Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+\ O(m)\times$ (time for decrease_key)

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.

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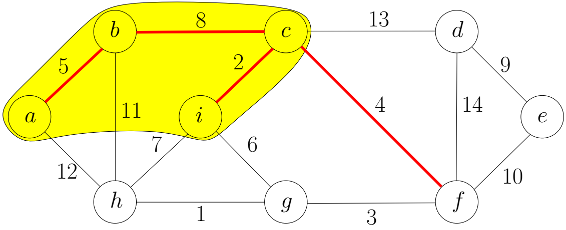

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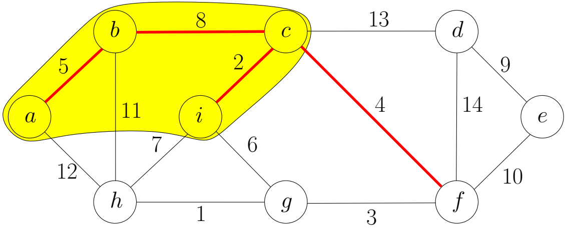

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

"Evidence" for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
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Thus, the minimum spanning tree is unique with assumption.

Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
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algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

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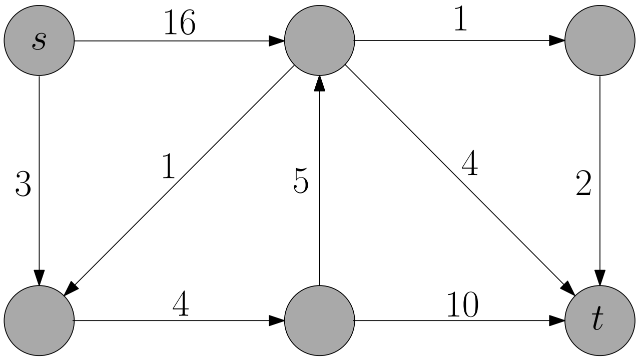

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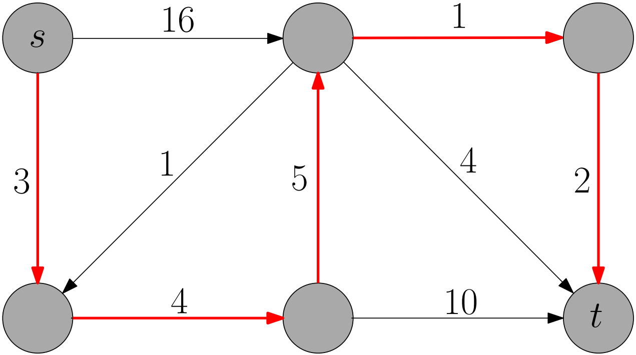

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Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

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Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi[v], v \in V \setminus s$: the parent of v in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from s to v

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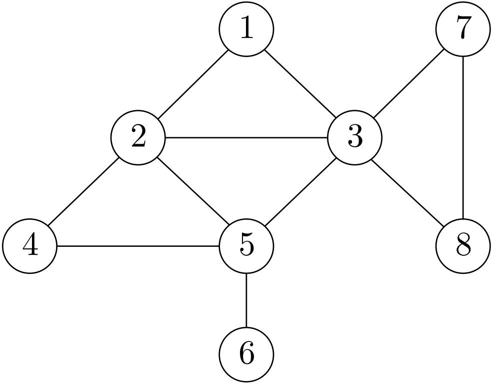

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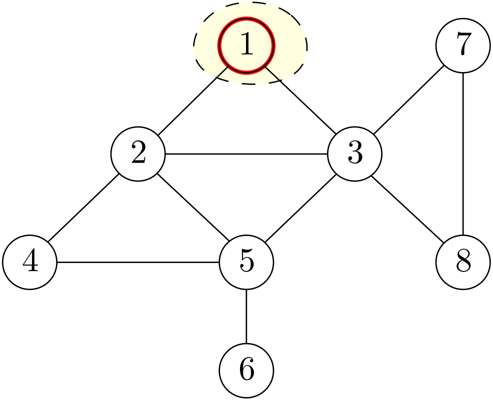

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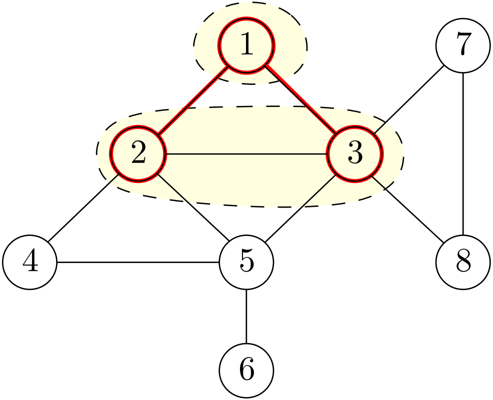

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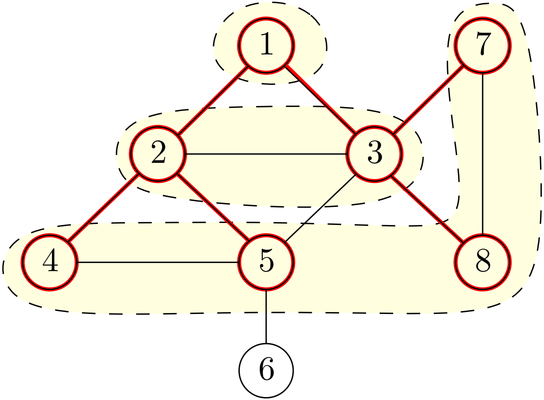

