Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree: $opt[i, j]$ = value of instances defined by matrices i to j

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Important notations/algorithms

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  - Asymptotic analysis: O, Ω, Θ, compare the orders

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  - Topological Ordering problem: topological-sort algorithm

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  - Exercise problems: Job scheduling with deadline, clustering problem, Coin Problem, Weighted scheduling problem

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CSE 431/531: Algorithm Analysis and Design (Spring 2024)

Graph Algorithms

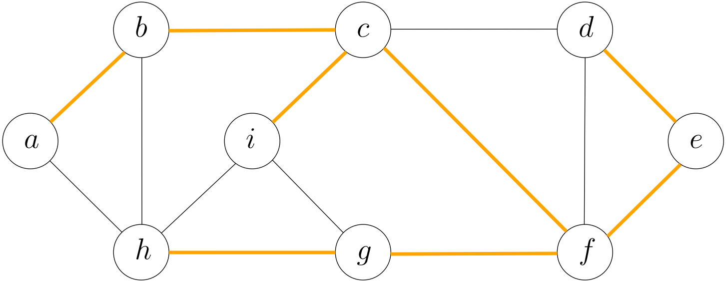
Lecturer: Kelin Luo

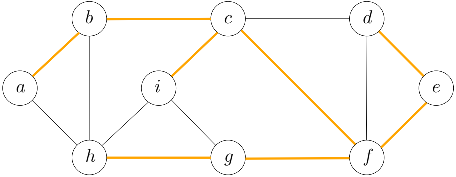
Department of Computer Science and Engineering
University at Buffalo

Outline

Spanning Tree

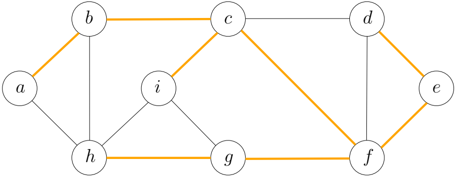
Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;



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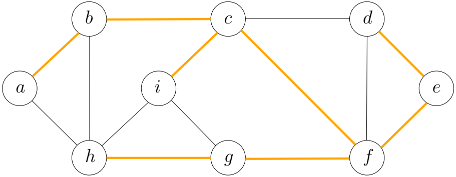

Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

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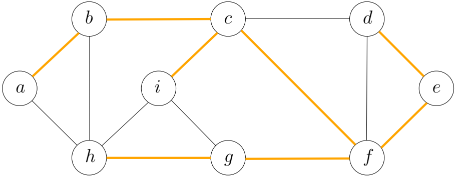

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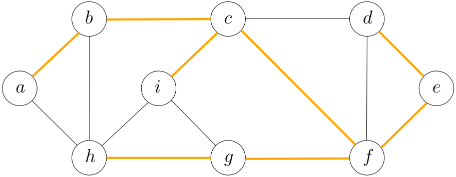

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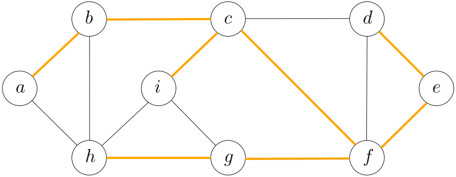

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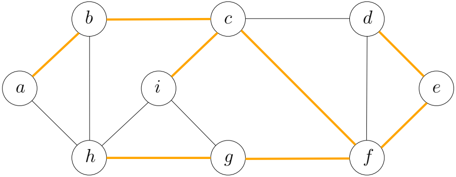

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- T has a unique simple path between every pair of nodes.

- How to find a spanning tree?
  - BFS

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Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

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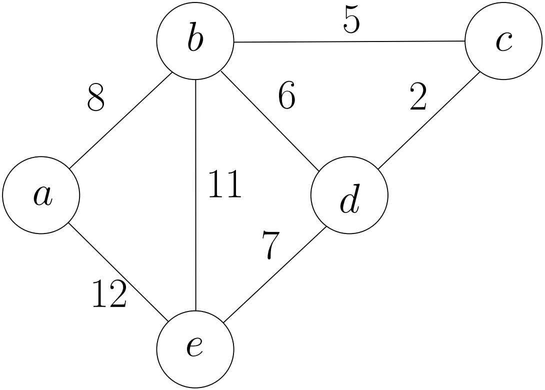

