Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2: **while** $A \neq \emptyset$ **do**
3: $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4: $\quad S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5: **return** S

Running time of algorithm?

- Naive implementation: $O(n^2)$ time

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Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

sort jobs according to f values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of f_j **do**
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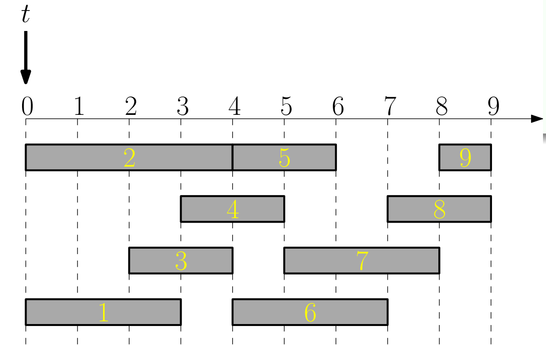

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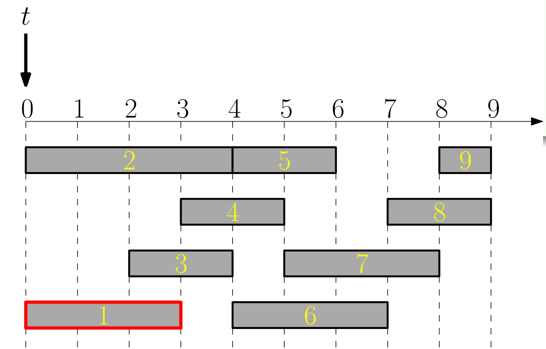

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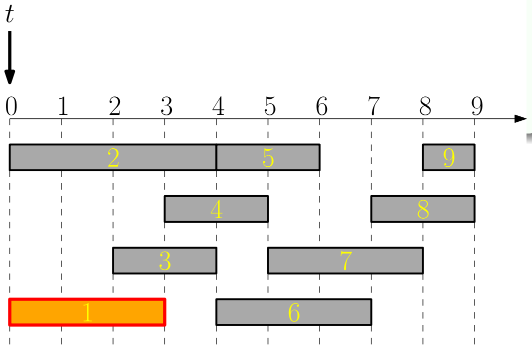

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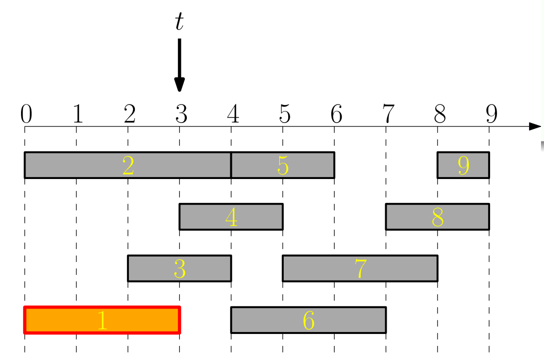

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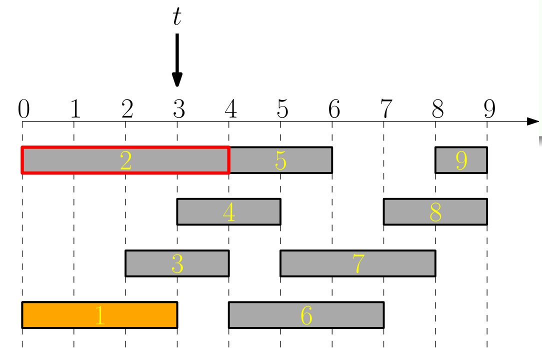

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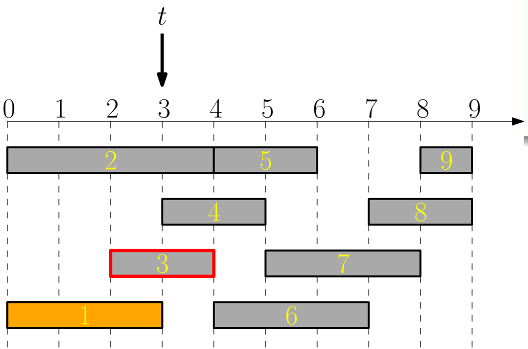

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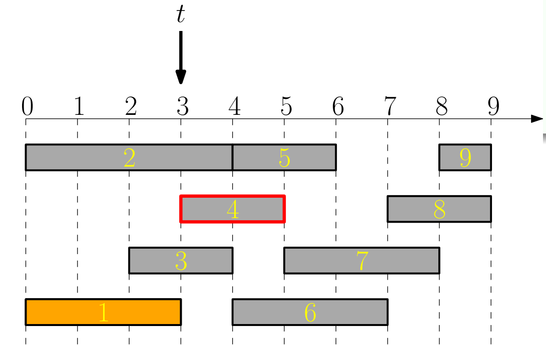

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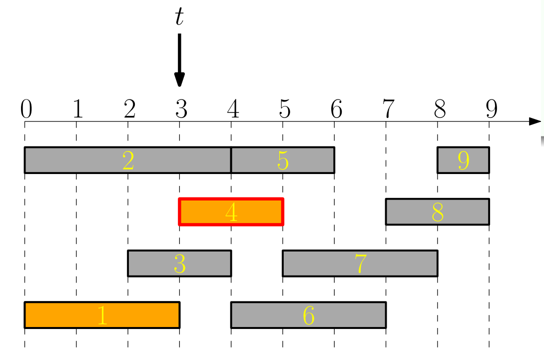

