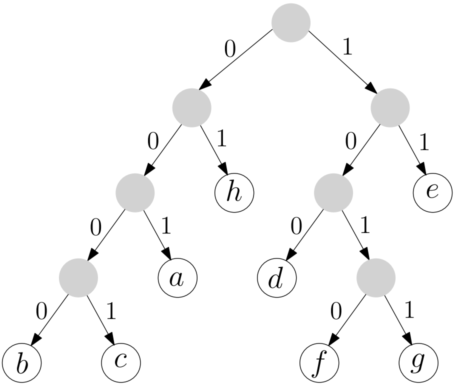


Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some letter
- If coding scheme is not wasteful: a non-leaf has exactly two children

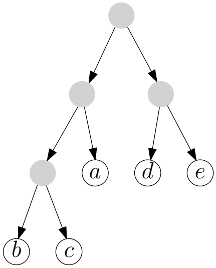
Best Prefix Codes

Input: frequencies of letters in a message

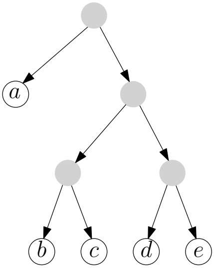
Output: prefix coding scheme with the shortest encoding for the message

example

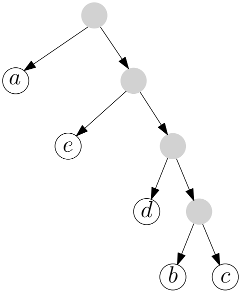
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



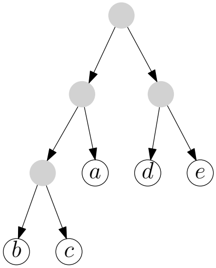
scheme 2



scheme 3

example

letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total = 89
scheme 2 length	1	3	3	3	3	total = 87
scheme 3 length	1	4	4	3	2	total = 84



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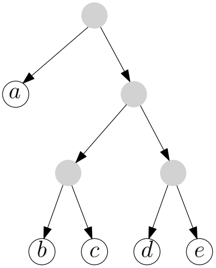

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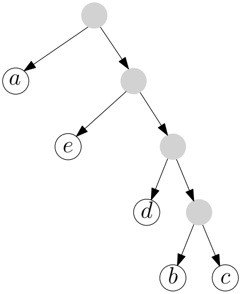

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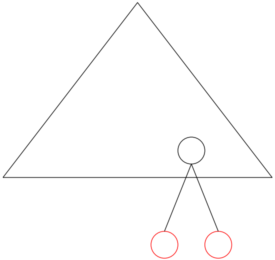
A: We can choose two letters and make them brothers in the tree.

Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree



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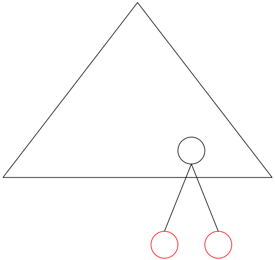

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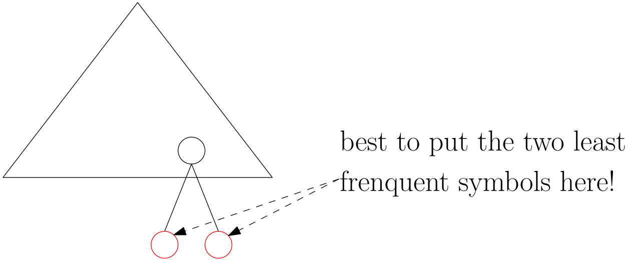

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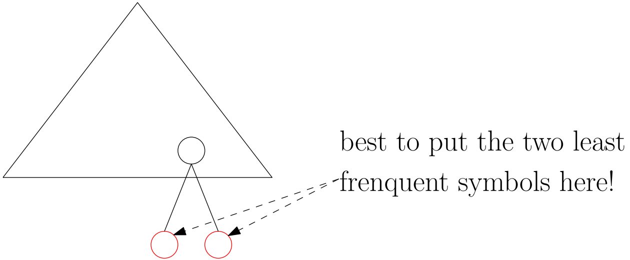

Lemma It is safe to make the two least frequent letters brothers.

Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

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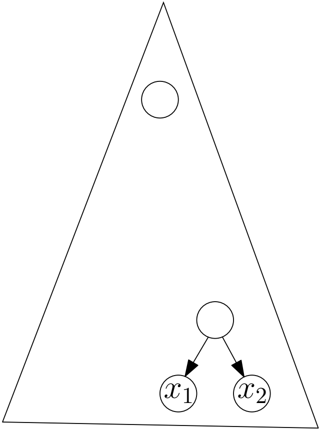
Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.

- f_x: the frequency of the letter x in the support.
- x_1 and x_2: the two letters we decided to put together.
- d_x the depth of letter x in our output encoding tree.



$$\sum_{x \in S} f_x d_x$$
$$= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2}$$
$$= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}$$

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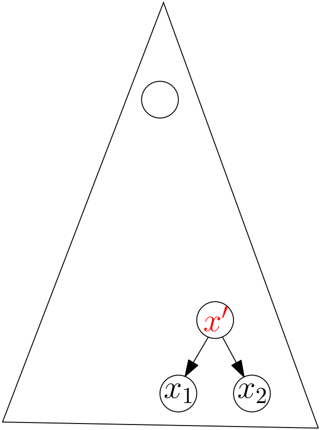

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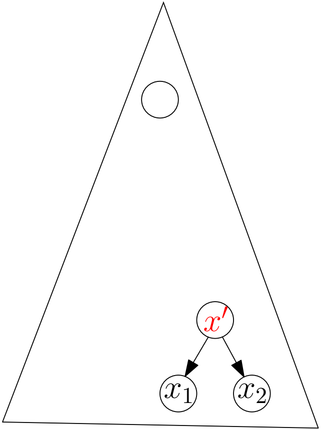

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Def: $f_{x'} = f_{x_1} + f_{x_2}$

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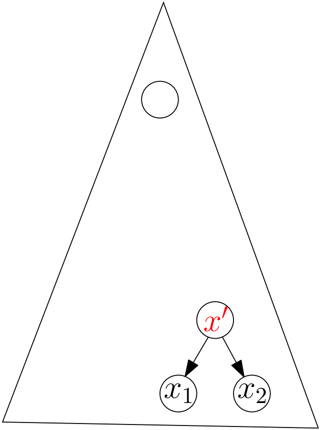

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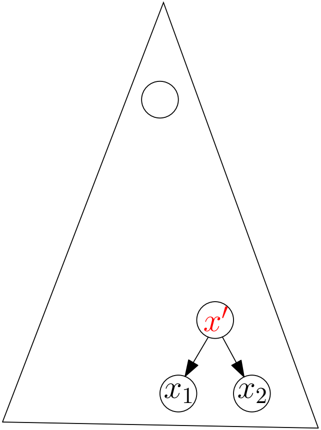

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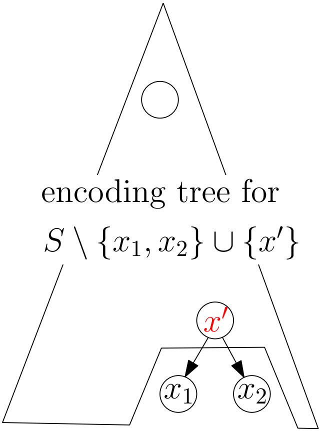

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\end{aligned}
$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Example

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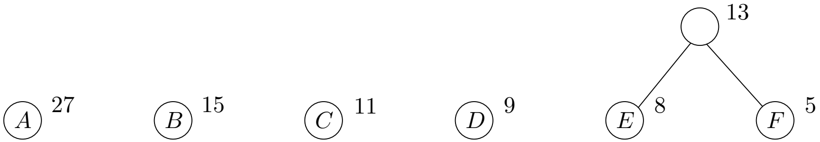