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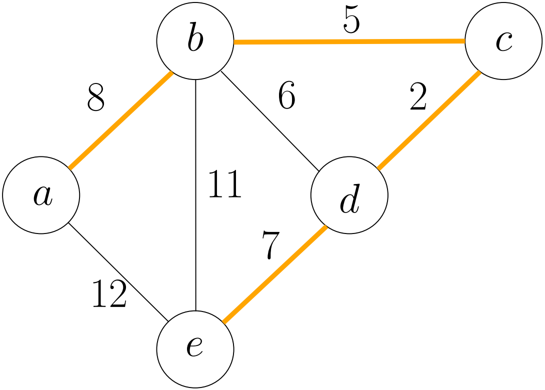

Recall: Steps of Designing A Greedy Algorithm

- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

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Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

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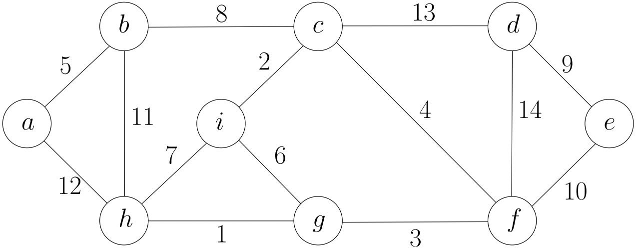
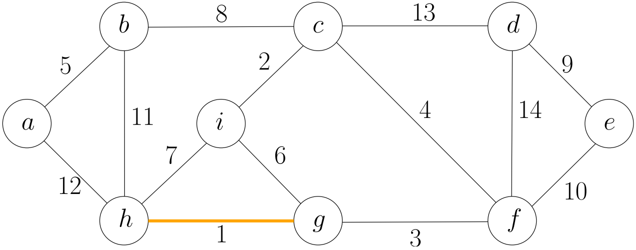


Q: Which edge can be safely included in the MST?



Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

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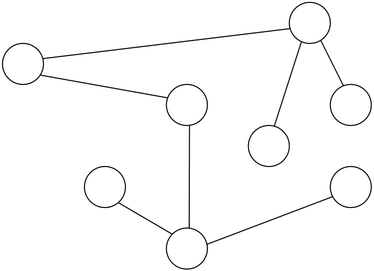

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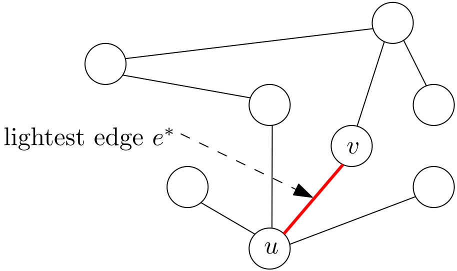
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T



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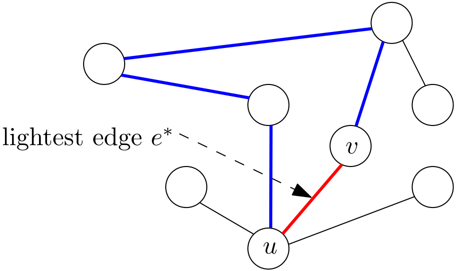

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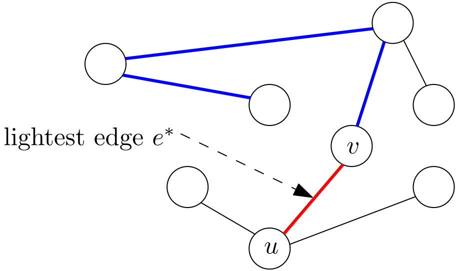

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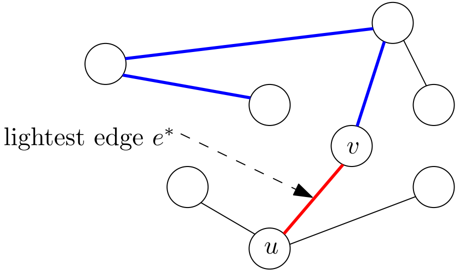
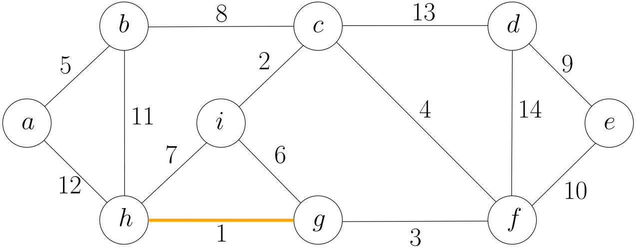
- $w(e^*) \le w(e) \implies w(T') \le w(T)$: T' is also a MST $\square$



Is the Residual Problem Still a MST Problem?