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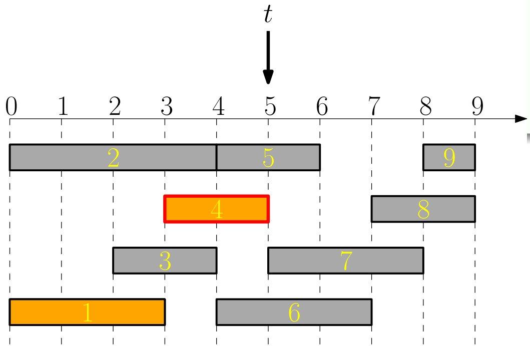

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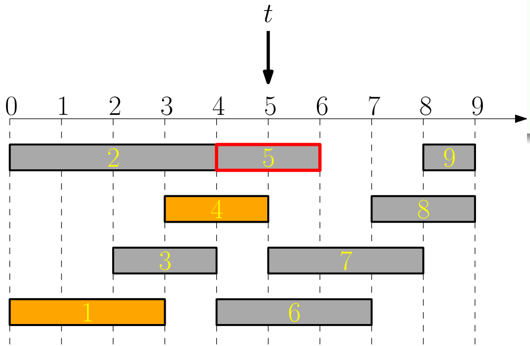

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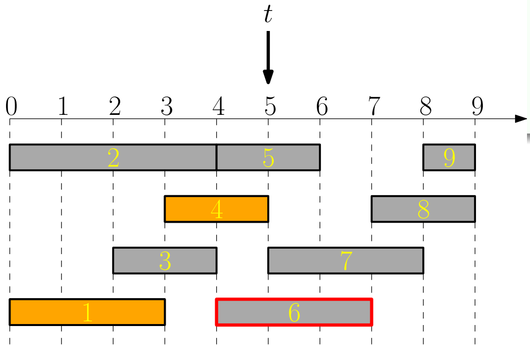

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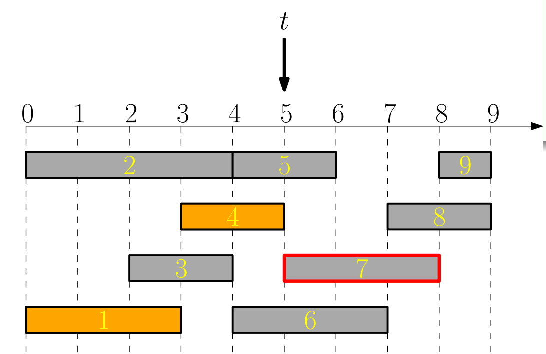

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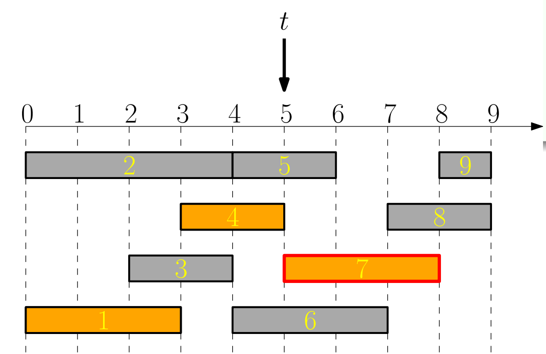

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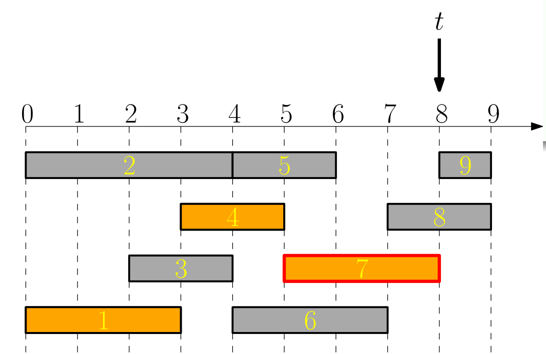

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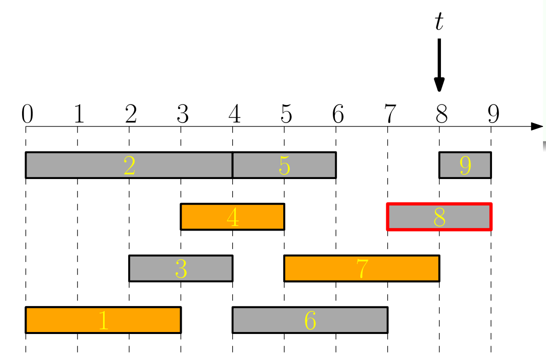

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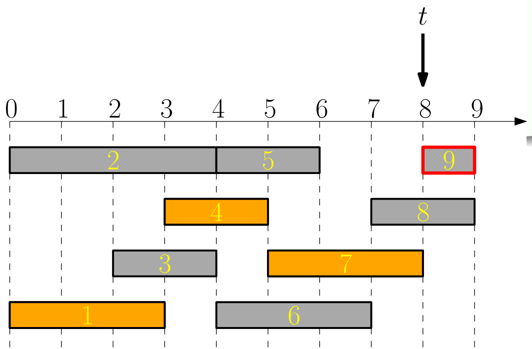

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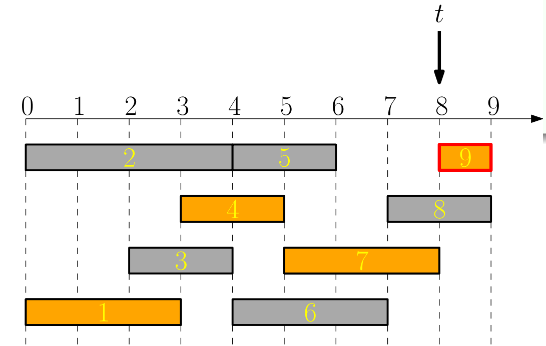

