Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

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Example:

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: $45 \to 60, 20 \to 40, 19 \to 25$

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?

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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.

Analysis of Greedy Algorithm

- Safety: Prove that the reasonable strategy is "safe"
- Self-reduce: Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

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- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:

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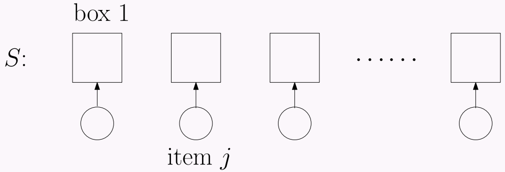

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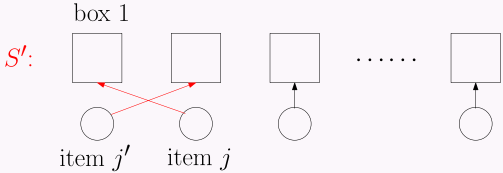

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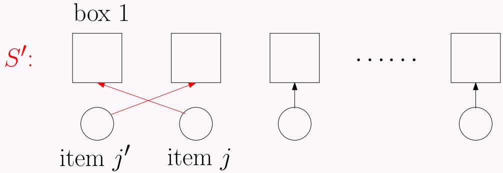

- $s_{j'} \leq s_j$, and swapping gives another solution S'
- S' is also an optimum solution. In S', j is put into Box 1. □

- Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.

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- Trivial: we decided to put Item j into Box 1, and the remaining instance is obtained by removing Item j and Box 1.

Generic Greedy Algorithm

while the instance is non-trivial **do**
    make the choice using the greedy strategy
    reduce the instance

Greedy Algorithm for Box Packing

$T \leftarrow \{1, 2, 3, \cdots, m\}$
for $i \leftarrow 1$ to n **do**
    if some item in T can be put into box i **then**
        $j \leftarrow$ the largest item in T that can be put into box i
        print("put item j in box i")
        $T \leftarrow T \setminus \{j\}$

Why "Safety" + "Self-reduce" $\Longrightarrow$ Optimality?

- Let BP(B, T) denote a box-packing instance.
- $\phi(1, 2, ..., m) \mapsto \{1, 2, ..., n, \text{NULL}\}$ denote packing strategy. e.g., $\phi(2) = 3$ means item 2 is put into box 3.
- val(ϕ) := the number of items packed by ϕ.
- ϕ_g: the packing strategy obtained by greedy algorithm.

Proof.

- Base case: When $|B| = 1$ or $|T| = 1$.
- Inductive case: (Hypothesis) Assume Greedy alg solves BP(B', T') optimally for $|B'| = n - 1$ and $|T'| = m - 1$.

□

Why "Safety" + "Self-reduce" $\implies$ Optimality?

Proof.

- (Induction) Wlog, let π be the optimal solution matches our greedy sol on BP(B, T), saying $\pi(j) = 1$.
- By self-reduce: BP$(B \setminus \{1\}, T \setminus \{j\})$ is a smaller BP instance.
- π and ϕ_g onto BP$(B \setminus \{1\}, T \setminus \{j\})$, denoted as π' and ϕ'_g.
- By Inductive hypothesis, ϕ'_g is the optimal sol for BP$(B \setminus \{1\}, T \setminus \{j\})$.
- val$(\pi) \geq$ val$(\phi_g) = 1+$ val$(\phi'_g) \geq 1+$val(π') =val(π).

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Running time

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- With sorted item-sizes and box-capacities, running time is $O(\max\{n, m\})$.

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Generic Greedy Algorithm

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Lemma Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy stretegy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.

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Def. A strategy is "safe" if there is always an optimum solution that is "consistent" with the decision made according to the strategy.

