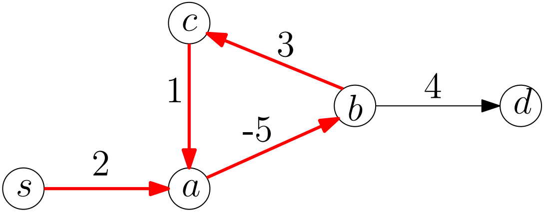

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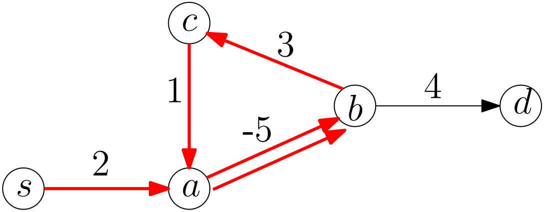

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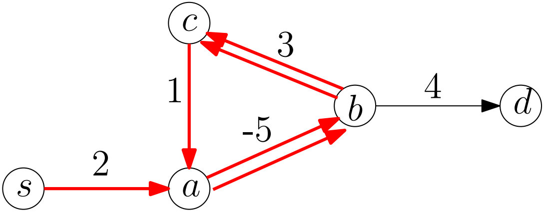

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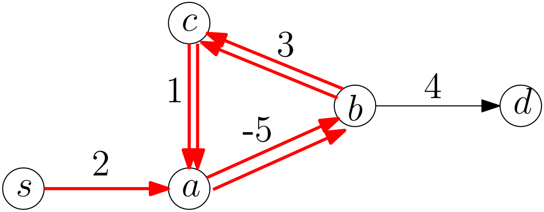

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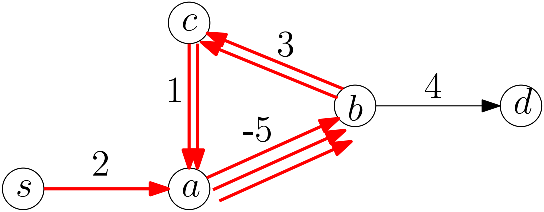

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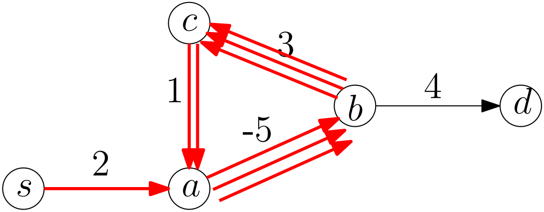

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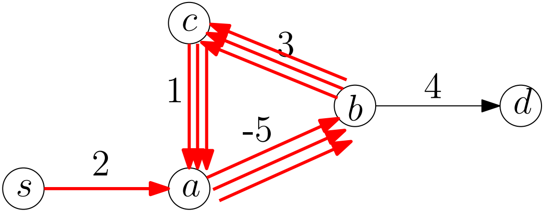

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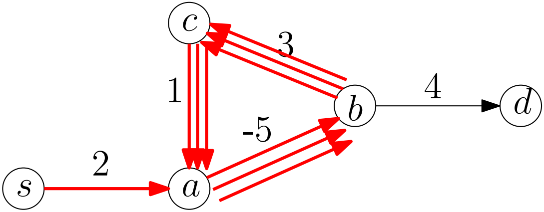

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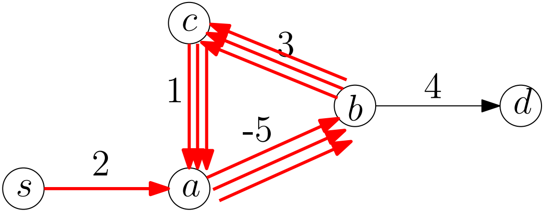
Def. A negative cycle is a cycle in which the total weight of edges is negative.



