Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

```
if n = 0 return 0
if n = 1 return 1
return Fib(n − 1) + Fib(n − 2)
```

Q: Is the running time of the algorithm polynomial or exponential in n?

A: Exponential

- Running time is at least $\Omega(F_n)$

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2: if $n = 1$ return 1
3: return Fib$(n-1)$ + Fib$(n-2)$

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A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n: Reasonable Algorithm

Fib(n)

$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for $i \leftarrow 2$ to n **do**
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

- Dynamic Programming

Computing F_n: Reasonable Algorithm

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F[0] ← 0
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- Dynamic Programming
- Running time = ?

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- Dynamic Programming
- Running time $= O(n)$

Computing F_n: Even Better Algorithm

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\cdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

power(n)

```
if n = 0 then return ( 1 0 ; 0 1 )
R ← power(⌊n/2⌋)
R ← R × R
if n is odd then R ← R × ( 1 1 ; 1 0 )
return R
```

if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$R \leftarrow$ power$(\lfloor n/2 \rfloor)$
$R \leftarrow R \times R$
if n is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
return R

Fib(n)

if $n = 0$ then return 0
$M \leftarrow$ power$(n - 1)$
return $M[1][1]$

power(n)

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- Recurrence for running time?

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- Even printing $F(n)$ requires time much larger than $O(\lg n)$

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Fixing the Problem

To compute F_n, we need $O(\lg n)$ basic arithmetic operations on integers

Summary: Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

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- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, $\cdots$:
  $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \lg n)$

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- To improve running time, design better algorithm for "combine" step, or reduce number of recursions, ...

CSE 431/531: Algorithm Analysis and Design (Fall 2023)
Dynamic Programming

Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo

Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many independent sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Recall: Computing the n-th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

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- Store each $F[i]$ for future use.

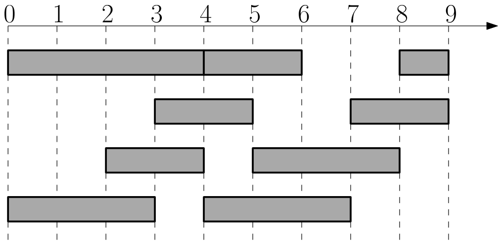
Outline

Recall: Interval Schduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs



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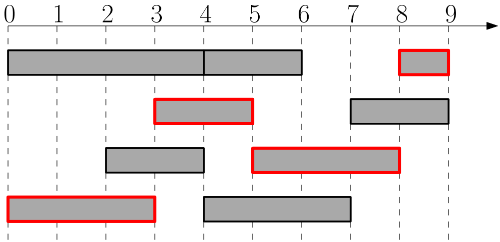

Weighted Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

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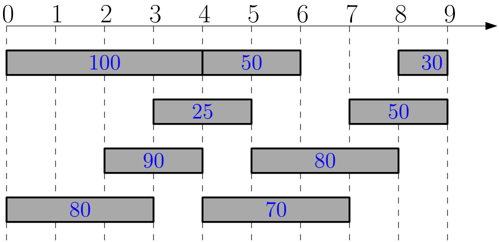

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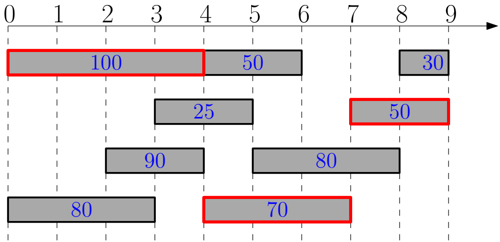

Optimum value $= 220$

Hard to Design a Greedy Algorithm

Q: Which job is safe to schedule?

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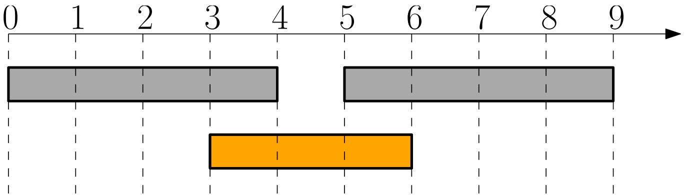

Designing a Dynamic Programming Algorithm

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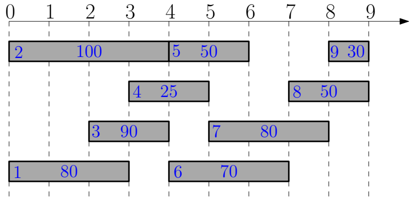

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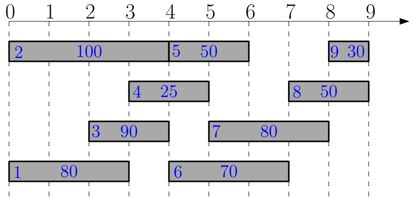
Designing a Dynamic Programming Algorithm



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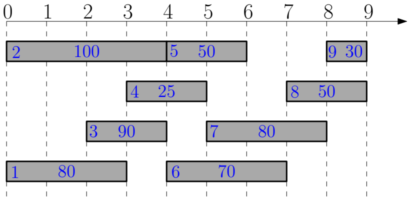

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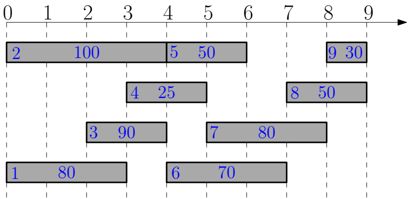
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Designing a Dynamic Programming Algorithm



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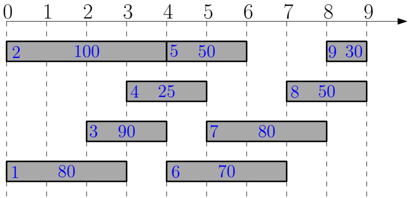

i	$opt[i]$
0	0
1	80
2	
3	
4	
5	
6	
7	
8	
9	

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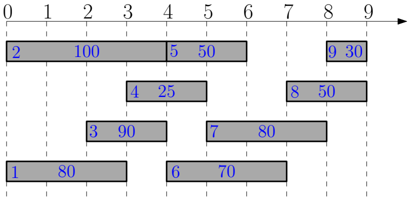

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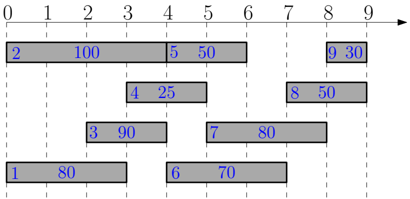

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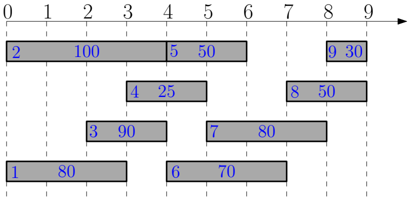

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i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220