Terminologies

When we talk about upper bound on running time:

- Logarithmic time: $O(\log n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
  - $O(n \log n) \subseteq O(n^{1.1})$. So, an $O(n \log n)$-time algorithm is also a polynomial time algorithm.
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.

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- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

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- For "natural" algorithms, constants are not so big!

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A:

- Sometimes yes
- However, when n is big enough, $1000n < 0.1n^2$
- For "natural" algorithms, constants are not so big!
- So, for reasonably large n, algorithm with lower order running time beats algorithm with higher order running time.

CSE 431/531B: Algorithm Analysis and Design (Fall 2023)

Graph Basics

Lecturer: Kelin Luo

Department of Computer Science and Engineering
University at Buffalo

Outline

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

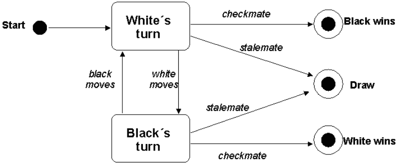
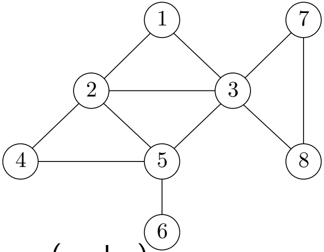


Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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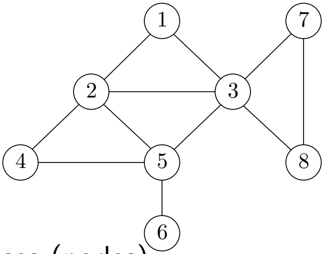
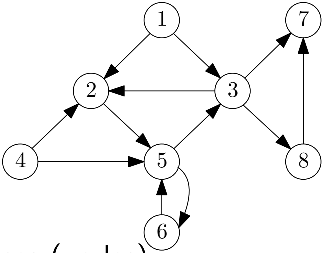


- V: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$

Directed Graph $G = (V, E)$



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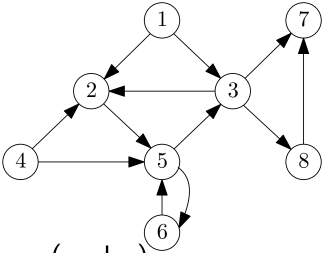

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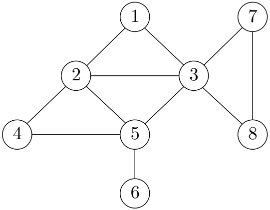
Abuse of Notations

- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



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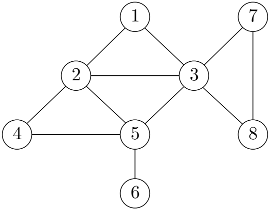
- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

Representation of Graphs



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected

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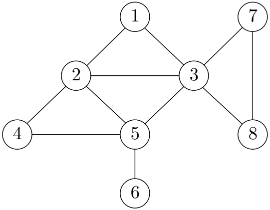

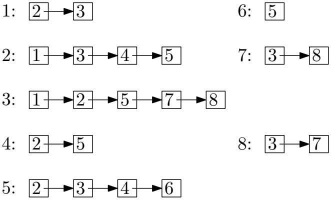

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  - For every vertex v, there is a linked list containing all neighbors of v.

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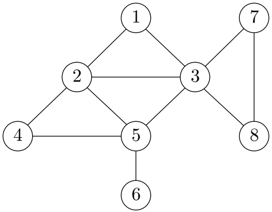

1: $\boxed{2}\boxed{3}$

2: $\boxed{1}\boxed{3}\boxed{4}\boxed{5}$

3: $\boxed{1}\boxed{2}\boxed{5}\boxed{7}\boxed{8}$

4: $\boxed{2}\boxed{5}$

5: $\boxed{2}\boxed{3}\boxed{4}\boxed{6}$

6: $\boxed{5}$

7: $\boxed{3}\boxed{8}$

8: $\boxed{3}\boxed{7}$

$d : (2, 4, 5, 2, 4, 1, 2, 2)$

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  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbors of v.
  - When graph is static, can use array of variant-length arrays.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
time to list all neighbors of v		

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	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u,v) \in E$	$O(1)$	
time to list all neighbors of v		

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Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

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  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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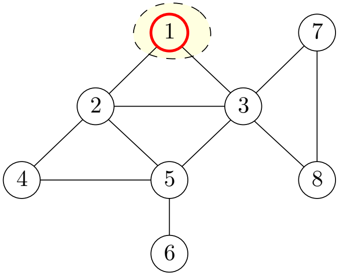