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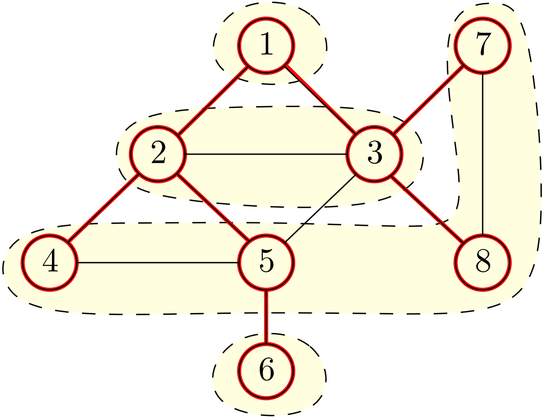

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Shortest Path Algorithm by Running BFS

1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS
3: $\pi[v] \leftarrow$ vertex from which v is visited
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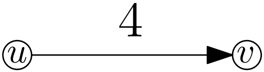
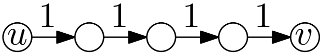




Shortest Path Algorithm by Running BFS

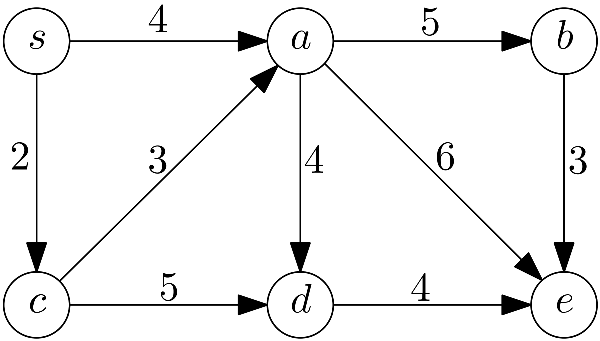
1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS virtually
3: $\pi[v] \leftarrow$ vertex from which v is visited
4: $d[v] \leftarrow$ index of the level containing v

- Problem: $w(u, v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

$S \leftarrow \{s\}, d(s) \leftarrow 0$
while $|S| \leq n$ **do**
    find a $v \notin S$ that minimizes $\min_{u \in S:(u,v) \in E} \{d[u] + w(u,v)\}$
    $S \leftarrow S \cup \{v\}$
    $d[v] \leftarrow \min_{u \in S:(u,v) \in E} \{d[u] + w(u,v)\}$

Virtual BFS: Example



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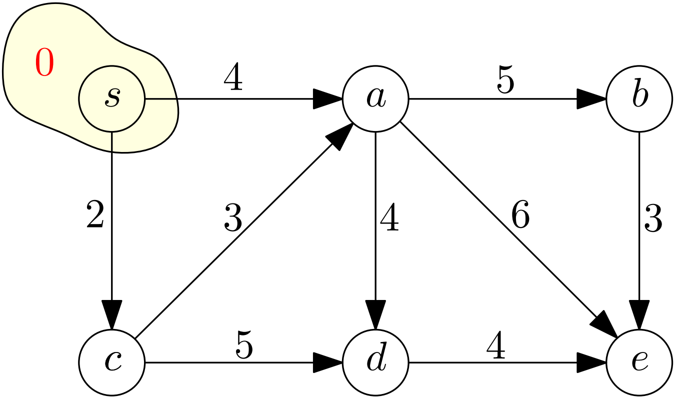
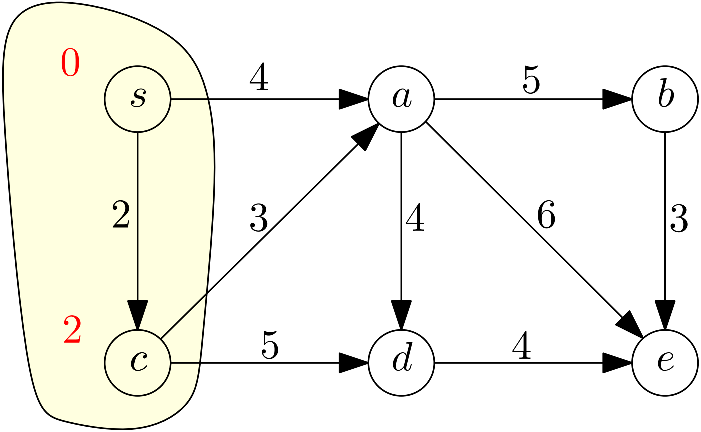