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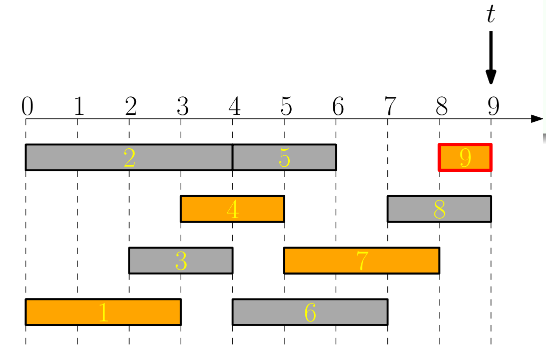


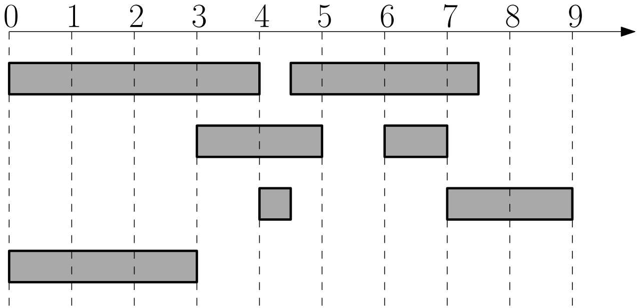
Outline

Interval Partitioning

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A minimum number of machines to schedule all jobs so that all jobs on a single machine are compatible.



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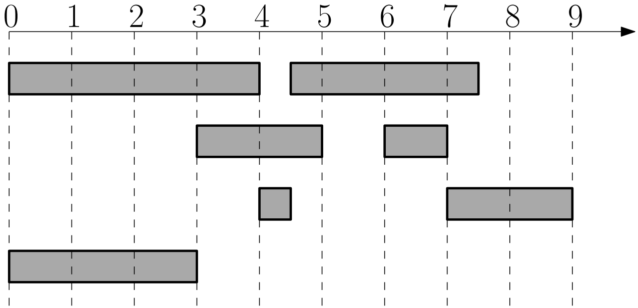

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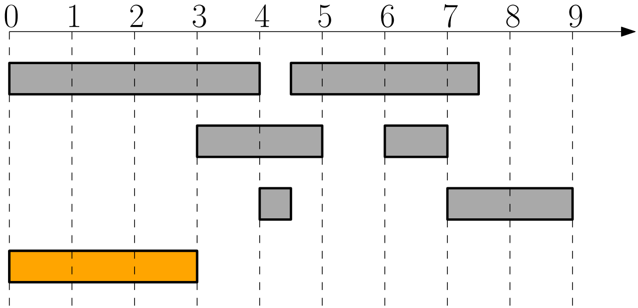

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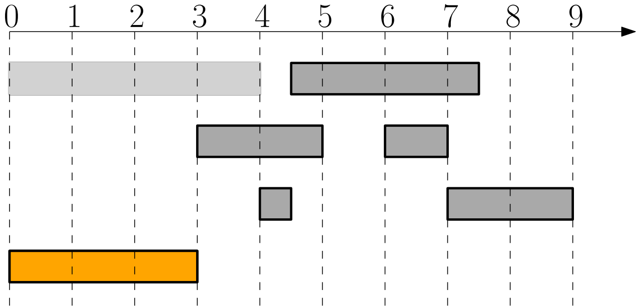

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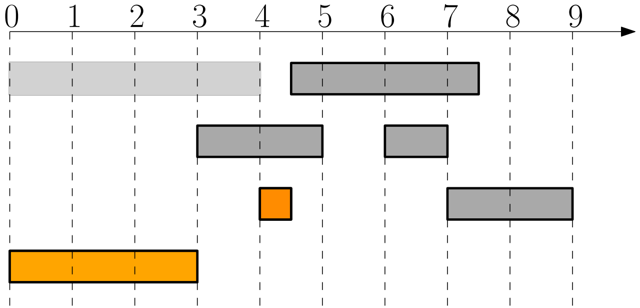

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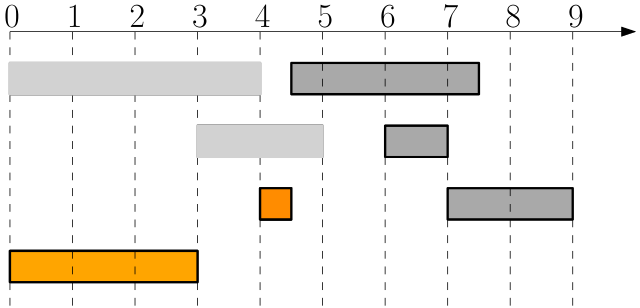

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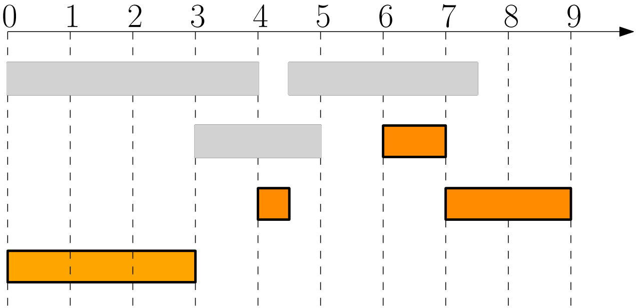