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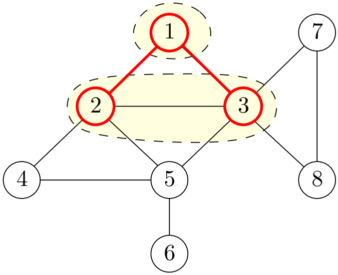

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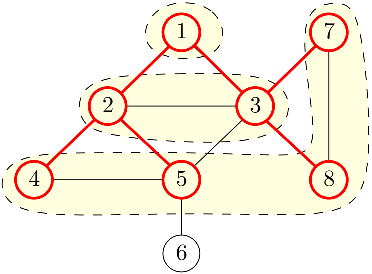

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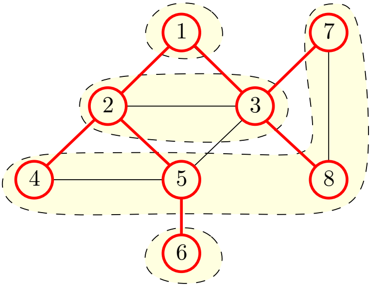


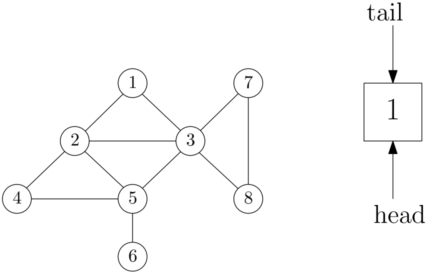
Implementing BFS using a Queue

BFS(s)

```
head ← 1, tail ← 1, queue[1] ← s
mark s as "visited" and all other vertices as "unvisited"
while head ≤ tail do
    v ← queue[head], head ← head + 1
    for all neighbors u of v do
        if u is "unvisited" then
            tail ← tail + 1, queue[tail] = u
            mark u as "visited"
```

- Running time: $O(n+m)$.

Example of BFS via Queue



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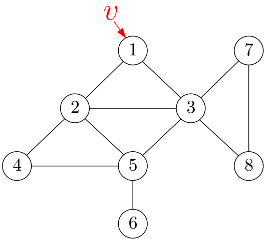

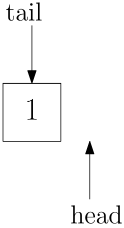

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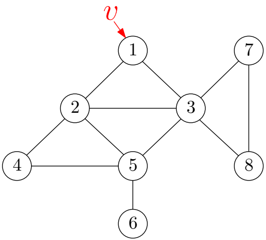

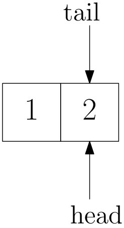

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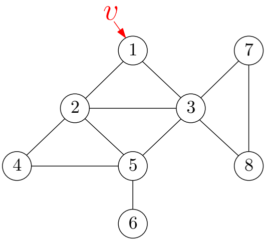

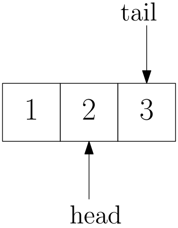

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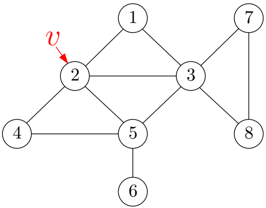

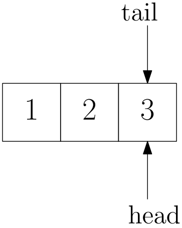

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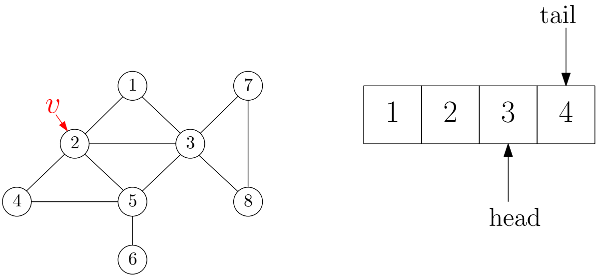

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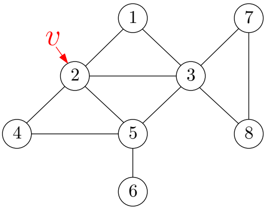

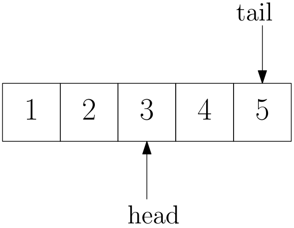

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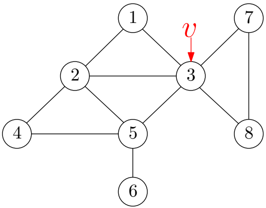