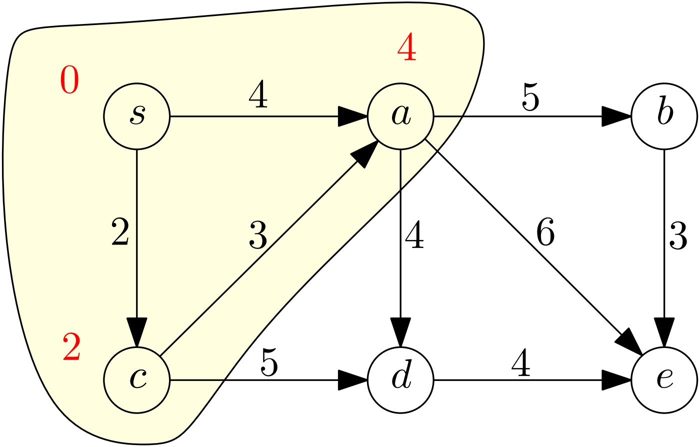
Time 0

Virtual BFS: Example



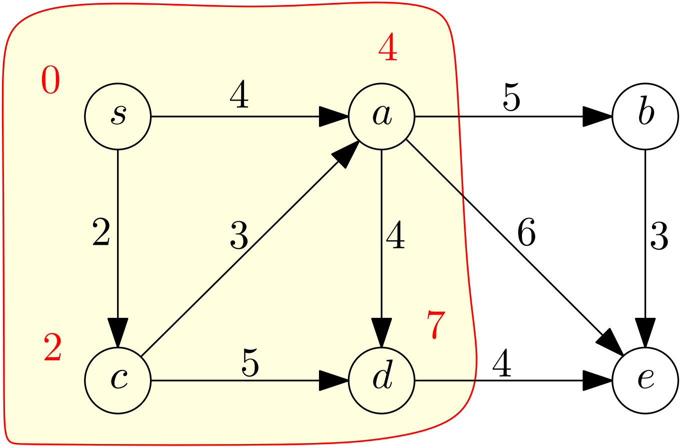
Time 2

Virtual BFS: Example



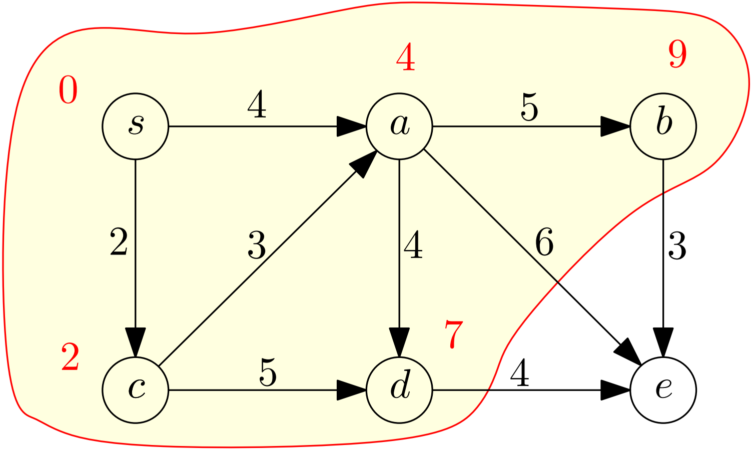
Time 4

Virtual BFS: Example



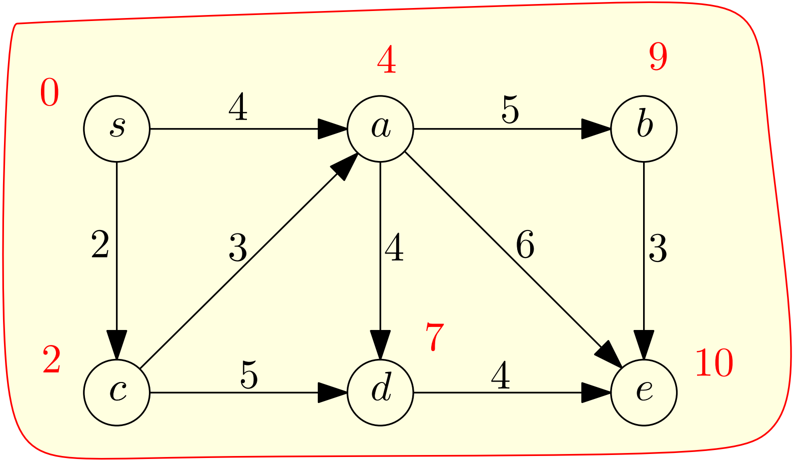
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

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Dijkstra's Algorithm

Dijkstra(G, w, s)

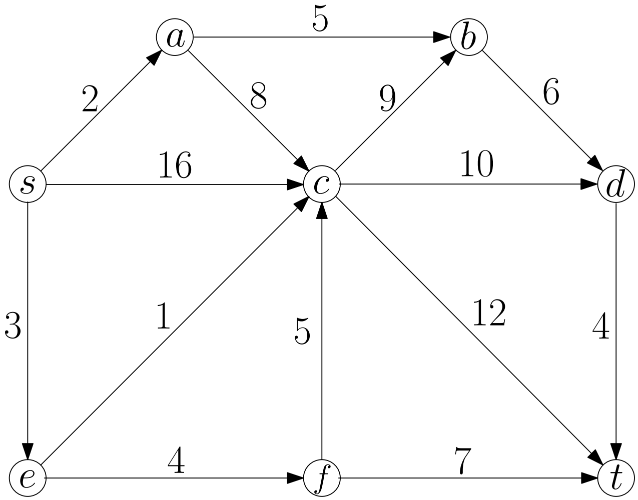
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    add u to S
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
return (d, π)

Dijkstra's Algorithm

Dijkstra(G, w, s)

```
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
while S ≠ V do
    u ← vertex in V \ S with the minimum d[u]
    add u to S
    for each v ∈ V \ S such that (u, v) ∈ E do
        if d[u] + w(u, v) < d[v] then
            d[v] ← d[u] + w(u, v)
            π[v] ← u
return (d, π)
```

- Running time $= O(n^2)$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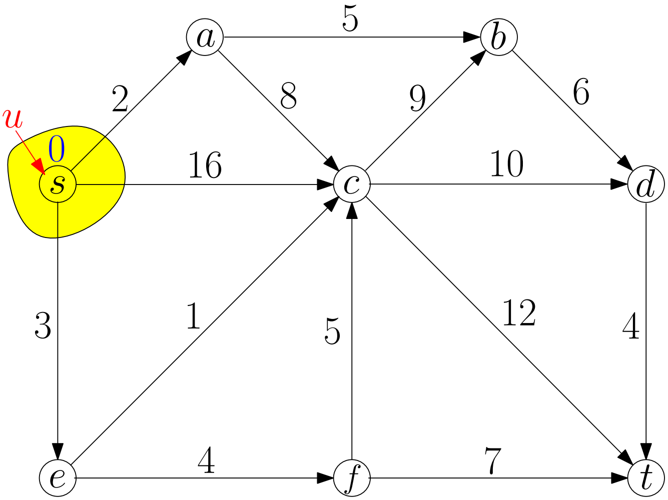