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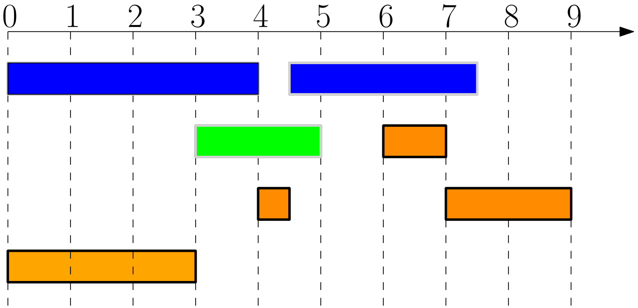

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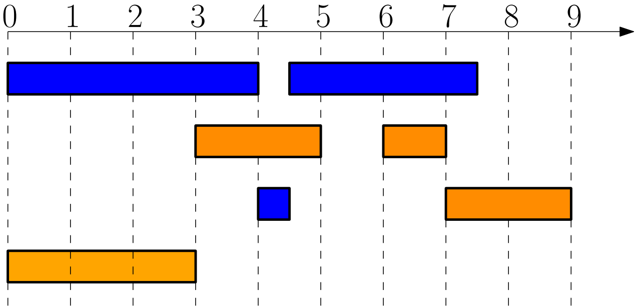

Greedy Algorithm for Interval Partitioning

Lemma It is safe to schedule the job j with the earliest starting time to a earliest-finished machine: There exists an optimum solution where job j with the earliest starting time is scheduled first on the earliest-finished machine that is compatible with all jobs in that machine if applicable; otherwise, it can be scheduled by opening a new machine.

Proof.

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Lemma It is safe to schedule the job j with the earliest starting time to a earliest-finished machine: There exists an optimum solution where job j with the earliest starting time is scheduled first on the earliest-finished machine that is compatible with all jobs in that machine if applicable; otherwise, it can be scheduled by opening a new machine.

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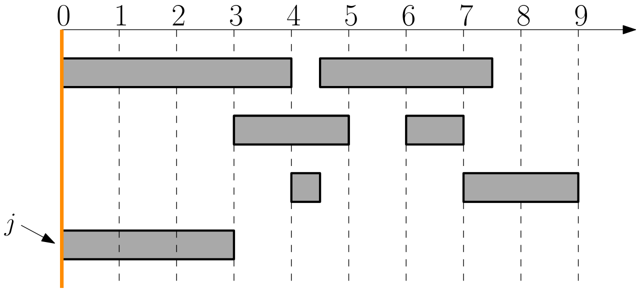
Proof.

- Take an arbitrary optimum solution S
- If it schedules j to the earliest-finished machine i, done
- Otherwise, replace all the jobs scheduled to the earliest-finished machine i in S with j and its subsequent jobs to obtain another optimum schedule S'. □

Greedy Algorithm for Interval Partitioning

- What is the remaining task after we decided to schedule j?
- Is it another instance of interval partitioning problem?



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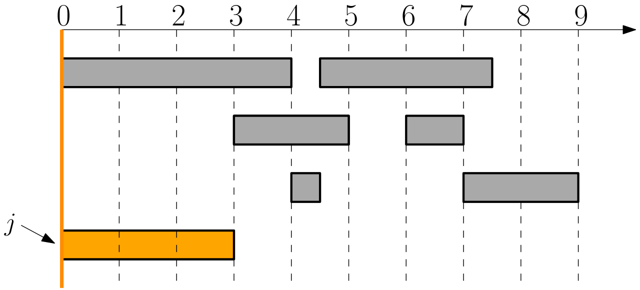

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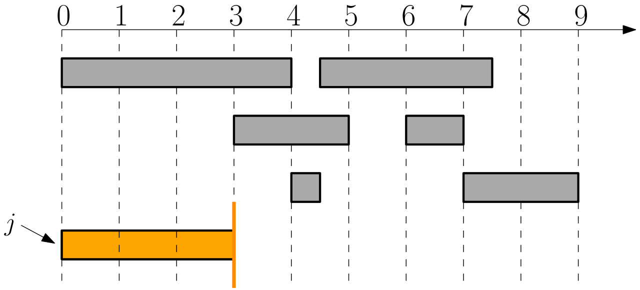

Greedy Algorithm for Interval Partitioning

Partition(s, f, n)

$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \{1\}$, $t_1 = 0$
while $A \neq \emptyset$ **do**
    $j \leftarrow \arg\min_{j' \in A} s_{j'}$, $S_j \leftarrow \{i'\}_{i' \in S, t_{i'} \leq s_j}$
    If $S_j \neq \emptyset$, then schedule j to machine $i \leftarrow \arg\min_{i' \in S_j} t_{i'}$ and $t_i = f_j$
    Otherwise, schedule j to machine $|S| + 1$, $S \leftarrow S \cup \{|S| + 1\}$ and $t_{|S|} = f_j$
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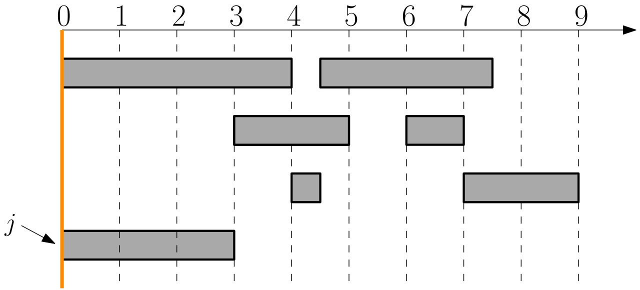
Greedy Algorithm for Interval Partitioning