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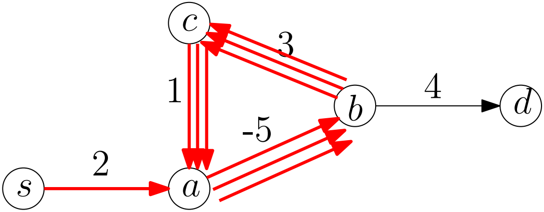
Dealing with Negative Cycles



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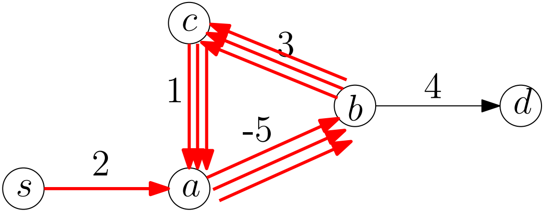
Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or



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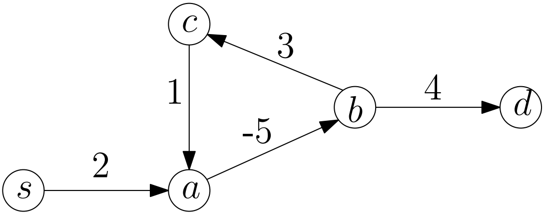
Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"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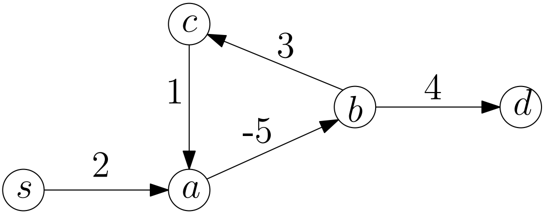

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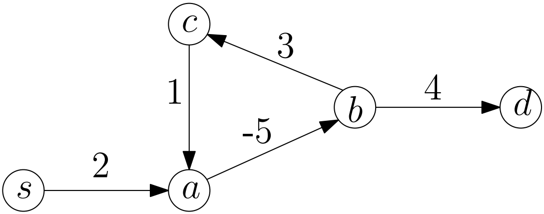
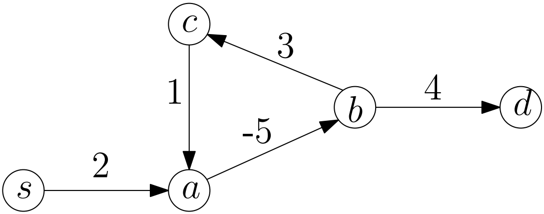


Q: What is the length of the shortest simple path from s to d?

A: 1



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A: 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n\log n+m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Defining Cells of Table

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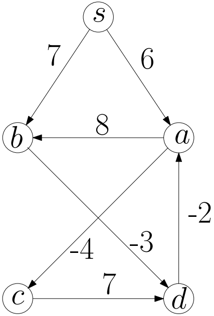
- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges



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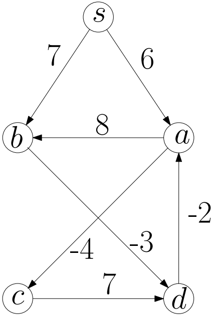

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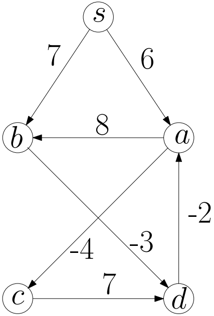
- $f^2[a] =$



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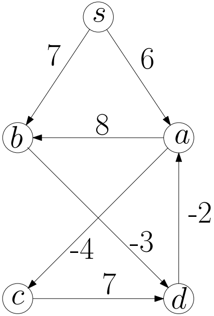

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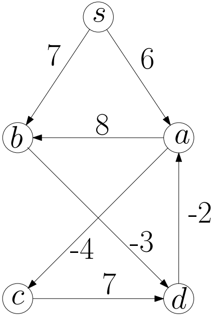

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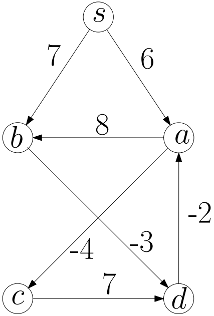

