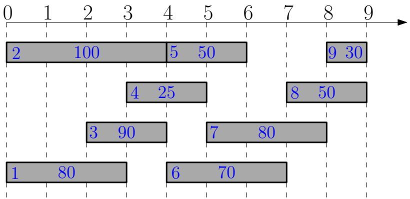
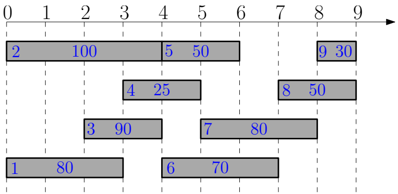
Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

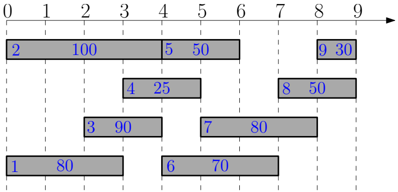
- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

Designing a Dynamic Programming Algorithm



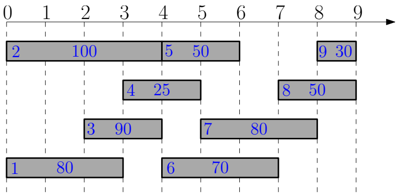
- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

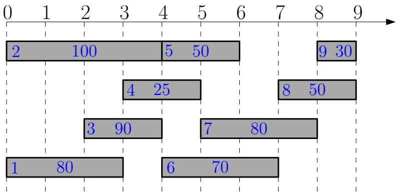
Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

Designing a Dynamic Programming Algorithm

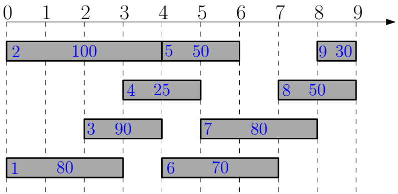


- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Designing a Dynamic Programming Algorithm



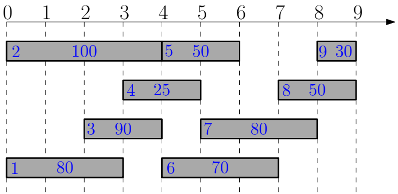
- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

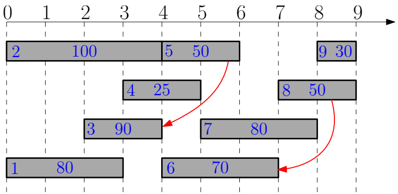
Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, p_i = the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \le s_i$

Designing a Dynamic Programming Algorithm

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \le s_i$

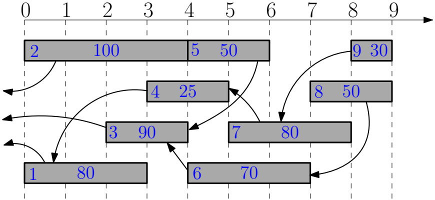
Recursion for $opt[i]$:

$$opt[i] = \max\left\{opt[i-1], v_i + opt[p_i]\right\}$$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

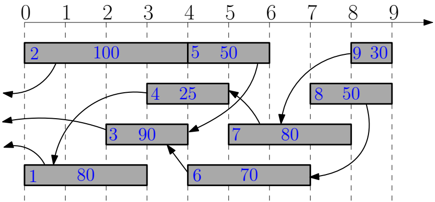


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] =$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

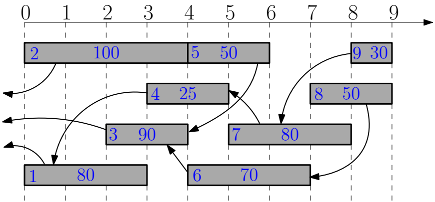


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] =$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

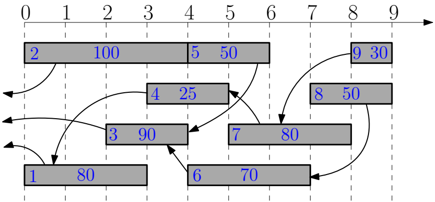


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\}$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

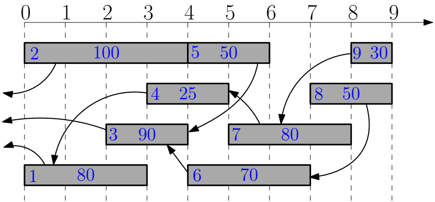


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

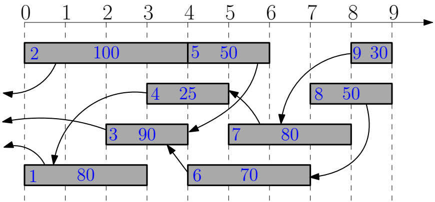


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\}$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