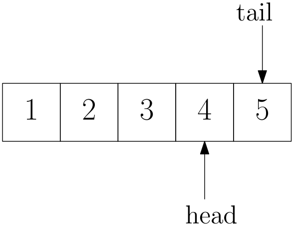

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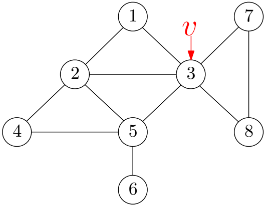

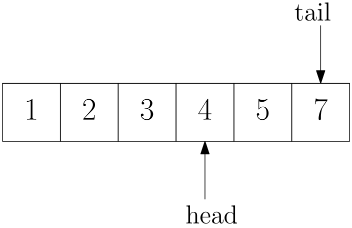

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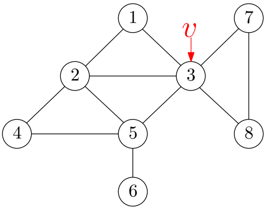

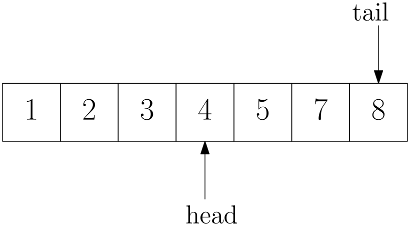

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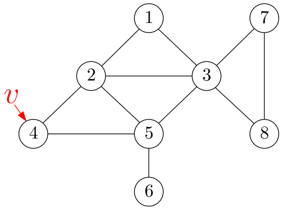

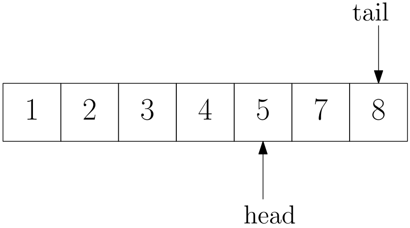

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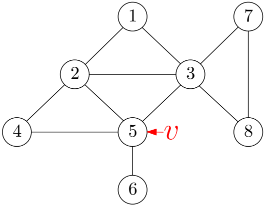

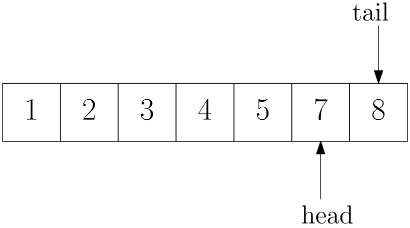

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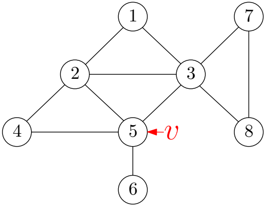

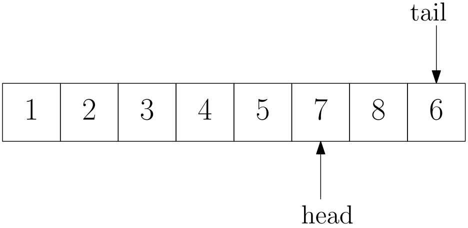

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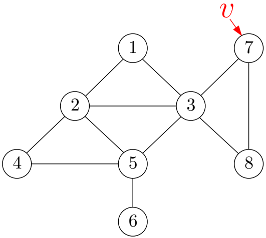
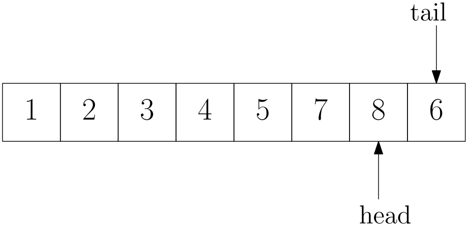
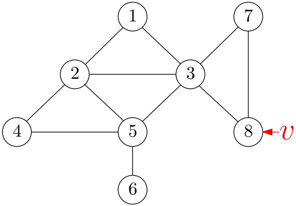
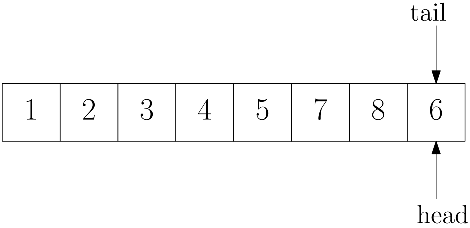




Example of BFS via Queue





Example of BFS via Queue