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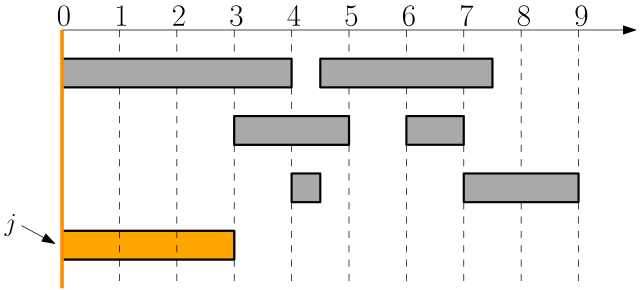

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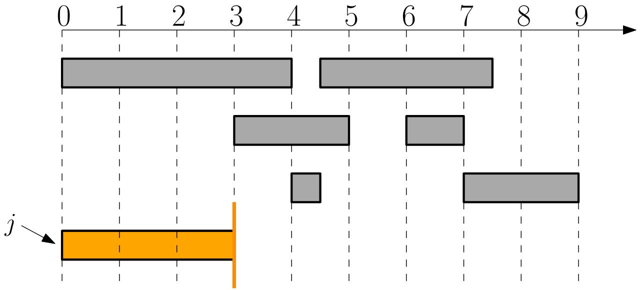

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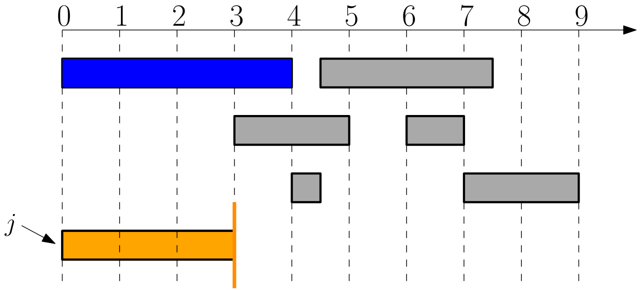

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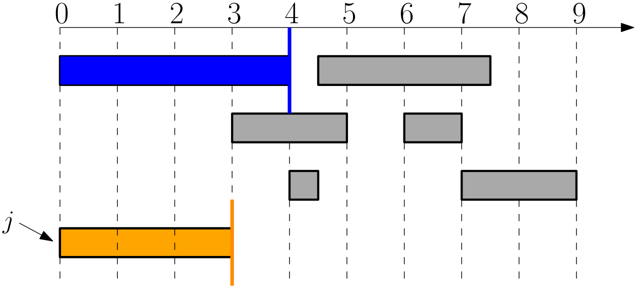

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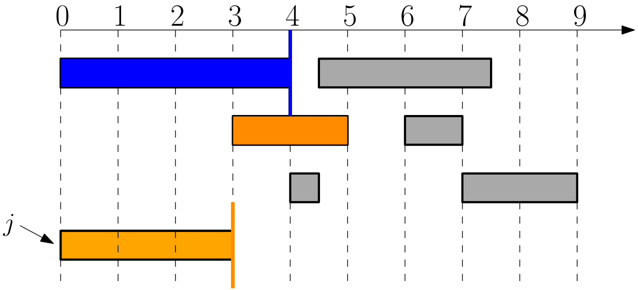

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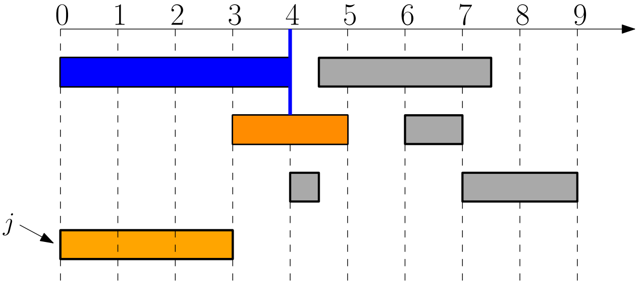

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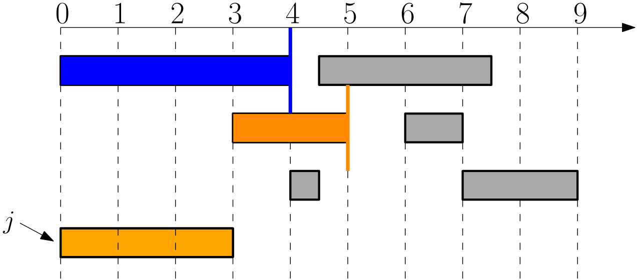

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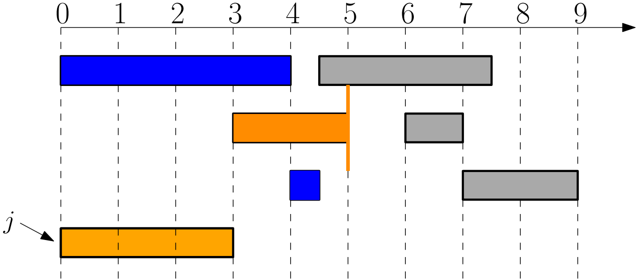

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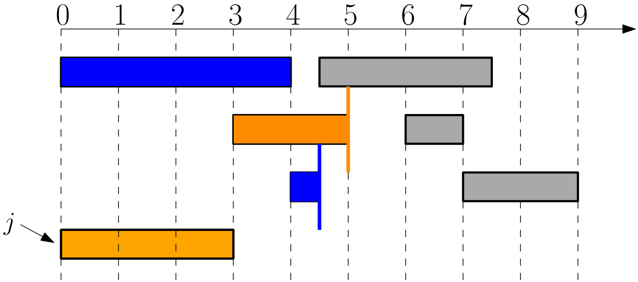