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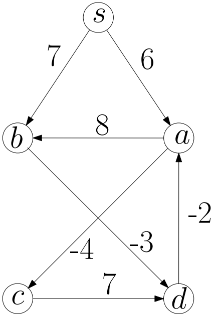
$$f^\ell[v] = \begin{cases} & \ell = 0, v = s \\ & \ell = 0, v \neq s \\ & \ell > 0 \end{cases}$$



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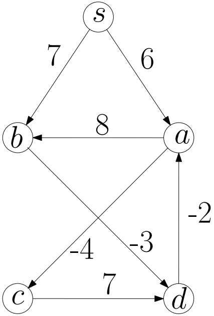

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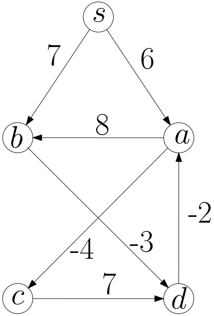

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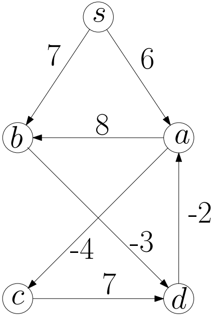

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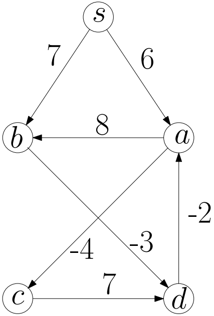

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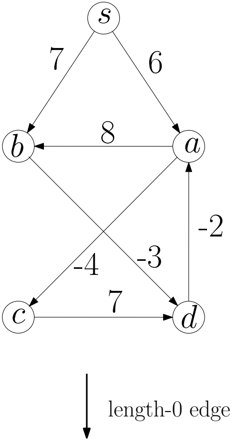
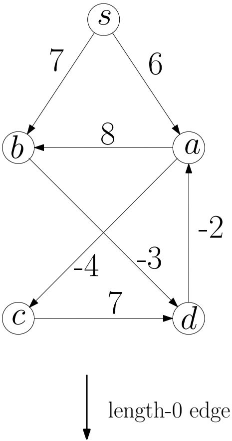
$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \begin{cases} f^{\ell-1}[v] \\ \min_{u:(u,v)\in E} \left(f^{\ell-1}[u] + w(u,v)\right) \end{cases} & \ell > 0 \end{cases}$$

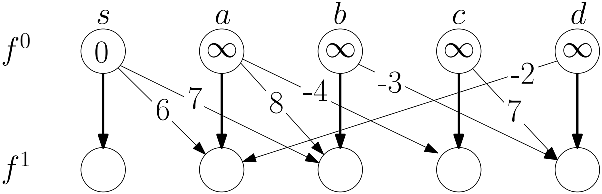
Dynamic Programming: Example



	s	a	b	c	d
f^0	0	∞	∞	∞	∞

Dynamic Programming: Example





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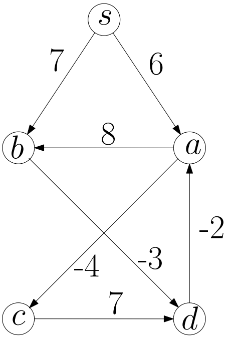
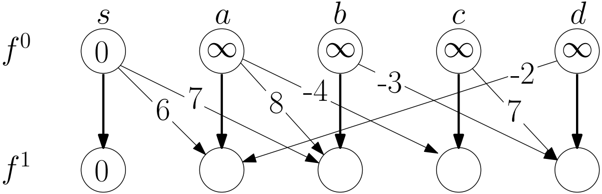
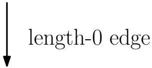
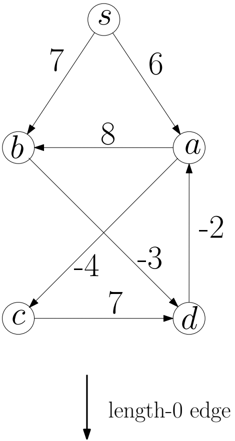
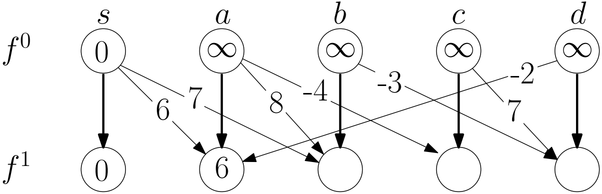