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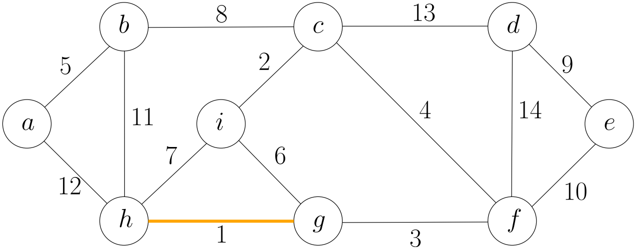

- Residual problem: find the minimum spanning tree that contains edge (g, h)

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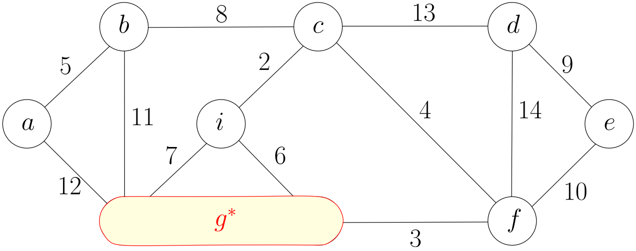

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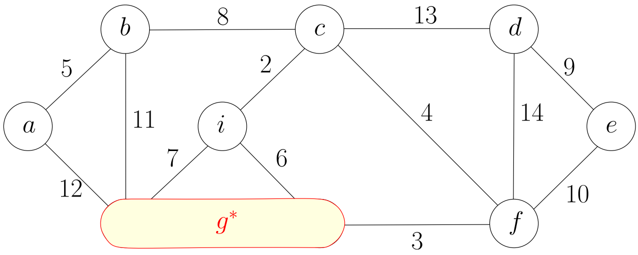
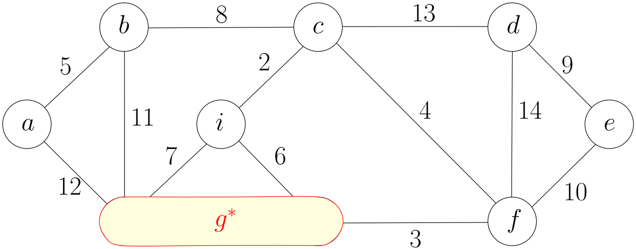


- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



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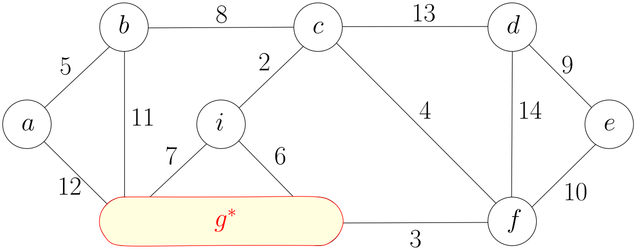

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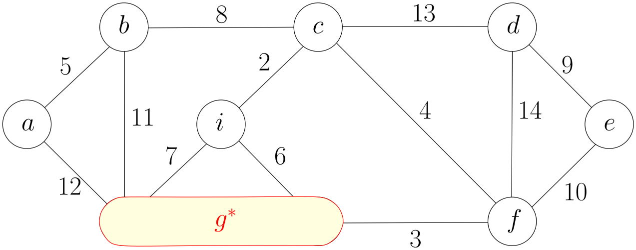

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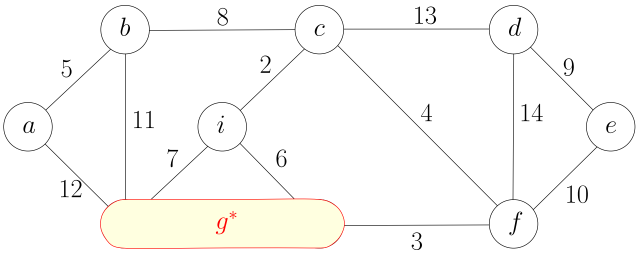