Exchange argument: Proof of Safety of a Strategy

- let S be an arbitrary optimum solution.
- if S is consistent with the greedy choice, done.
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Outline

1 Toy Example: Box Packing

2 Interval Scheduling

3 Offline Caching
- Heap: Concrete Data Structure for Priority Queue

4 Data Compression and Huffman Code

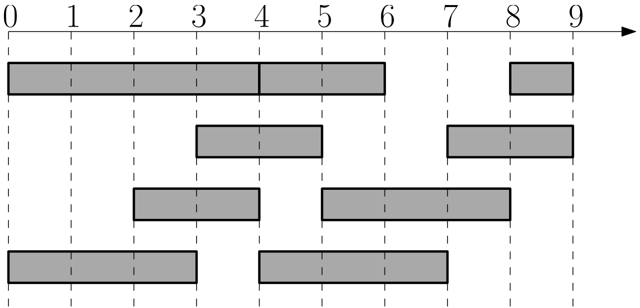
5 Summary

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



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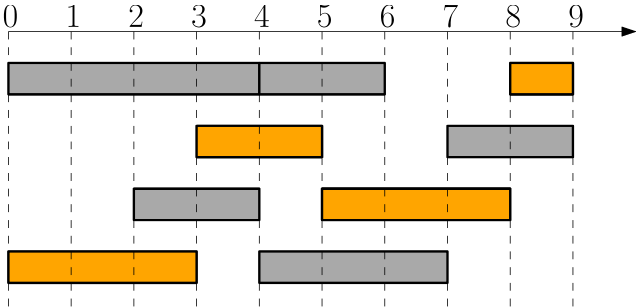

Greedy Algorithm for Interval Scheduling

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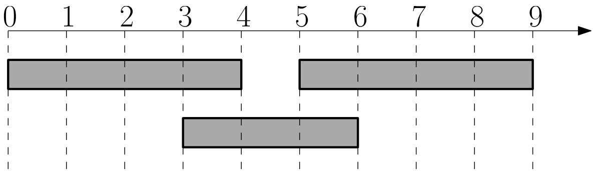

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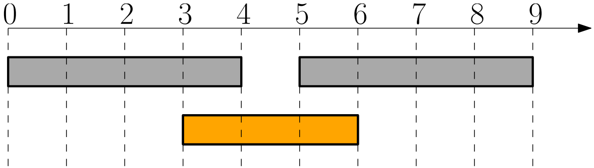

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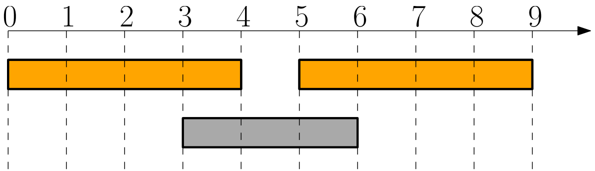

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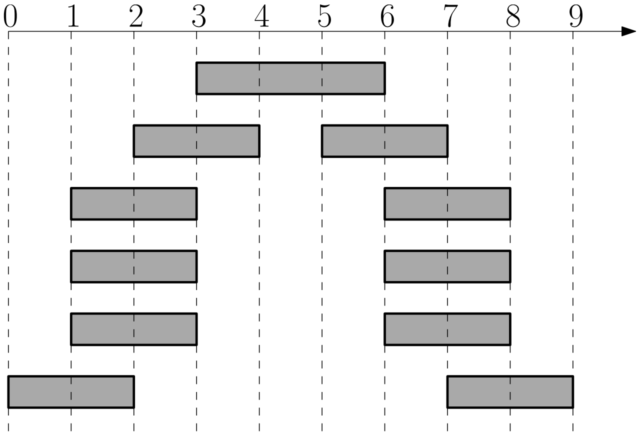