Dynamic Programming: Example





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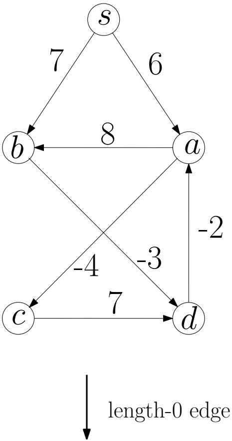

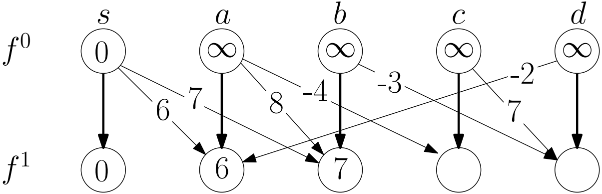

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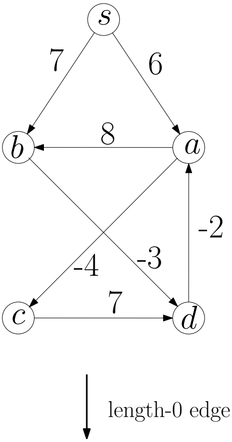

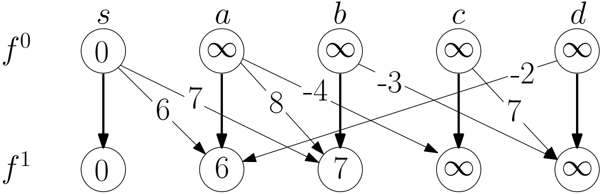

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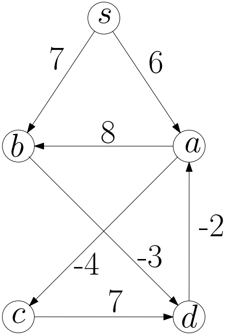

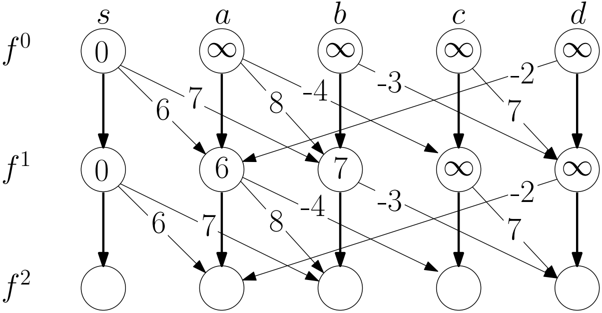

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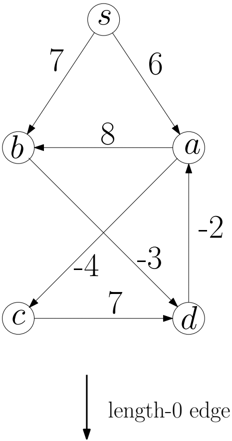

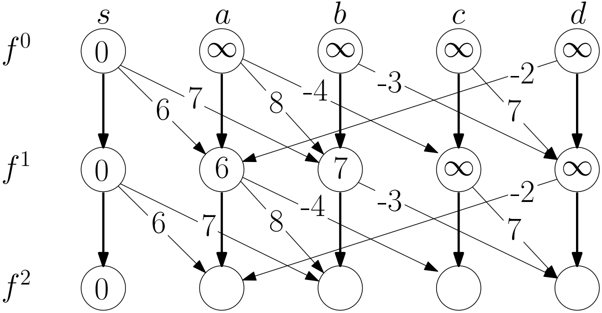

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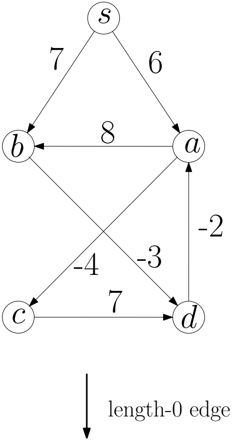