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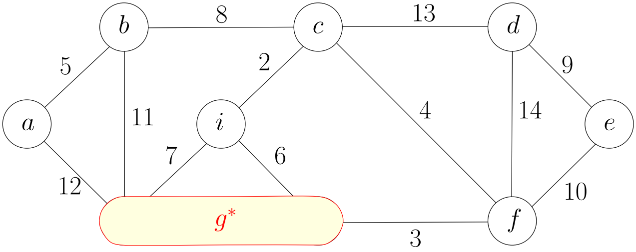

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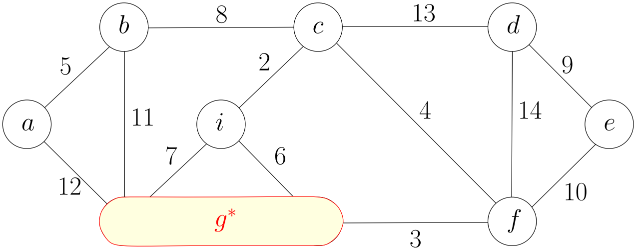

- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

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Q: What edges are removed due to contractions?

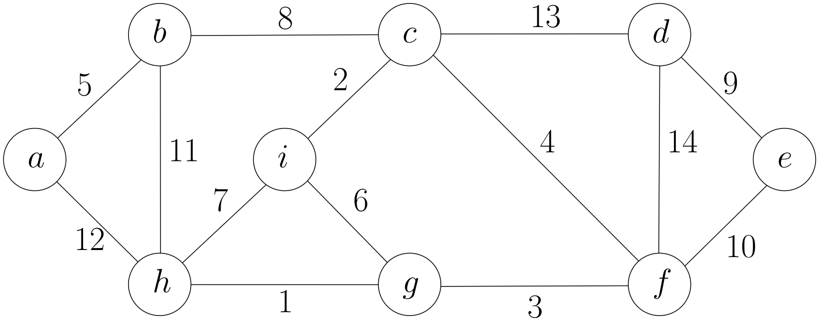
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

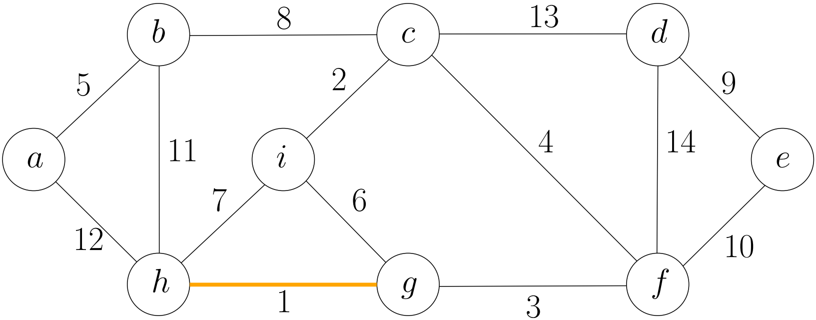
```
F ← ∅
sort edges in E in non-decreasing order of weights w
for each edge (u, v) in the order do
    if u and v are not connected by a path of edges in F then
        F ← F ∪ {(u, v)}
return (V, F)
```

Kruskal's Algorithm: Example



Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

Kruskal's Algorithm: Example



Sets: $\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\},\{h\},\{i\}$

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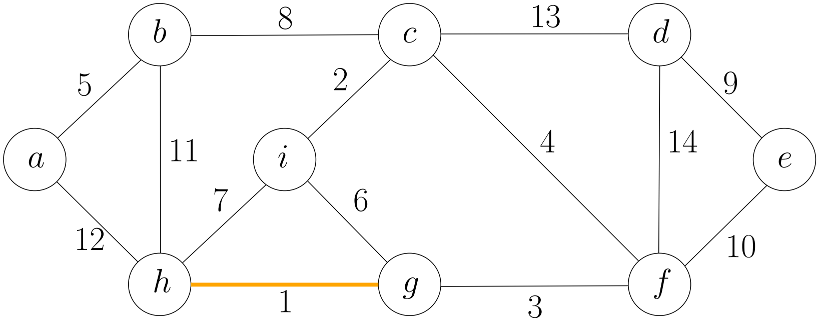