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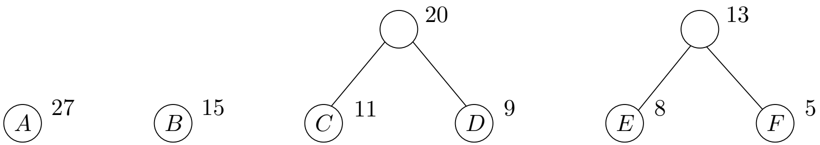

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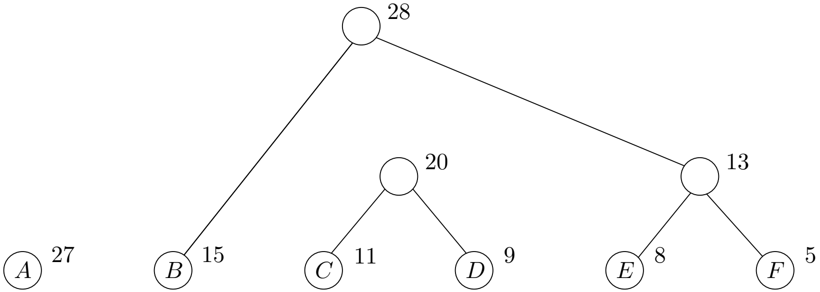

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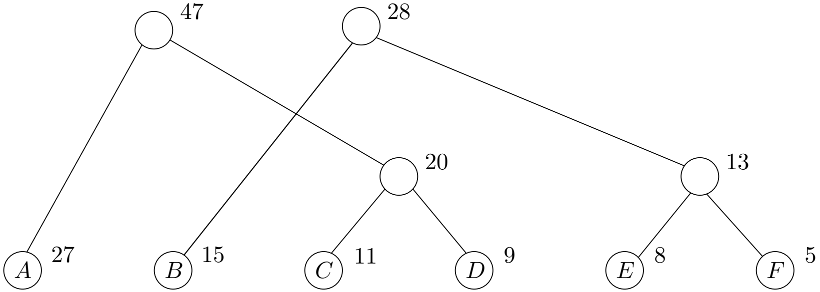

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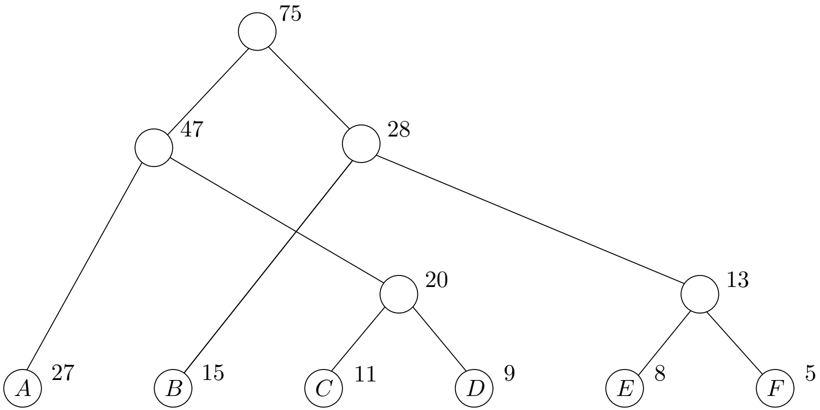

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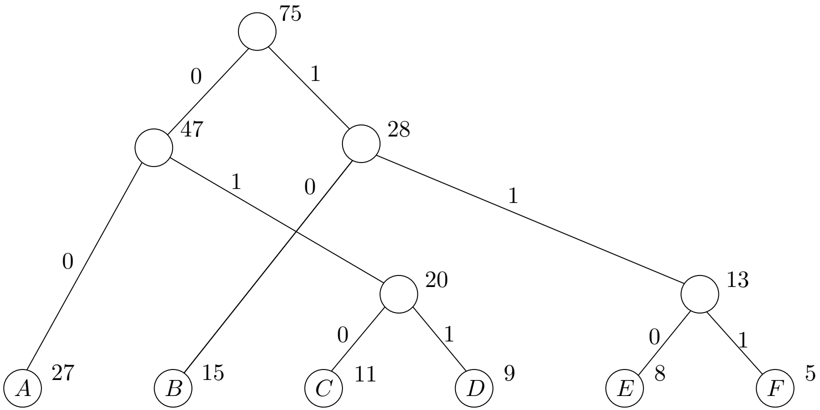