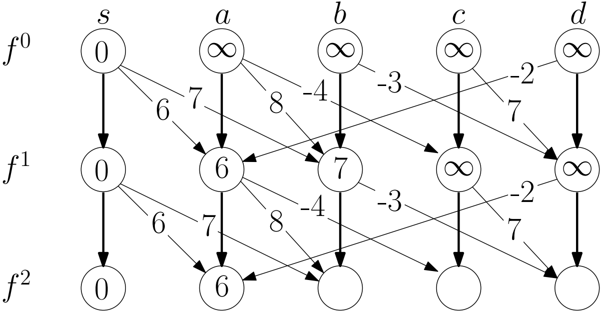

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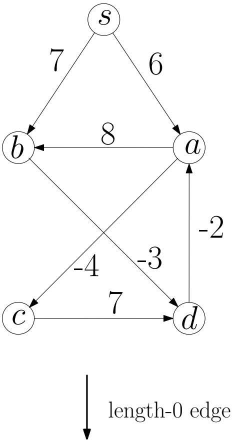

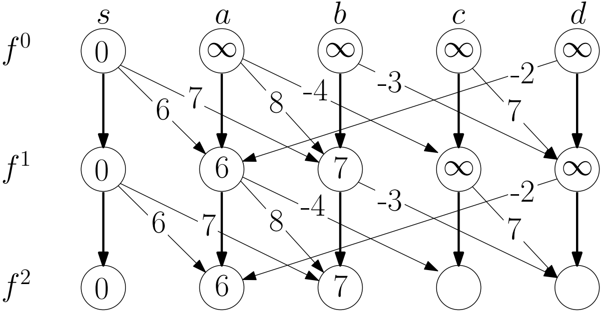

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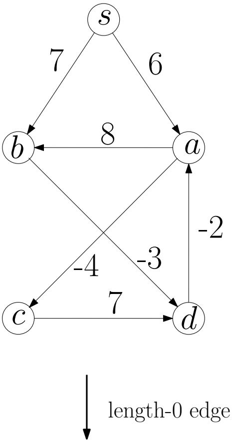