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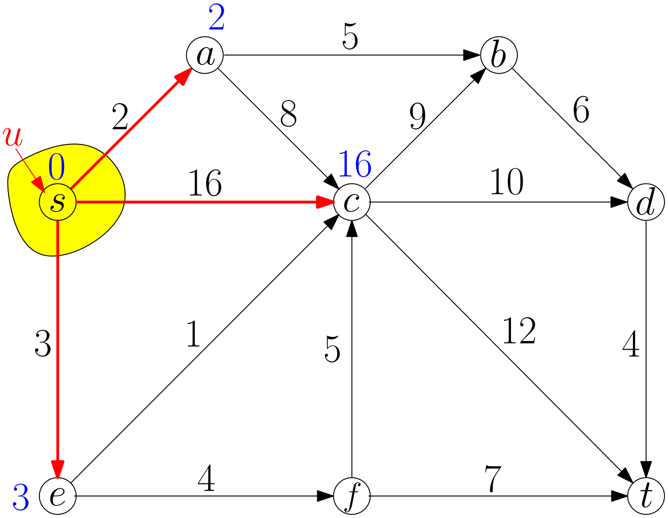

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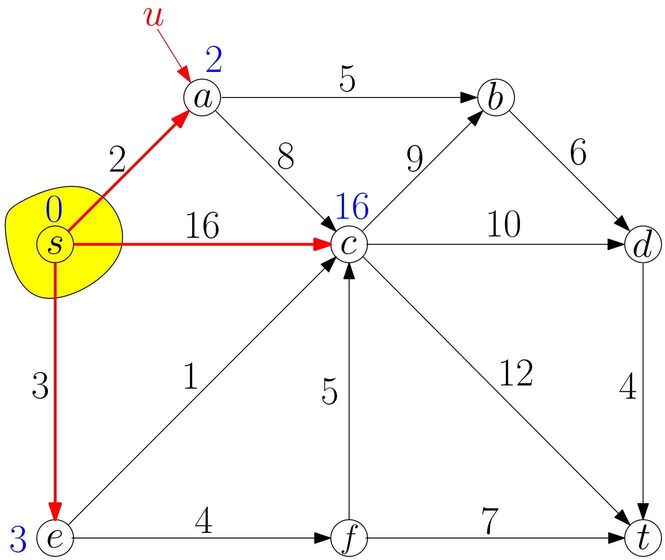
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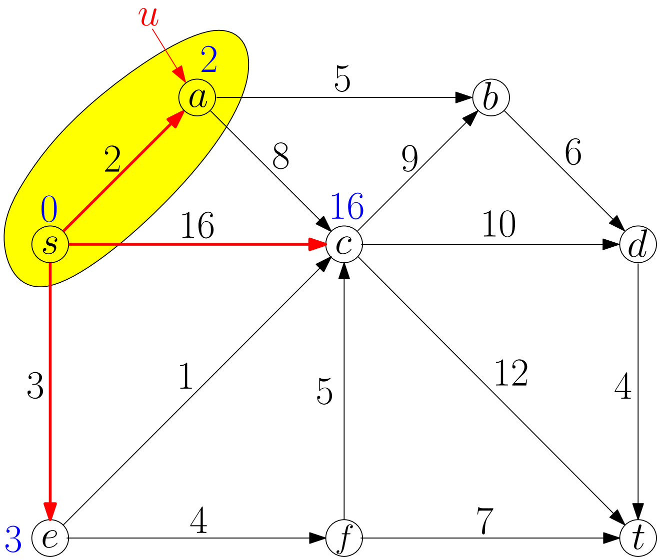

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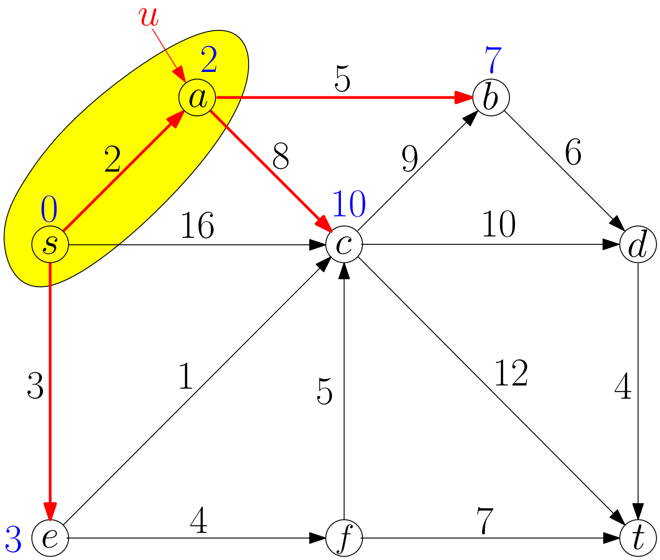
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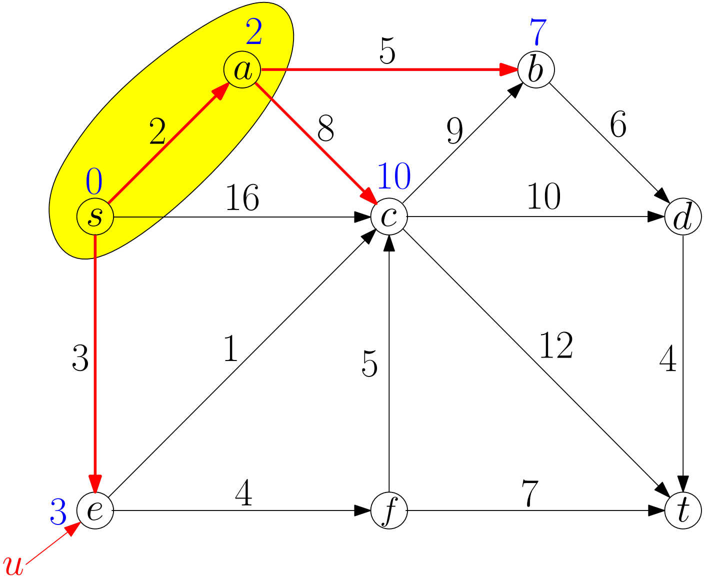