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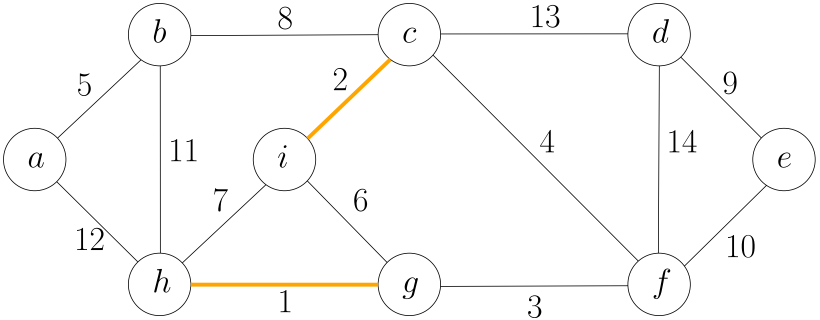

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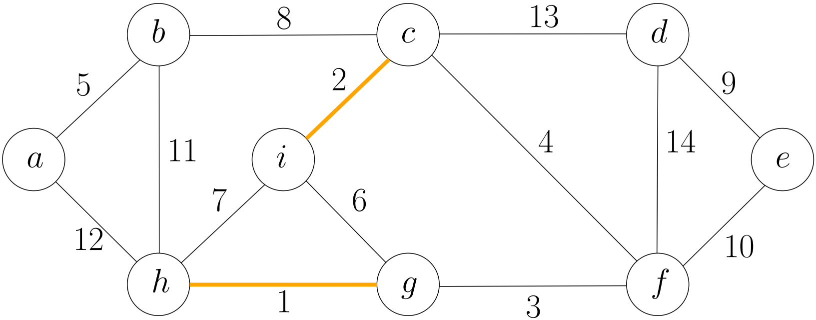

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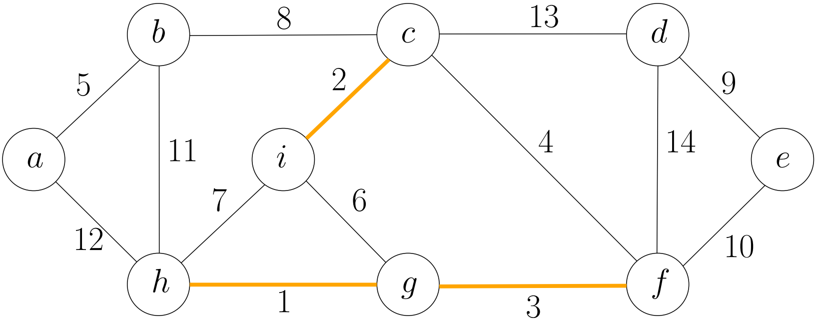

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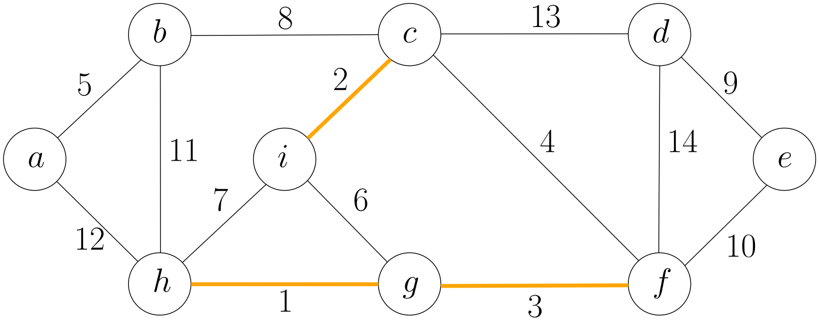

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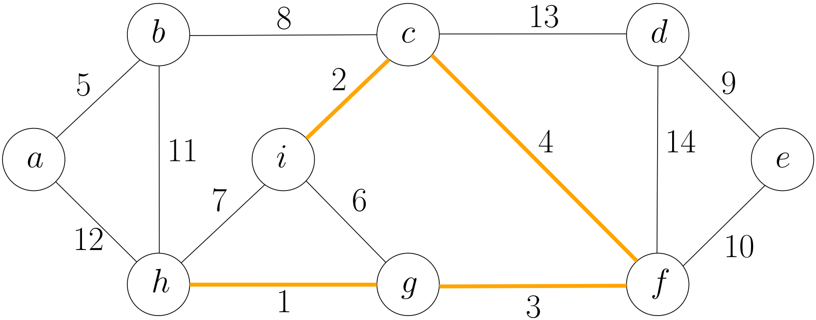