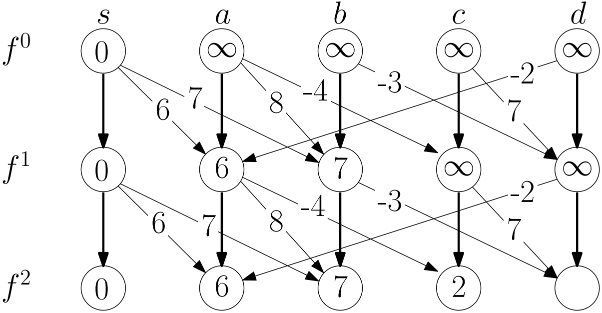

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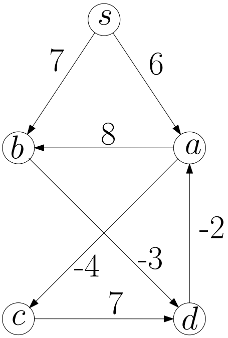
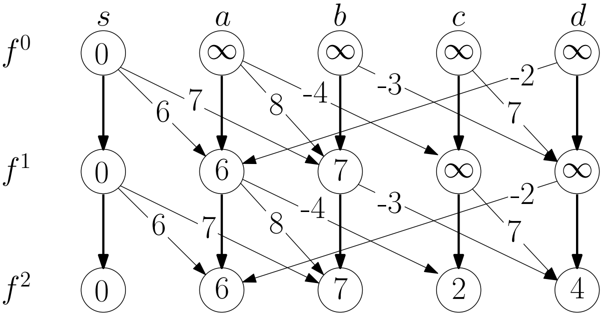
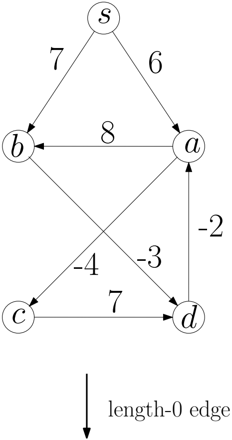
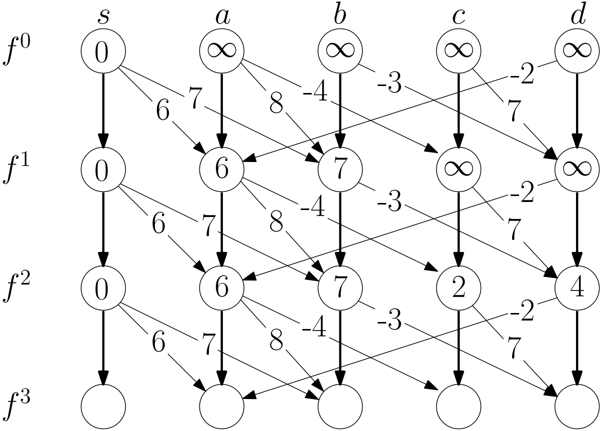




length-0 edge

Dynamic Programming: Example





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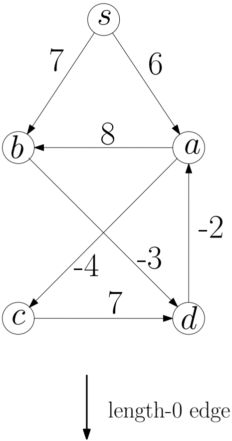

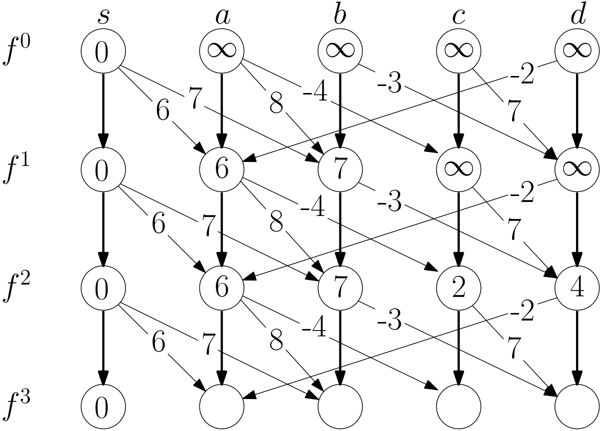

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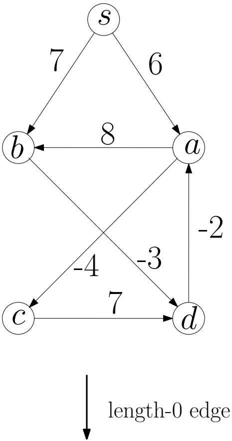

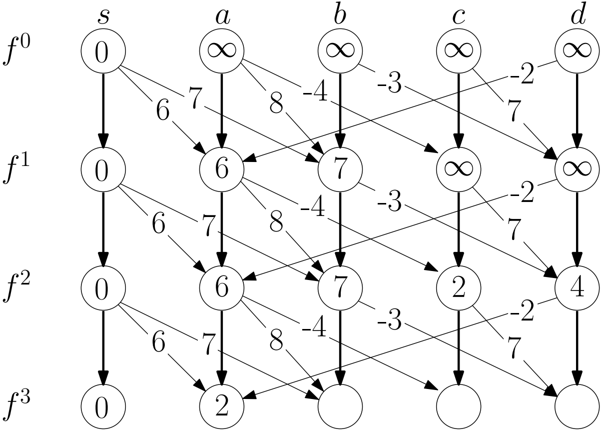

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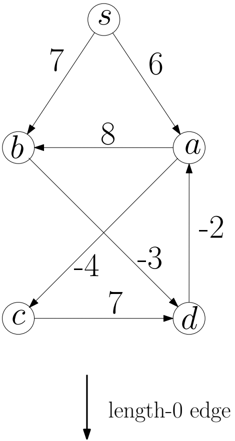