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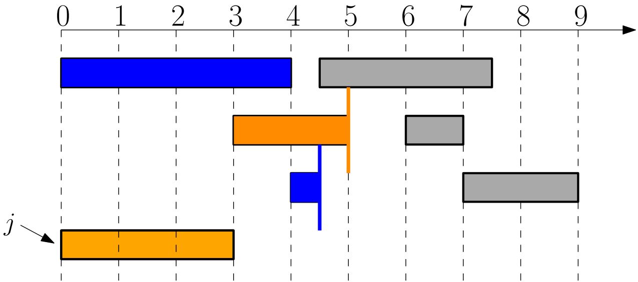

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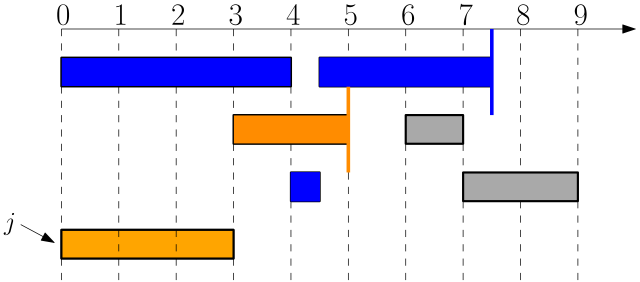

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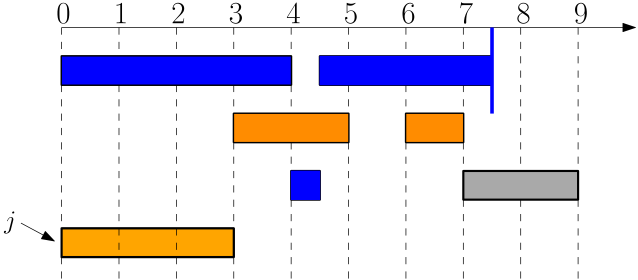

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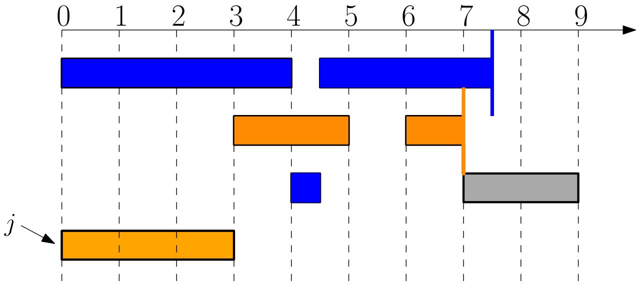

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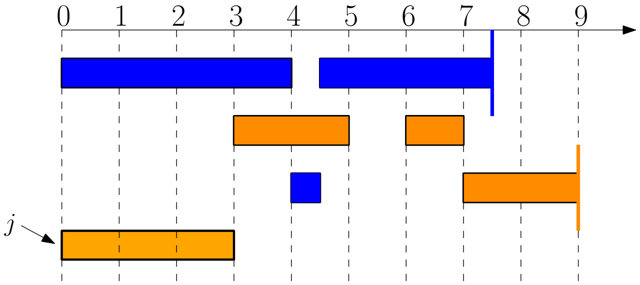

Greedy Algorithm for Interval Partitioning

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Obs. Greedy algorithm never schedules two incompatible jobs in the same machine.

Why "Greedy algorithm" is optimal?

Theorem Greedy algorithm is optimal.

Proof.

- Let d be the number of machines that greedy algorithm used.

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- By the Observation in the previous slide, an optimal solution $\geq d$. Thus the greedy algorithm is optimal.

□

Greedy Algorithm for Interval Partitioning

Partition(s, f, n)

$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \{1\}$, $t_1 = 0$
while $A \neq \emptyset$ **do**
    $j \leftarrow \arg\min_{j' \in A} s_{j'}$, $S_j \leftarrow \{i'\}_{i' \in S, t_{i'} \leq s_j}$
    If $S_j \neq \emptyset$, then schedule j to machine $i \leftarrow \arg\min_{i' \in S_j} t_{i'}$ and $t_i = f_j$
    Otherwise, schedule j to machine $|S| + 1$, $S \leftarrow S \cup \{|S| + 1\}$ and $t_{|S|} = f_j$
return S

Running time of algorithm?

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- Naive implementation: $O(n^2)$ time

Greedy Algorithm for Interval Partitioning

Partition(s, f, n)

```
A ← {1, 2, ⋯ , n}, S ← {1}, t_1 = 0
while A ≠ ∅ do
    j ← arg min_{j'∈A} s_{j'}, S_j ← {i'}_{i'∈S, t_{i'} ≤ s_j}
    If S_j ≠ ∅, then schedule j to machine i ← arg min_{i'∈S_j} t_{i'} and t_i = f_j
    Otherwise, schedule j to machine |S| + 1, S ← S ∪ {|S| + 1} and t_{|S|} = f_j
return S
```

Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time with Priority Queue.