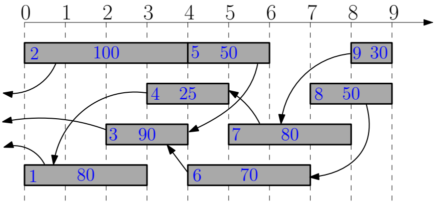


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

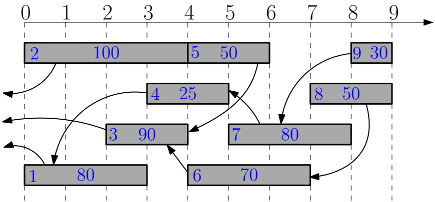


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\}$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

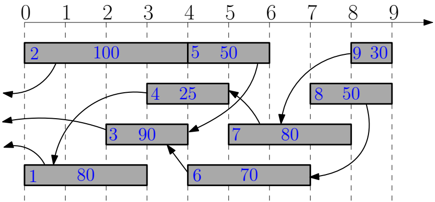


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

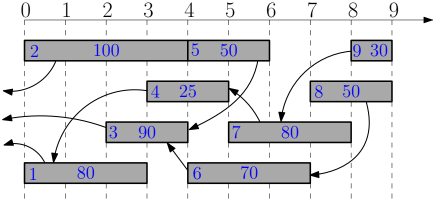


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\}$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

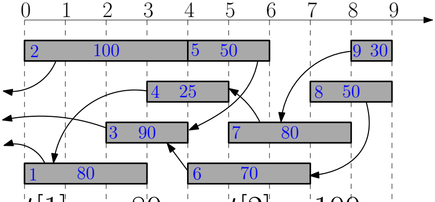


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

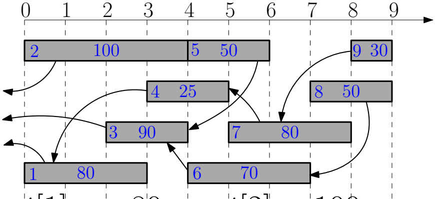


- $opt[0] = 0, \quad opt[1] = 80, \quad opt[2] = 100$
- $opt[3] = 100, \quad opt[4] = 105, \quad opt[5] = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



- $opt[0] = 0$, $opt[1] = 80$, $opt[2] = 100$
- $opt[3] = 100$, $opt[4] = 105$, $opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Dynamic Programming

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$

Dynamic Programming

```
sort jobs by non-decreasing order of finishing times
compute p_1, p_2, ..., p_n
opt[0] ← 0
for i ← 1 to n do
    opt[i] ← max{opt[i − 1], v_i + opt[p_i]}
```

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    if $opt[i-1] \geq v_i + opt[p_i]$ **then**
        $opt[i] \leftarrow opt[i-1]$
    else
        $opt[i] \leftarrow v_i + opt[p_i]$

How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    if $opt[i-1] \geq v_i + opt[p_i]$ **then**
        $opt[i] \leftarrow opt[i-1]$
        $b[i] \leftarrow$ N
    else
        $opt[i] \leftarrow v_i + opt[p_i]$
        $b[i] \leftarrow$ Y

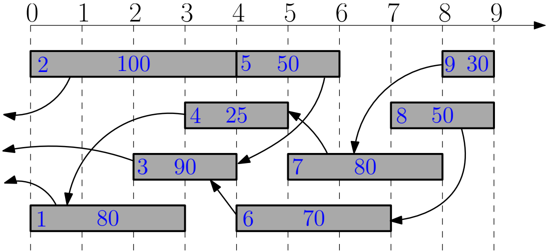
How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    if $opt[i-1] \geq v_i + opt[p_i]$ **then**
        $opt[i] \leftarrow opt[i-1]$
        $b[i] \leftarrow$ N
    else
        $opt[i] \leftarrow v_i + opt[p_i]$
        $b[i] \leftarrow$ Y

$i \leftarrow n, S \leftarrow \emptyset$
while $i \neq 0$ **do**
    if $b[i] =$ N **then**
        $i \leftarrow i - 1$
    else
        $S \leftarrow S \cup \{i\}$
        $i \leftarrow p_i$
return S