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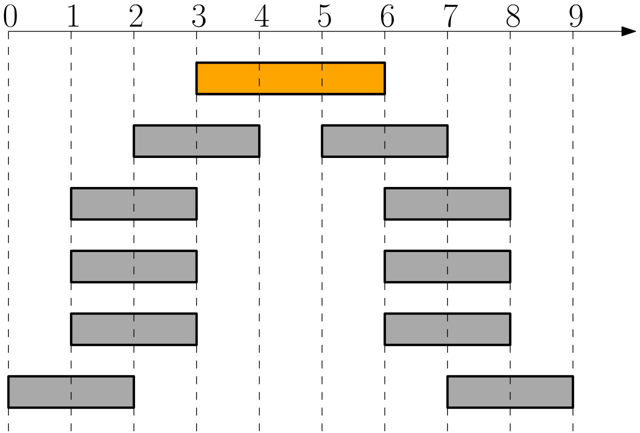

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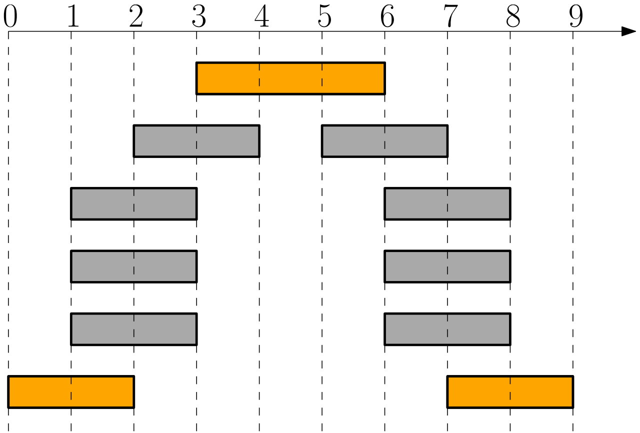

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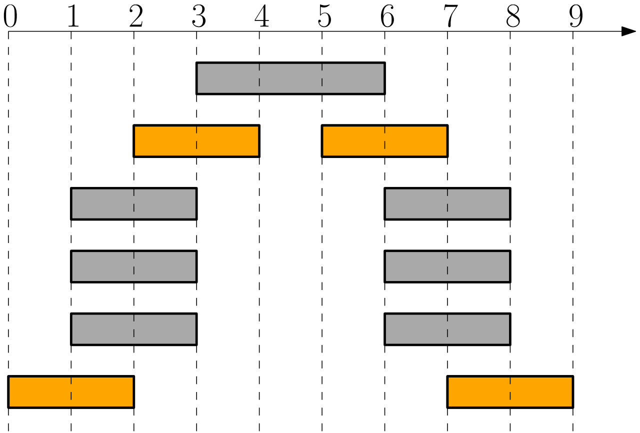

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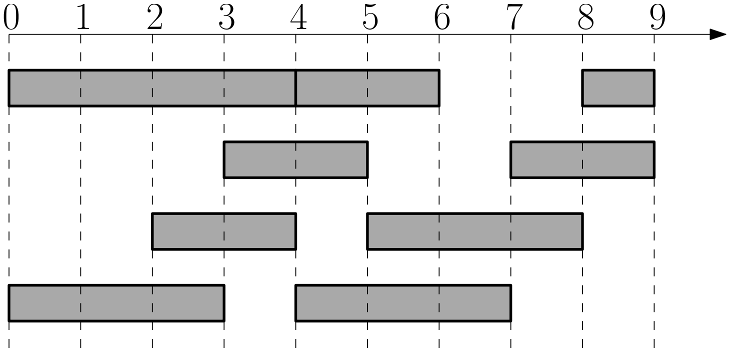

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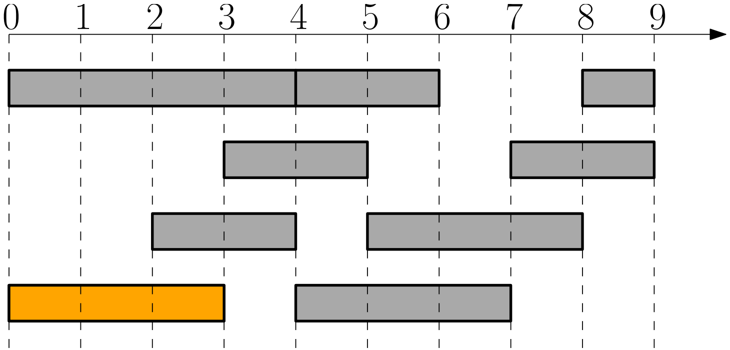