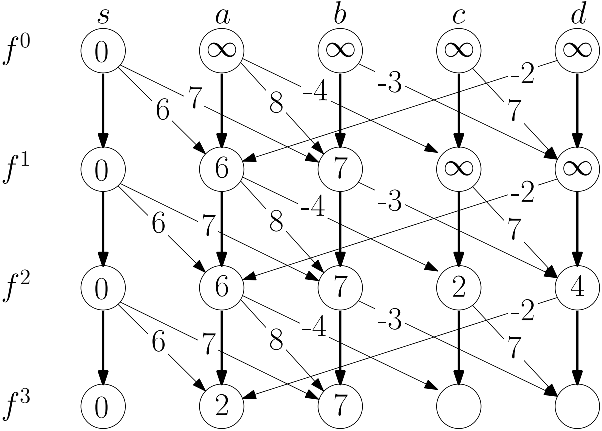

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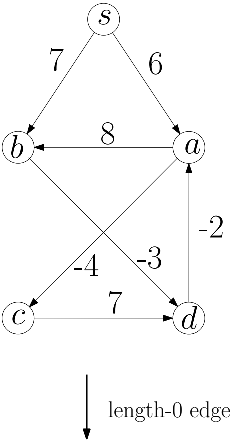

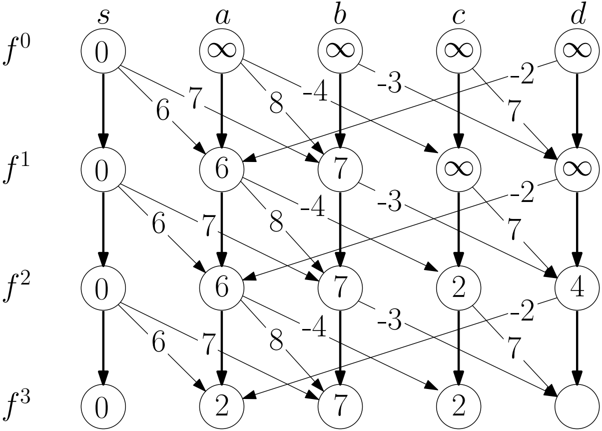

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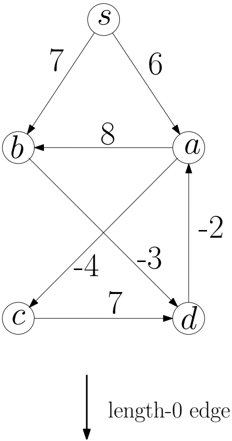