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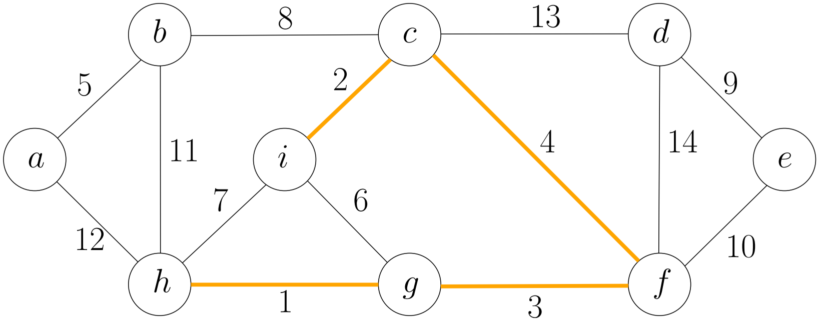

Sets: $\{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}$

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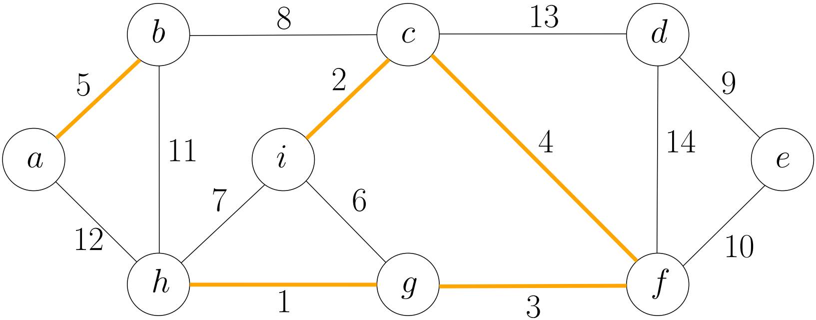

Sets: $\{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}$

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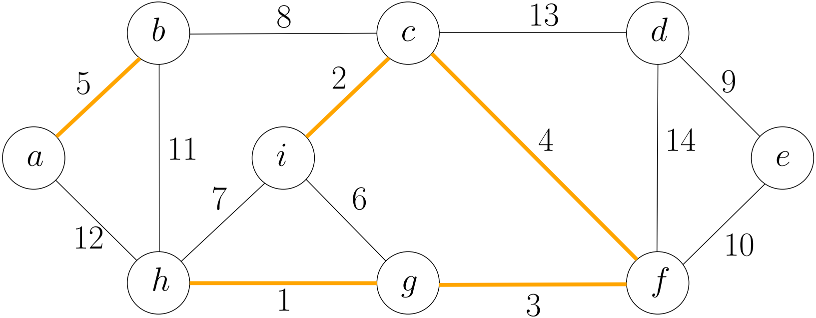

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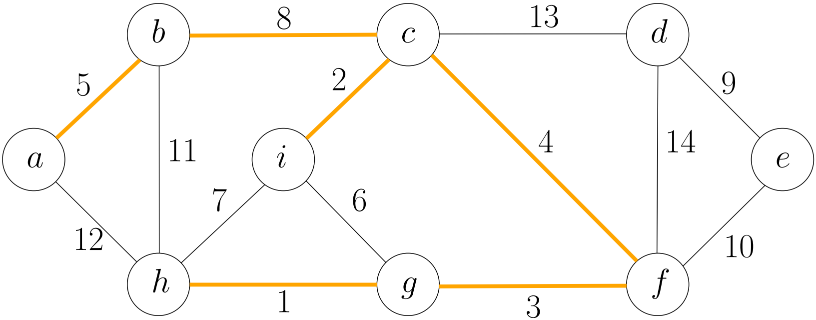

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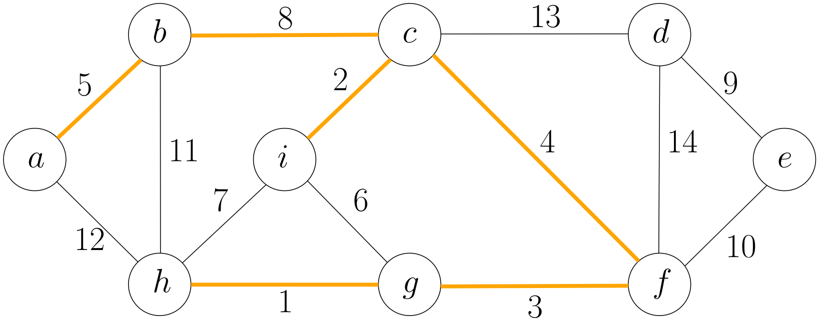