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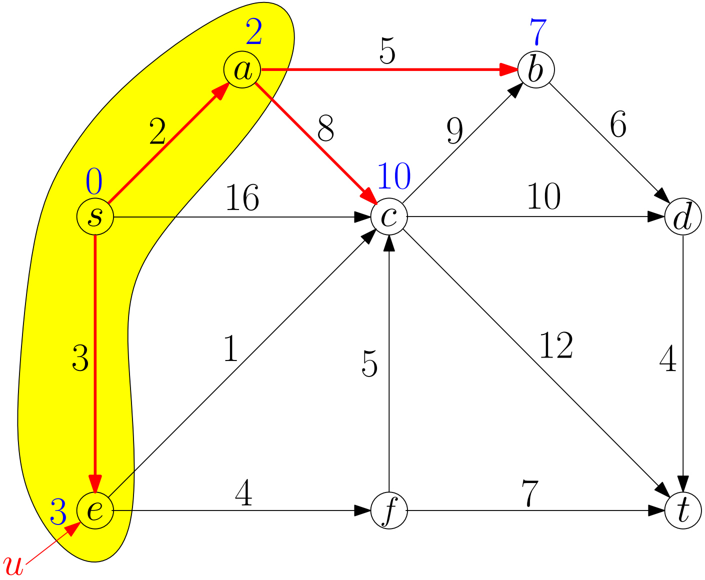
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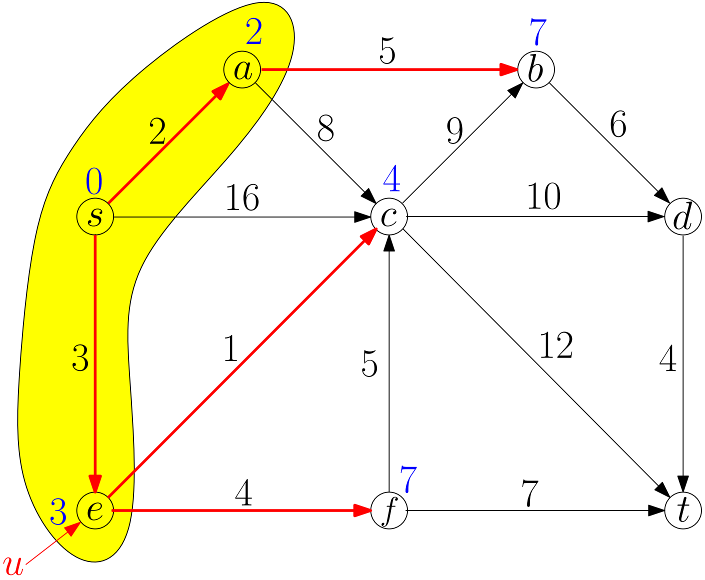

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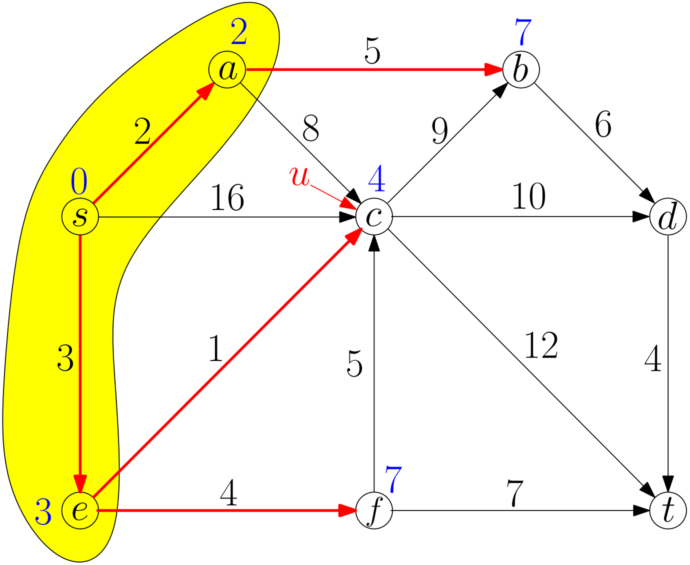
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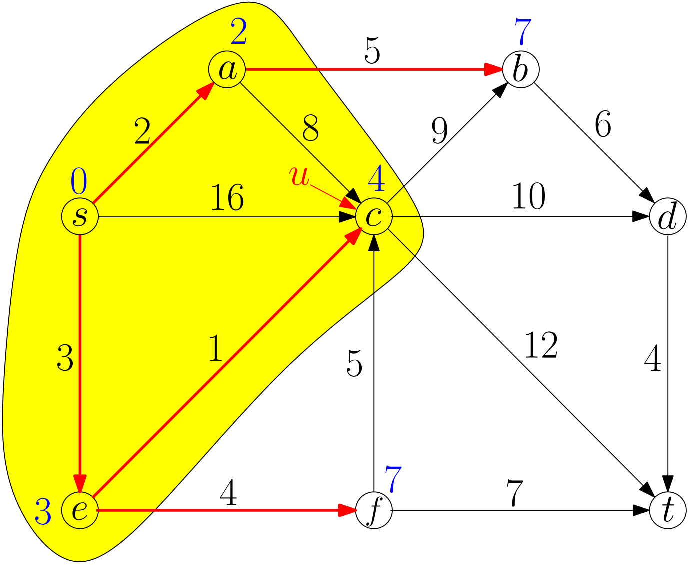