Outline

Offline Caching

- Cache that can store k pages
- Sequence of page requests

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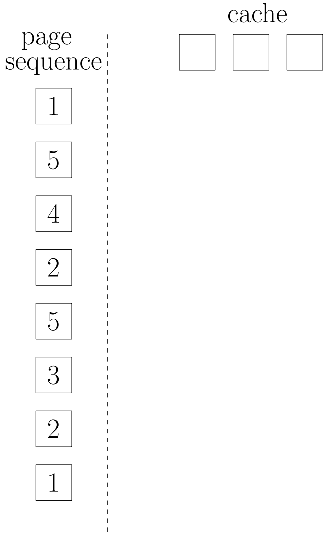

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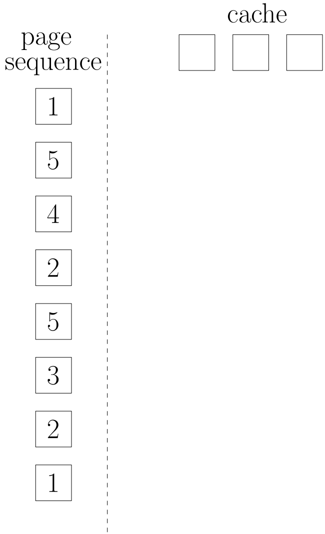

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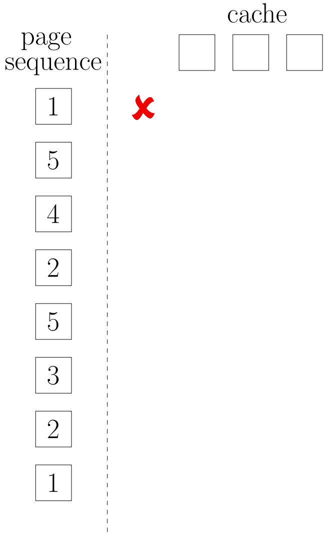

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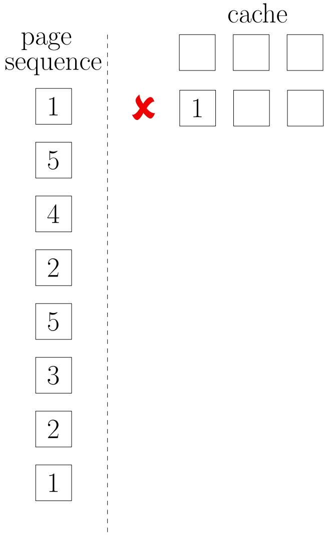

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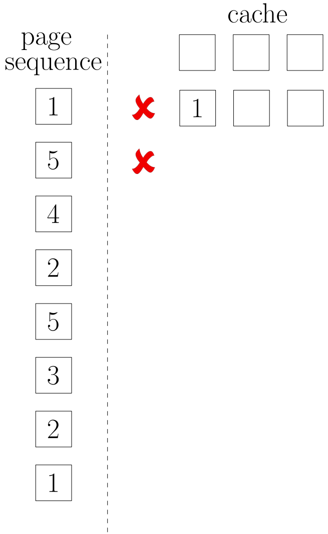

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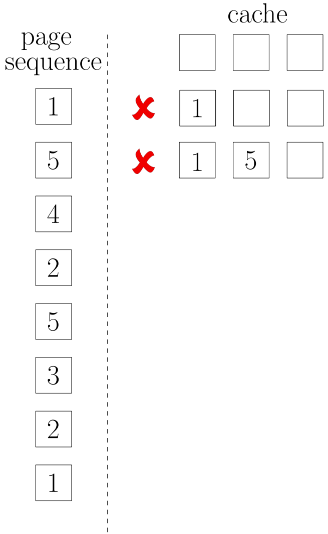

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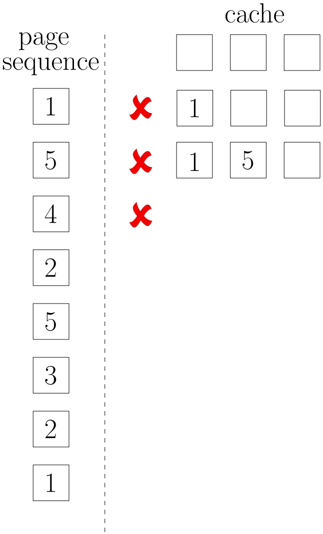

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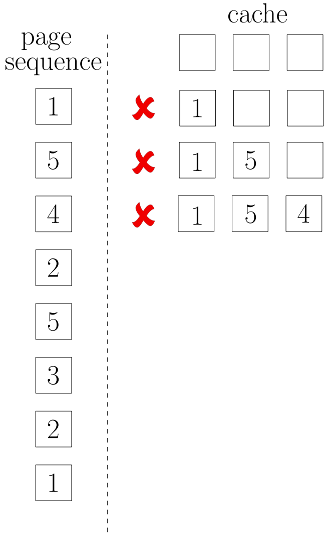

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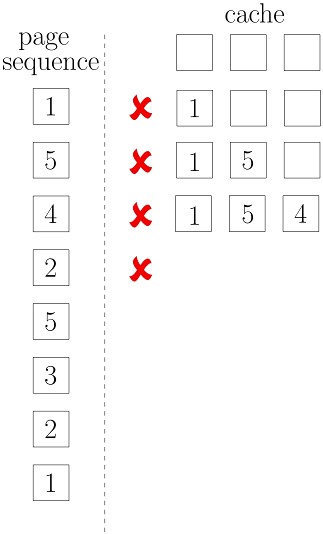

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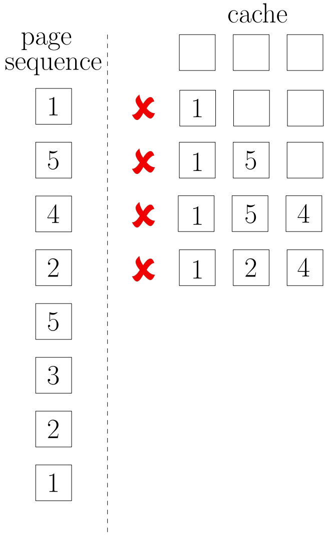