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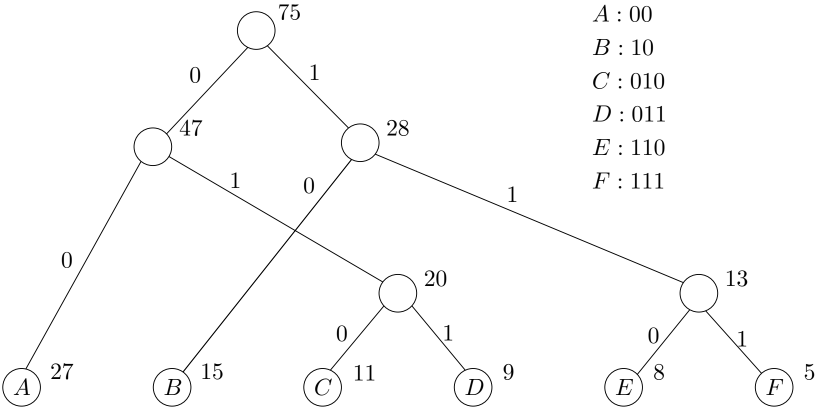

$A : 00$
$B : 10$
$C : 010$
$D : 011$
$E : 110$
$F : 111$

Def. The codes given the greedy algorithm is called the Huffman codes.

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Huffman(S, f)

```
while |S| > 1 do
    let x_1, x_2 be the two letters with the smallest f values
    introduce a new letter x' and let f_{x'} = f_{x_1} + f_{x_2}
    let x_1 and x_2 be the two children of x'
    S ← S \ {x_1, x_2} ∪ {x'}
return the tree constructed
```

Algorithm using Priority Queue

Huffman(S, f)

$Q \leftarrow$ build-priority-queue(S)
while $Q.\text{size} > 1$ **do**
    $x_1 \leftarrow Q.\text{extract-min}()$
    $x_2 \leftarrow Q.\text{extract-min}()$
    introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
    let x_1 and x_2 be the two children of x'
    $Q.\text{insert}(x', f_{x'})$
return the tree constructed

Outline

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Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

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Def. A strategy is "safe" if there is always an optimum solution that "agrees with" the decision made according to the strategy.

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- Interval scheduling problem: remove j^* and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one

Outline

Exercise: Fractional Knapsack Problem

Fractional Knapsack

Input: A knapsack of bounded capacity W;
n items, each of weight $\{w_1, w_2, ..., w_n\}$ and each item also has a value $\{v_1, v_2, ..., v_n\}$.

Output: Select a set of fractions $\{p_1, p_2, ..., p_n\}$ $(0 \leq p_i \leq 1)$ for all items to maximize the total value $p_1 v_1 + p_2 v_2 + ... + p_n v_n$ while $\sum_{i \in [n]} w_i p_i \leq W$.

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- Example: Given are a knapsack with capacity $W = 20$ and 5 items with the following weights and values:

	1	2	3	4	5
weight	10	6	5	8	12
value	15	10	10	10	10

Exercise: Scheduling Problem with Min Weighted Completion Time

Scheduling Problem

Input: Given are n jobs each $i \in [n]$ has a weight (or the importance) w_i and the length (or the time required) l_j. We define the completion time c_j of job j to be the sum of the lengths of jobs in the ordering up to and including l_j.

Output: An ordering of jobs that minimizes the weighted sum of completion times $\sum_{i \in [n]} w_i c_i$.

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	1	2	3	4	5
weight	2	6	5	4	2
length	5	4	10	8	3