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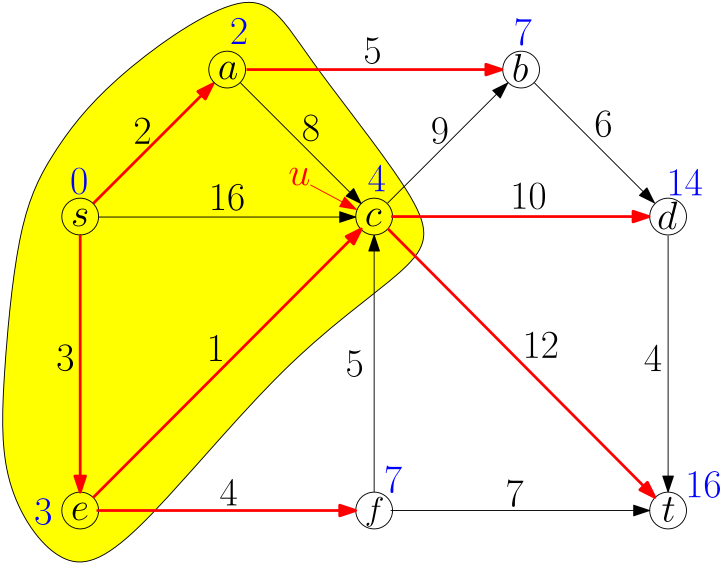
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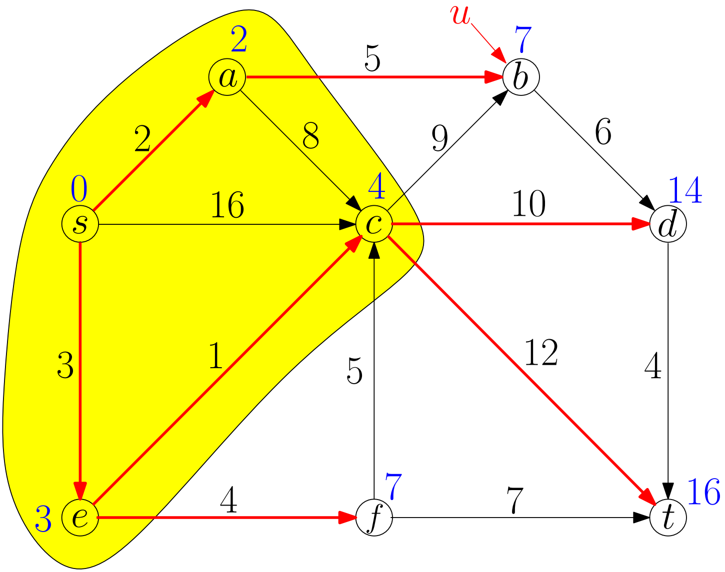
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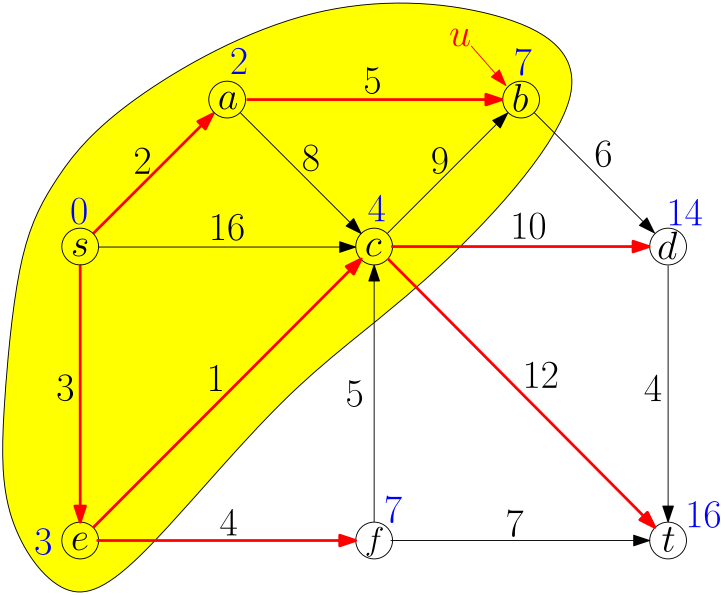
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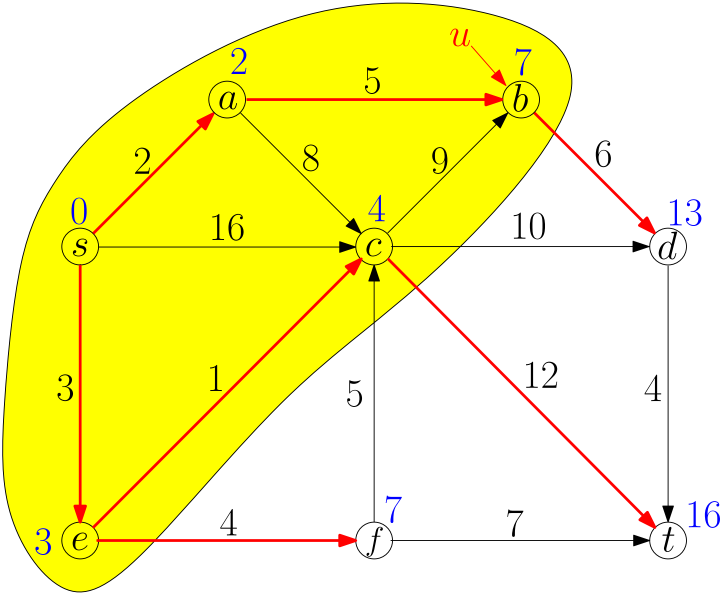