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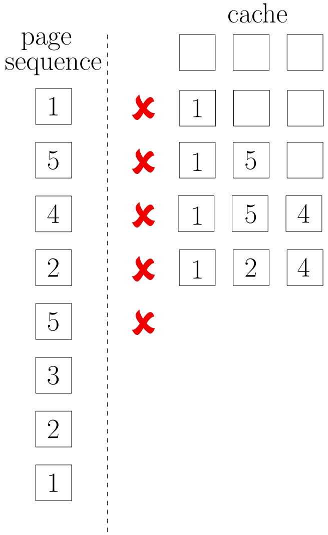

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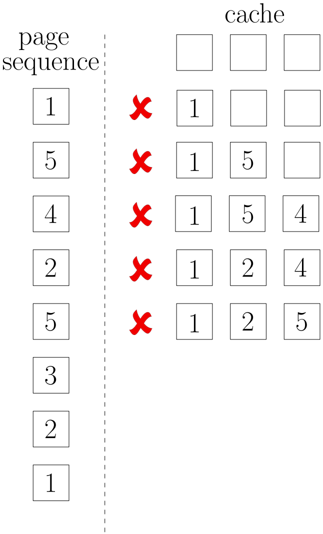

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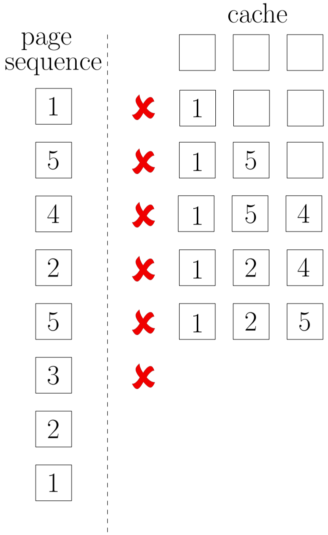

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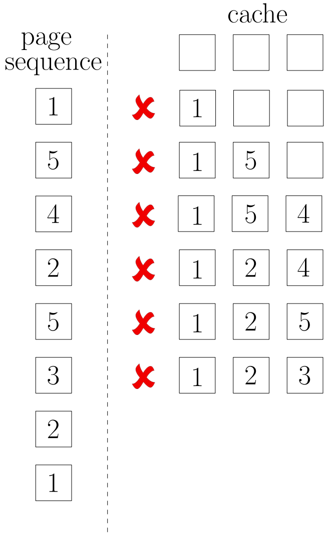

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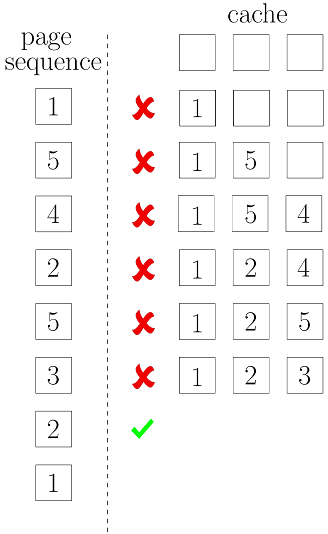

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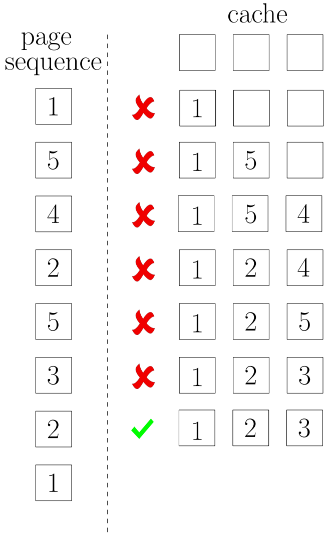

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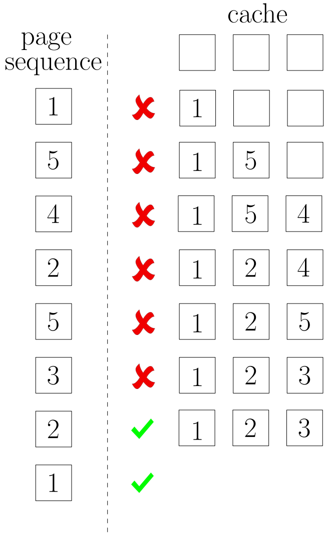

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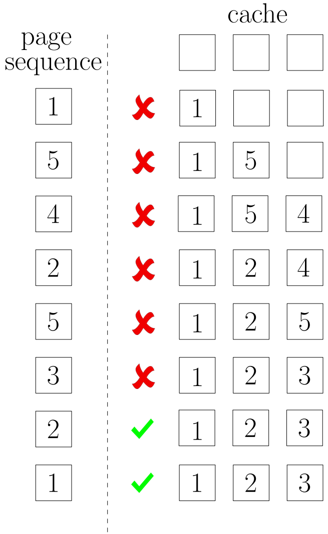

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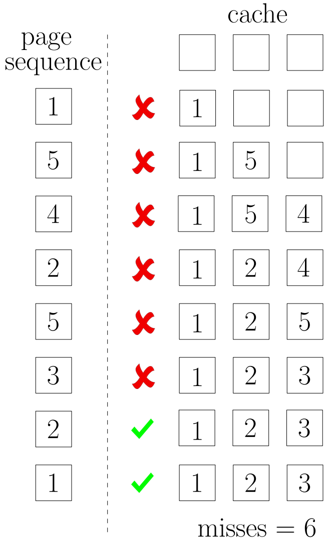

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- Goal: minimize the number of cache misses.