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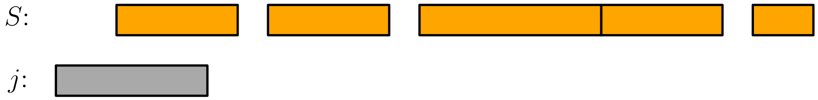

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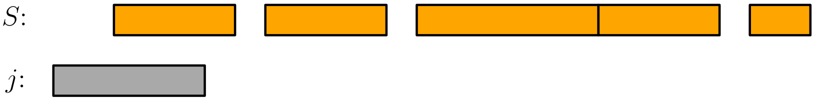

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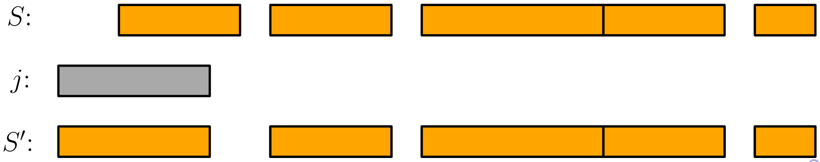

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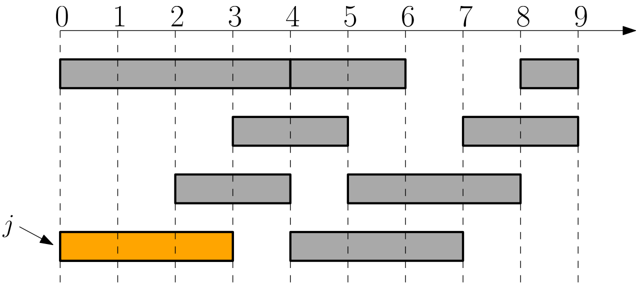
Lemma It is safe to schedule the job j with the earliest finish time: There is an optimum solution where the job j with the earliest finish time is scheduled.

- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem?



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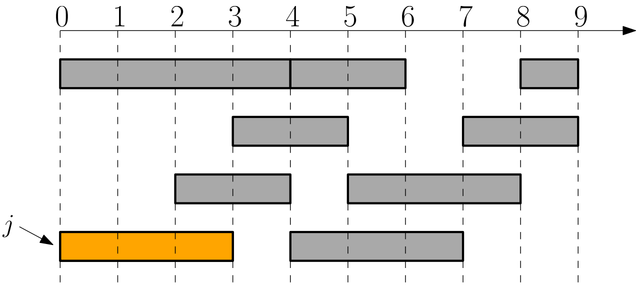

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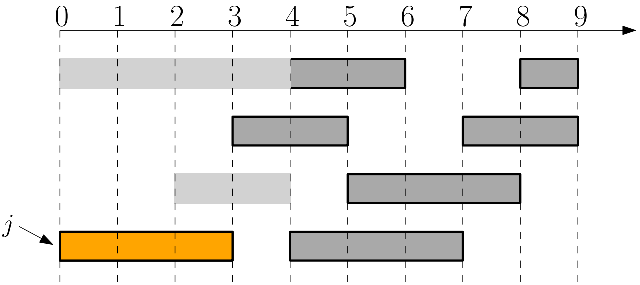