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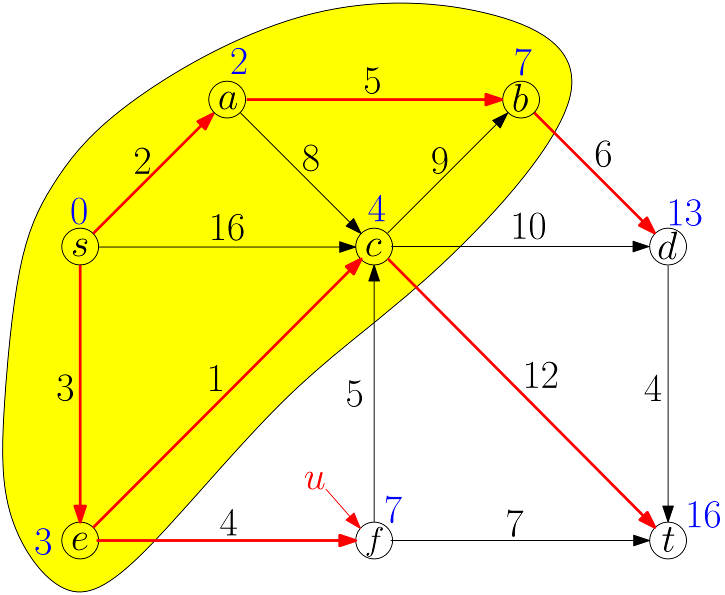

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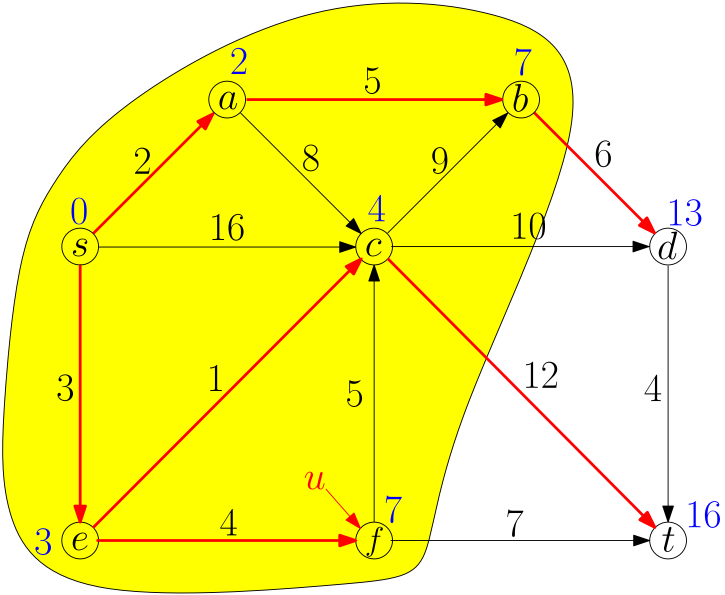