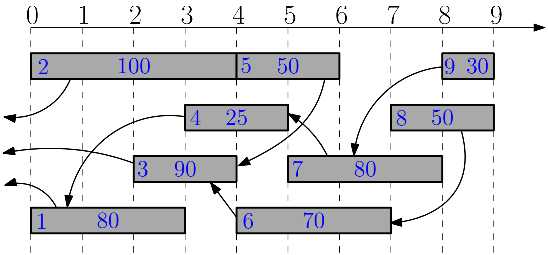
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	
2	100	
3	100	
4	105	
5	150	
6	170	
7	185	
8	220	
9	220	



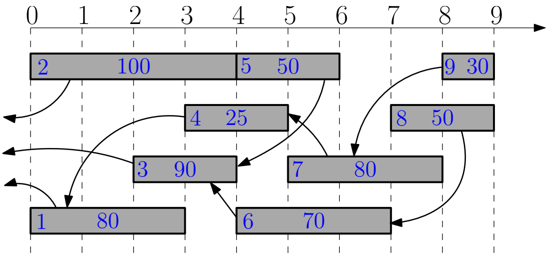
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	⊥
1	80	Y
2	100	
3	100	
4	105	
5	150	
6	170	
7	185	
8	220	
9	220	



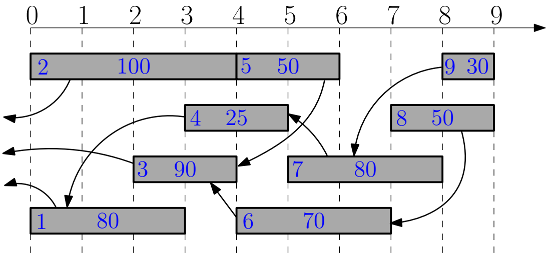
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	
4	105	
5	150	
6	170	
7	185	
8	220	
9	220	



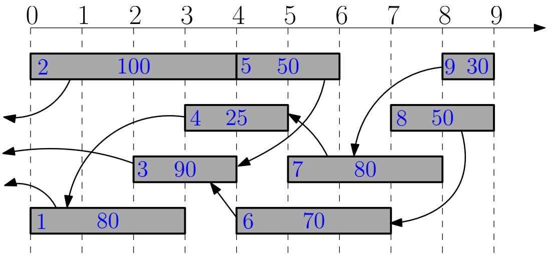
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	
5	150	
6	170	
7	185	
8	220	
9	220	



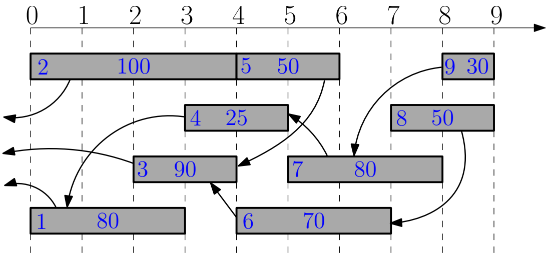
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	
6	170	
7	185	
8	220	
9	220	



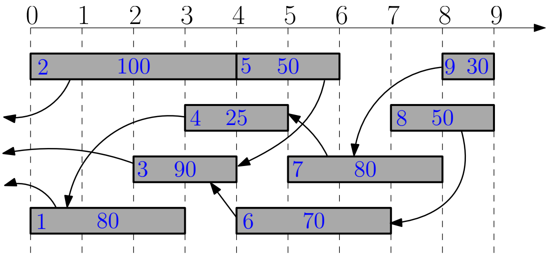
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	
7	185	
8	220	
9	220	



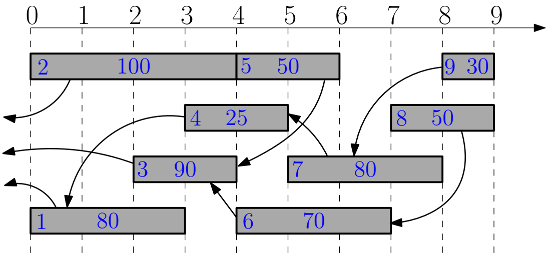
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	
8	220	
9	220	



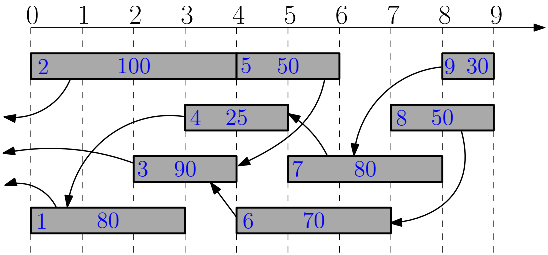
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	
9	220	