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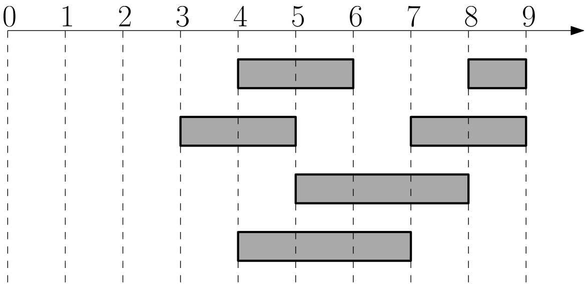

Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$ **do**
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
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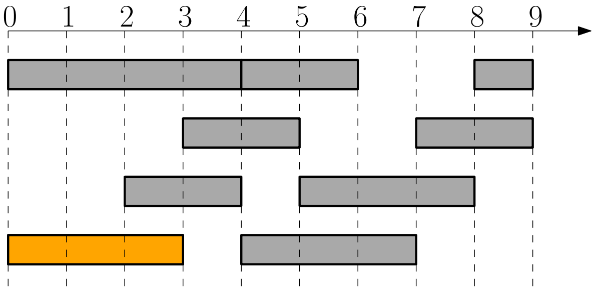

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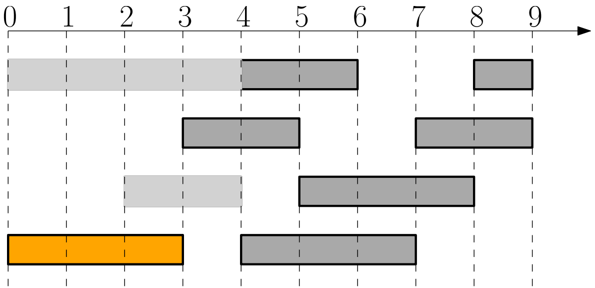

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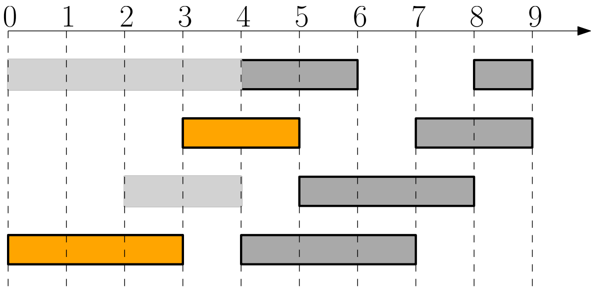

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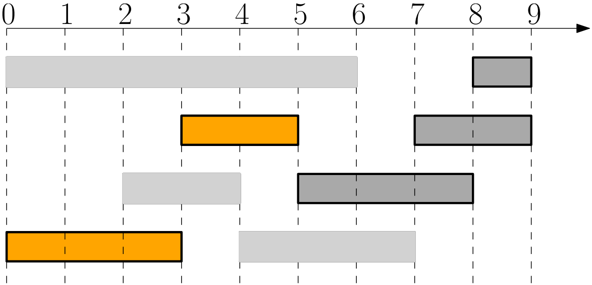

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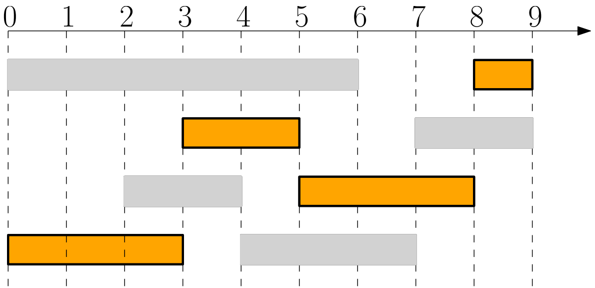

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```
A ← {1, 2, ··· , n}, S ← ∅
while A ≠ ∅ do
    j ← arg min_{j'∈A} f_{j'}
    S ← S ∪ {j}; A ← {j' ∈ A : s_{j'} ≥ f_j}
return S
```

Running time of algorithm?

Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2: **while** $A \neq \emptyset$ **do**
3: $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4: $\quad S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5: **return** S

Running time of algorithm?

- Naive implementation: $O(n^2)$ time