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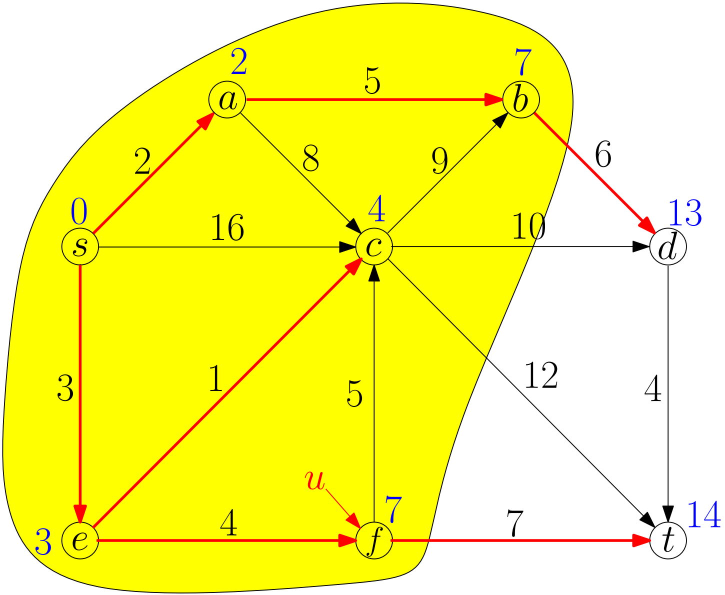
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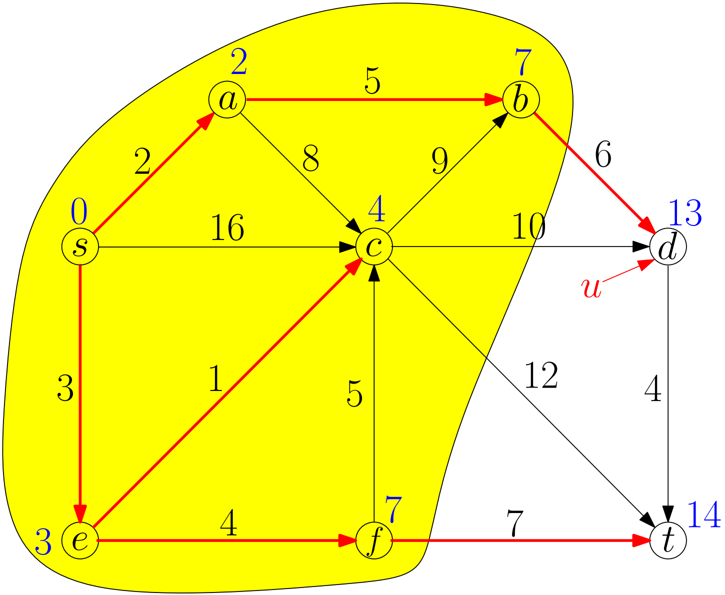
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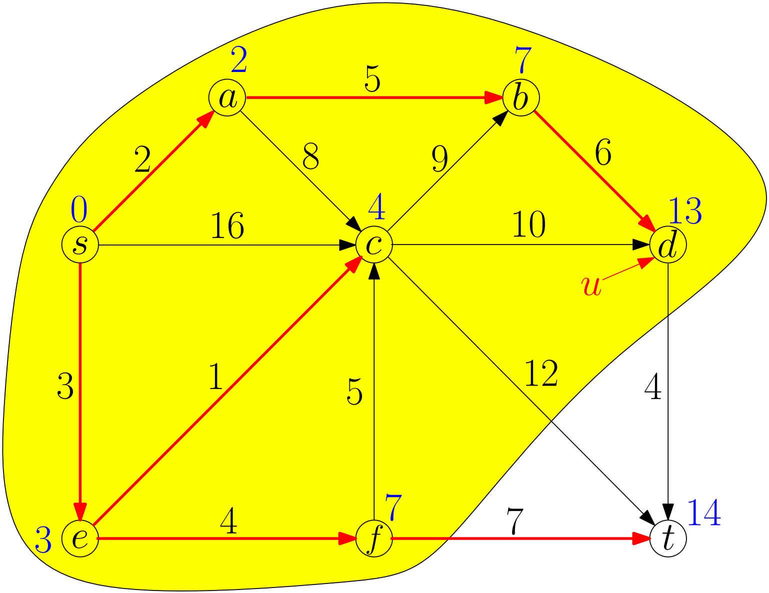
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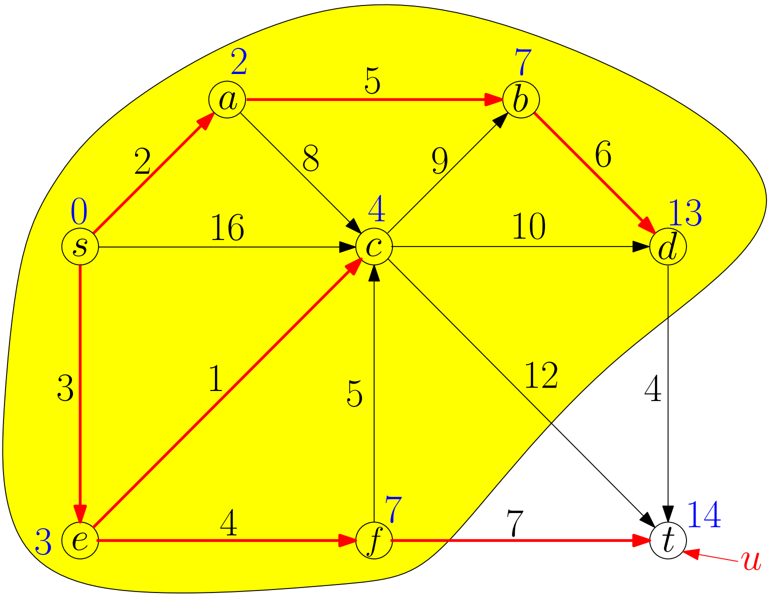

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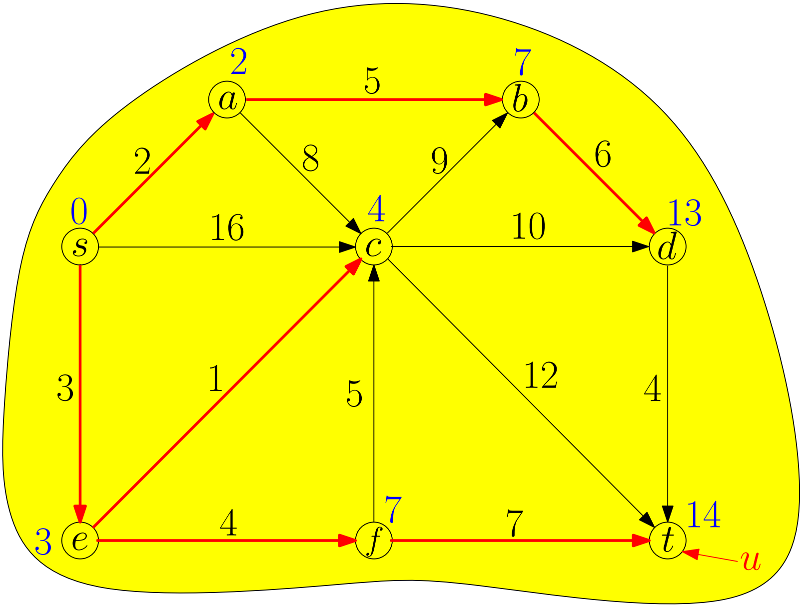
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Improved Running Time using Priority Queue

Dijkstra(G, w, s)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
Q ← empty queue, for each v ∈ V: Q.insert(v, d[v])
while S ≠ V do
    u ← Q.extract_min()
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if d[u] + w(u, v) < d[v] then
            d[v] ← d[u] + w(u, v), Q.decrease_key(v, d[v])
            π[v] ← u
return (π, d)
```

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
  $u \leftarrow Q$.extract_min()
  $S \leftarrow S \cup \{u\}$
  for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
    if $w(u, v) < d[v]$ **then**
      $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
      $\pi[v] \leftarrow u$
return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Improved Running Time

Running time:

$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$