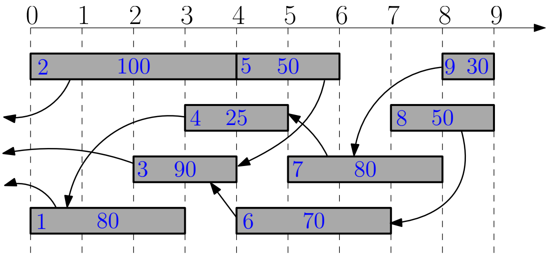
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	



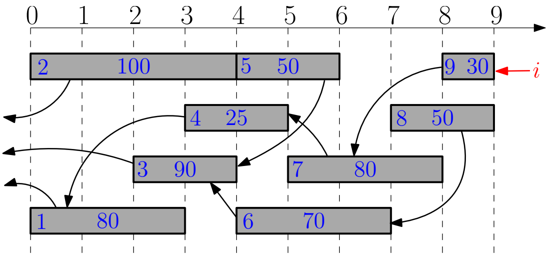
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



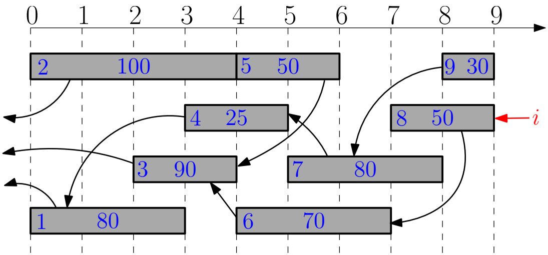
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



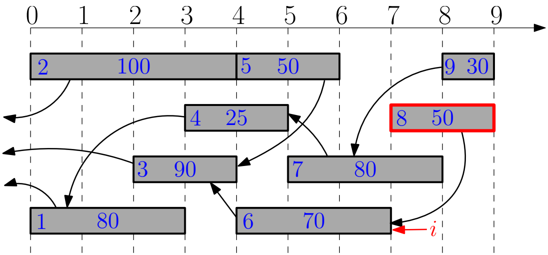
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



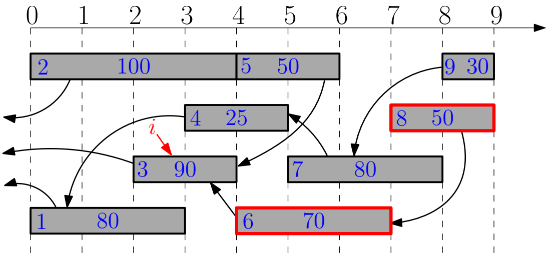
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



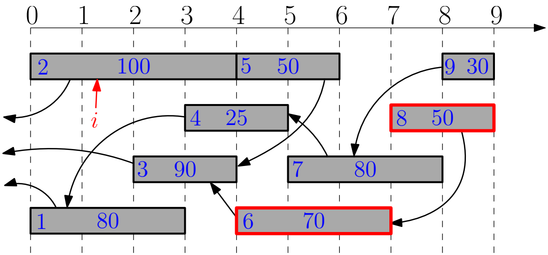
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



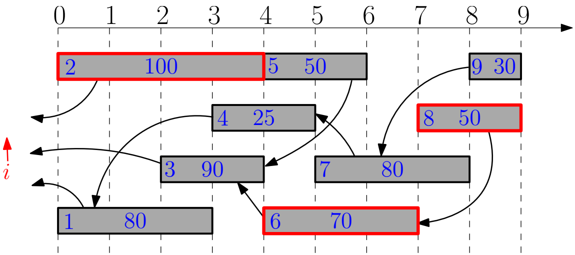
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Outline

1. Weighted Interval Scheduling
2. Subset Sum Problem
3. Knapsack Problem
4. Longest Common Subsequence
    - Longest Common Subsequence in Linear Space
5. Shortest Paths in Directed Acyclic Graphs
6. Matrix Chain Multiplication
7. Optimum Binary Search Tree
8. Summary

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \le W.$$

- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change "non-increasing" to "non-decreasing"?

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change "non-increasing" to "non-decreasing"?

A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

A: $opt[i-1, W' - w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \le W' \end{cases}$$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \le W' \end{cases}$$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ & i > 0, w_i \leq W' \end{cases}$$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W' - w_i] + w_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$