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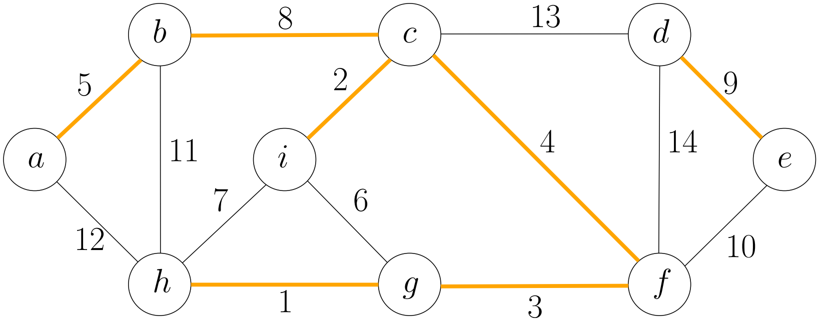

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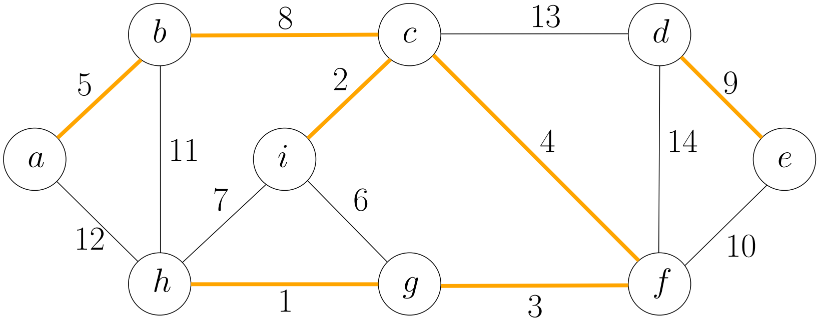

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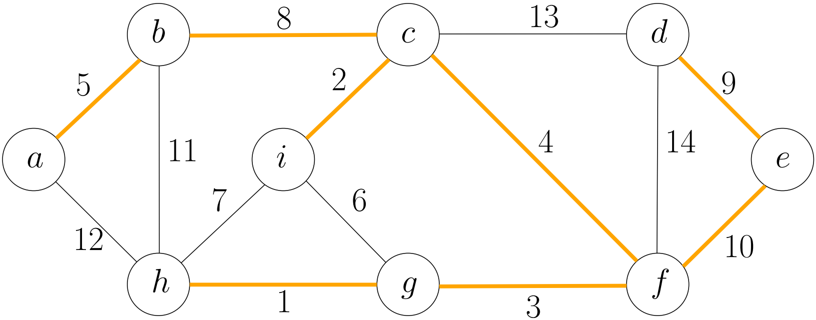

Sets: $\{a, b, c, i, f, g, h\}, \{d, e\}$

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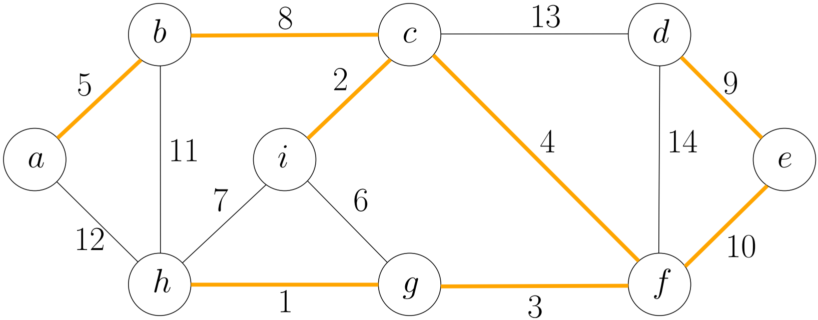

Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of E in non-decreasing order of weights w
for each edge $(u, v) \in E$ in the order **do**
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
    if $S_u \neq S_v$ **then**
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return (V, F)