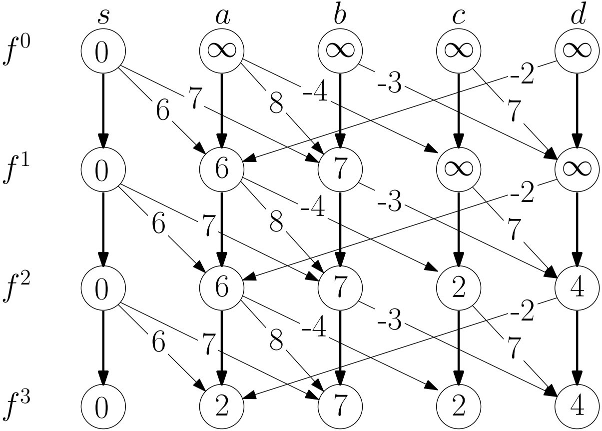

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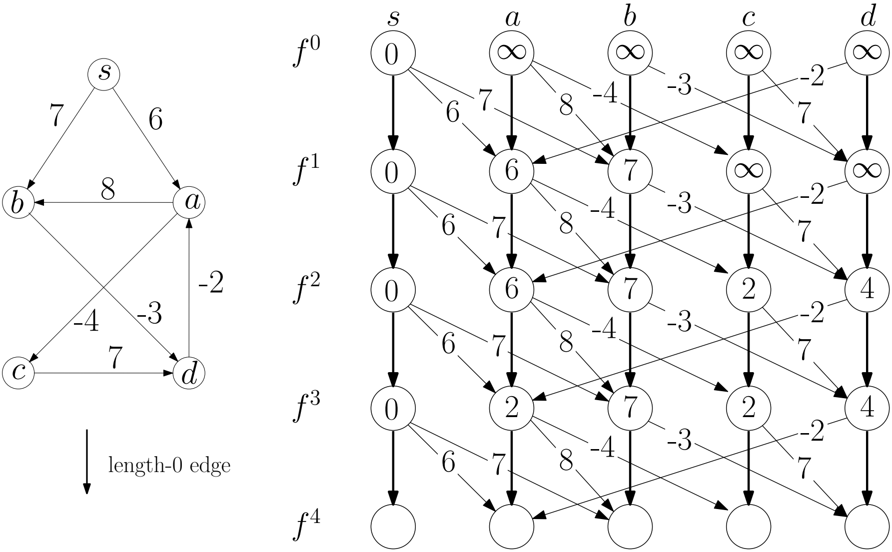

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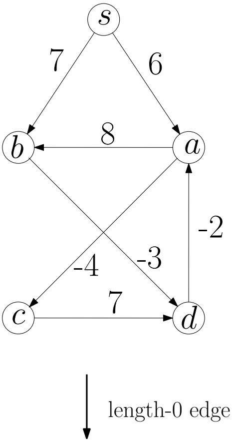

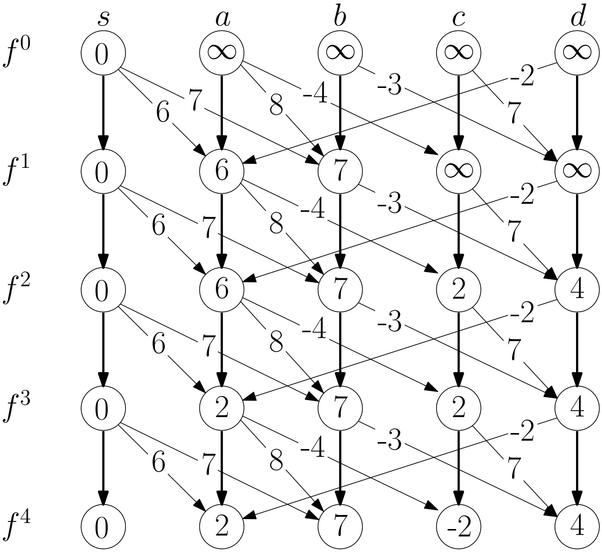

dynamic-programming(G, w, s)

```
f^0[s] ← 0 and f^0[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^{ℓ−1} → f^ℓ
    for each (u, v) ∈ E do
        if f^{ℓ−1}[u] + w(u, v) < f^ℓ[v] then
            f^ℓ[v] ← f^{ℓ−1}[u] + w(u, v)
return (f^{n−1}[v])_{v∈V}
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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. □

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

```
f^old[s] ← 0 and f^old[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^old → f^new
    for each (u, v) ∈ E do
        if f^old[u] + w(u, v) < f^new[v] then
            f^new[v] ← f^old[u] + w(u, v)
    copy f^new → f^old
return f^old
```

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

$f^{\text{old}}[s] \leftarrow 0$ and $f^{\text{old}}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\text{old}} \rightarrow f^{\text{new}}$
    for each $(u, v) \in E$ **do**
        if $f^{\text{old}}[u] + w(u,v) < f^{\text{new}}[v]$ **then**
            $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u,v)$
    copy $f^{\text{new}} \rightarrow f^{\text{old}}$
return f^{old}

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- only need 1 vector!

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Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

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- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: **for** $\ell \leftarrow 1$ to $n-1$ **do**
3:     **for** each $(u, v) \in E$ **do**
4:         **if** $f[u] + w(u, v) < f[v]$ **then**
5:             $f[v] \leftarrow f[u] + w(u, v)$
6: **return** f

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v